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TRACE FORMULA FOR THE STURM-LIOUVILLE OPERATOR WITH SINGULARITY AT $x=0$

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Abstract

Let $\mu_1, \mu_2, ..., \mu_n, ...$ be the Dirichlet spectrum of the operator $-d^2/dx^2 + q(x)$ acting on $L^2(0,\pi)$. In the special case where $q(x) \equiv 0$, $\mu_n = n^2$. In the [1] and others discovered the asymptotic formula

$$\mu_n = n^2 + \frac{1}{\pi}\int_0^\pi q(x)dx + O(n^{-2})$$

and the trace formula

$$\sum_n [\mu_n - n^2] = \frac{q(0)+q(\pi)}{4},$$

provided that $\int_0^\pi q(x)dx = 0$, where $q(x) \in C^2[0,\pi]$. These are beautiful formulas with many application for example in solving inverse problems. In this work, the above mentioned problem has been studied for a Sturm-Liouville operator with the potential $\frac{A}{x} + \frac{\delta}{x^p} + q(x)$ (A, δ is real and $p \in (1,2)$) singularity at $x = 0$.

Introduction. Let's take L differential operator which is generated by differential expression $\ell(y) = -y'' + \left(\frac{A}{x} + \frac{\delta}{x^p} + q(x)\right)y$ and boundary conditions $y(0) = 0$, $y'(\pi) - Hy(\pi) = 0$, where $A, \delta, H, p \in (1,2)$ real const., $q(x) \in L_2[0,\pi]$ is real valued function.

The domain of operator L is taken as $D(L) = \{y | y' \in AC[0,\pi], y(0) = 0, y'(\pi) - Hy(\pi) = 0\}$. Eigenfunctions and eigenvalues of the operator L are given in the [3].

Now, let $A: H \to H$ be an operator. H is a Hilbert space and as it is separable we can use orthonormal systems. $\{e_n\} \in H$ is an orthonormal system and $\|e_n\|_H = 1$.

If we take $\{e_n\}$ as the eigenfunctions of $A, (Ae_n = \lambda_n e_n)$, we find

$$\sum_{n=1}^\infty < Ae_n, e_n > = \sum_{n=1}^\infty < \lambda_n e_n, e_n > = \sum_{n=1}^\infty \lambda_n < e_n, e_n > = \sum_{n=1}^\infty \lambda_n .$$

If $\sum_{n=1}^\infty \lambda_n < +\infty$ then it is the trace of A, shortly $tr(A) = \sum_{n=1}^\infty \lambda_n$.

As $L: L_2[0,\pi] \to L_2[0,\pi]$ and is a Hilbert space, $\{y_n\}$ can be thought of an orthonormal system. Taking $Ly_n = \lambda_n y_n$, one obtains

$$\sum_{n=1}^{\infty} < Ly_n, y_n > = \sum_{n=1}^{\infty} < \lambda_n y_n, y_n > = \sum_{n=1}^{\infty} \lambda_n < y_n, y_n > = \sum_{n=1}^{\infty} \lambda_n .$$

However, this series is not convergent. Thus, we search for the regularized trace. These traces are important in solving inverse problems with two spectrums.

Calculation of the regularized trace of Sturm-Liouville operator with Coulomb singularities.

Given the differential equation

$$-y'' + \left(\frac{A}{x} + \frac{\delta}{x^p} + q(x)\right) y = \lambda y , \qquad (1)$$

$$y(0) = 0 , \qquad (2)$$

$$y'(\pi) - H_1 y(\pi) = 0 , \qquad (3)$$

$$(y'(\pi) - H_2 y(\pi) = 0), \qquad (3')$$

where $q(x) \in C^2[0,\pi]; A, H_1$ and H_2 real numbers. Let $\{\lambda_n\}$ and $\{\mu_n\}$ respectively represent the spectrum of (1)-(2)-(3) and (1)-(2)-(3'). Then,

$$\lambda_n = (n+1/2)^2 + 2c_1(n+1/2)^{p-1} + \frac{A}{\pi}\ln(n+1/2) + 2c_2 + \frac{2c_3}{(n+1/2)^{3-2p}} + 2c_4\frac{\ln(n+1/2)}{(n+1/2)^{2-p}} +$$

$$+ \frac{2c_5}{(n+1/2)^{2-p}} + 2c_9\frac{\ln(n+1/2)}{(n+1/2)} + \frac{2c_{10}}{(n+1/2)} + \frac{2c_6}{(n+1/2)^{5-3p}} + 2c_7\frac{\ln(n+1/2)}{(n+1/2)^{4-2p}} +$$

$$+ \frac{\gamma_1}{(n+1/2)^{4-2p}} - \frac{A^2\delta c_p}{2\pi}\frac{\ln^2(n+1/2)}{(n+1/2)^{3-p}} + \gamma_2\frac{\ln(n+1/2)}{(n+1/2)^{3-p}} + \frac{\gamma_3}{(n+1/2)^{3-p}} - \frac{A_3}{12\pi}\frac{\ln^3(n+1/2)}{(n+1/2)^2} +$$

$$+ \gamma_4\frac{\ln^2(n+1/2)}{(n+1/2)^2} + \gamma_5\frac{\ln(n+1/2)}{(n+1/2)^2} + \frac{\gamma_6}{(n+1/2)^2} + O\left(\frac{1}{n^{7-4p}}\right),$$

$$\mu_n = (n+1/2)^2 + 2c_1(n+1/2)^{p-1} + \frac{A}{\pi}\ln(n+1/2) + 2c_2' + \frac{2c_3}{(n+1/2)^{3-2p}} + 2c_4\frac{\ln(n+1/2)}{(n+1/2)^{2-p}} +$$

$$+ \frac{2c_5'}{(n+1/2)^{2-p}} + 2c_9\frac{\ln(n+1/2)}{(n+1/2)} + \frac{2c_{10}'}{(n+1/2)} + \frac{2c_6}{(n+1/2)^{5-3p}} + 2c_7\frac{\ln(n+1/2)}{(n+1/2)^{4-2p}} +$$

$$+ \frac{\gamma_1'}{(n+1/2)^{4-2p}} - \frac{A^2\delta c_p}{2\pi}\frac{\ln^2(n+1/2)}{(n+1/2)^{3-p}} + \gamma_2'\frac{\ln(n+1/2)}{(n+1/2)^{3-p}} + \frac{\gamma_3'}{(n+1/2)^{3-p}} - \frac{A^3}{12\pi}\frac{\ln^3(n+1/2)}{(n+1/2)^2} +$$

$$+ \gamma_4'\frac{\ln^2(n+1/2)}{(n+1/2)^2} + \gamma_5'\frac{\ln(n+1/2)}{(n+1/2)^2} + \frac{\gamma_6'}{(n+1/2)^2} + O\left(\frac{1}{n^{7-4p}}\right).$$

Series of $\sum_{n=0}^{\infty}(\lambda_n - \mu_n)$ is called regularized trace of L operator. In this study, the following theorem has been proved about regularized trace of L operator.

Theorem. *If* $q(x) \in C^2[0,\pi]$ *then*

$$\sum_{n=0}^{\infty}(\lambda_n - \mu_n) = \frac{q(\pi) - q(0)}{4} + \frac{A}{4\pi} + \frac{\delta}{4\pi^p} - \frac{H}{2}\int_0^{\pi} q(t)dt + K .$$

Here K *is the known constant,* $H = H_1 - H_2$.

Proof. To find them we notice that the above summation is convergent and $\varphi(x,\lambda)$ is an entire function of $\frac{1}{2}$ order with respect to λ parameter. Weierstrass theorem gives

$$\varphi'(\pi,\lambda)-H\varphi(\pi,\lambda)=A\Phi(\lambda).$$

With $\lambda_1,\lambda_2,...,\lambda_n,...$ are the zeros of $\Phi(\lambda)$ and $\Phi(\lambda)$ is an entire function, $\Phi(\lambda)=$ $=\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right)$ holds.

If $\lambda=-\mu$ and $\mu>0$ is satisfied then we can use the equation $\varphi'(\pi,-\mu)-H\varphi(\pi,-\mu)=A\Phi(-\mu)$. First, we find the asymptotic of $\Phi(-\mu)$

$$\Phi(\lambda)=\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right)\Rightarrow\Phi(-\mu)\prod_{n=0}^{\infty}\left(1+\frac{\mu}{\lambda_n}\right)=\frac{\prod_{n=0}^{\infty}\left(1+\frac{\mu}{\lambda_n}\right)}{\prod_{n=0}^{\infty}\left(1+\frac{\mu}{(n+1/2)^2}\right)}\cosh\sqrt{\mu}\pi=$$

$$=\prod_{n=0}^{\infty}\left(\frac{(n+1/2)^2}{\lambda_n}\right)\prod_{n=0}^{\infty}\left[\frac{\lambda_n+\mu}{\mu+(n+1/2)^2}\right]\cosh\sqrt{\mu}\pi,$$

$\Phi(-\mu)=c\Psi(\mu)\cosh\sqrt{\mu}\pi$, where $c=\prod_{n=0}^{\infty}\left(\frac{(n+1/2)^2}{\lambda_n}\right)$. $\Psi(\mu)$ is given by

$$\Psi(\mu)=\prod_{n=0}^{\infty}\left[\frac{\lambda_n+\mu}{\mu+(n+1/2)^2}\right]=\prod_{n=0}^{\infty}\left[1-\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right]$$

and $\Psi(\mu)$ is convergent.

Therefore,

$$\ln\Psi(\mu)=\sum_{n=0}^{\infty}\ln\left[1-\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right]=\ln\left(1-\frac{1/4-\lambda_0}{\mu+1/4}\right)+\sum_{n=0}^{\infty}\ln\left[1-\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right].$$

Combining these, we find

$$\ln\Psi(\mu)=\frac{\lambda_0-1/4}{\mu}+\sum_{n=0}^{\infty}\ln\left[1-\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right]+O\left(\frac{1}{\mu^2}\right).$$

As $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ is convergent, above series is convergent too. For $n\to\infty$, general sum goes to zero, enabling us to perform the following operators. We obtain

$$\ln\Psi(\mu)=\frac{\lambda_0-1/4}{\mu}-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right)^k+O\left(\frac{1}{\mu^2}\right)=$$

$$=\frac{\lambda_0-1/4}{\mu}-\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\left(\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right)^k+O\left(\frac{1}{\mu^2}\right).$$

Let us take the expression $\left(\frac{(n+1/2)^2-\lambda_n}{\mu+(n+1/2)^2}\right)^k$. Adding and substracting the terms $2c_1(n+1/2)^{p-1}$ and $\frac{A}{\pi}\ln(n+1/2)$.

We obtain

$$\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n-2c_1(n+1/2)^{p-1}-\frac{A}{\pi}\ln(n+1/2)}{\mu+(n+1/2)^2}\right]^k=$$

$$=\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}--2c_1\frac{(n+1/2)^{p-1}}{\mu+(n+1/2)^2}-\right.$$

$$\left.\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right]^k=\sum_{j=0}^{k}\binom{k}{j}\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-j}\times$$

$$\times\left(-2c_1\frac{(n+1/2)^{p-1}}{\mu+(n+1/2)^2}-\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right)^j.$$

Inserting its place in $\ln\Psi(\mu)$, we have

$$\ln\Psi(\mu)=-\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-j}\times$$

$$\times\left(-2c_1\frac{(n+1/2)^{p-1}}{\mu+(n+1/2)^2}-\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right)^j+\frac{\lambda_0-1/4}{\mu}+O\left(\frac{1}{\mu^2}\right)=$$

$$=\frac{\lambda_0-1/4}{\mu}-\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^k-$$

$$-\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k}k\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-1}\times$$

$$\times\left(-2c_1\frac{(n+1/2)^{p-1}}{\mu+(n+1/2)^2}-\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right)-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=2}^{\infty}\binom{k}{j}\times$$

$$\times\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-j}\left(-2c_1\frac{(n+1/2)^{p-1}}{\mu+(n+1/2)^2}-\right.$$

$$-\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\Bigg)^j+O\left(\frac{1}{\mu^2}\right).$$

Now take the expression

$$I=\sum_{n=1}^{\infty}\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-1}\times$$

$$\times\left(-2c_1\frac{(n+1/2)^{p-1}}{\mu+(n+1/2)^2}-\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right).$$

From the eigenvalue expression we know that

$$\left|\lambda_n-(n+1/2)^2-2c_1(n+1/2)^{p-1}-\frac{A}{\pi}\ln(n+1/2)\right|\le a, \text{ (a constant)}$$

$$I\le\sum_{n=1}^{\infty}\frac{a^{k-1}}{\left[\mu+(n+1/2)^2\right]^{k-1}}\left(-2c_1\frac{(n+1/2)^{p-1}}{\mu+(n+1/2)^2}-\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right)=$$

$$=-2c_1a^{k-1}\sum_{n=1}^{\infty}\frac{(n+1/2)^{p-1}}{\left[\mu+(n+1/2)^2\right]^{k-1}}-\frac{A}{\pi}a^{k-1}\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{\left[\mu+(n+1/2)^2\right]^{k-1}}=$$

$$=-\frac{2c_1a^{k-1}}{\mu^k}\sum_{n=1}^{\infty}\frac{(n+1/2)^{p-1}}{\left[1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2\right]^k}-\frac{A}{\pi}\frac{a^{k-1}}{\mu^k}\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{\left[1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2\right]^k}.$$

That is,

$$\ln\Psi(\mu)=-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^k-$$

$$-2c_1\sum_{k=1}^{\infty}\frac{a^{k-1}}{\mu^k}\sum_{n=1}^{\infty}\frac{(n+1/2)^{p-1}}{\left[1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2\right]^k}+\frac{A}{\pi}\sum_{k=1}^{\infty}\frac{a^{k-1}}{\mu^k}\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{\left[1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2\right]^k}-$$

$$-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=2}^{k}\binom{k}{j}\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-j}\times$$

$$\times\left(-2c_1\frac{(n+1/2)^{p-1}}{\mu+(n+1/2)^2}-\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right)^j+\frac{\lambda_0-1/4}{\mu}+O\left(\frac{1}{\mu^2}\right)=$$

$$=-\sum_{n=1}^{\infty}\left(\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right)+\frac{2c_1}{\mu}\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2}+$$

$$+\frac{A}{\pi}\frac{1}{\mu}\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2}-\sum_{n=1}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k}\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln(n+1/2)-\lambda_n}{\mu+(n+1/2)^2}\right]^k+$$

$$+2c_1\sum_{k=2}^{\infty}\frac{a^{k-1}}{\mu^k}\sum_{n=1}^{\infty}\frac{(n+1/2)^{p-1}}{\left[1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2\right]^k}+\frac{A}{\pi}\sum_{k=2}^{\infty}\frac{a^{k-1}}{\mu^k}\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{\left[1+\left(\frac{n+1/2}{\sqrt{\mu}}\right)^2\right]^k}-$$

$$-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=2}^{k}\binom{k}{j}\left[\frac{(n+1/2)^2+2c_1(n+1/2)^{p-1}+\frac{A}{\pi}\ln\left(n+\frac{1}{2}\right)-\lambda_n}{\mu+(n+1/2)^2}\right]^{k-j}\times$$

$$\times\left(-2c_1\frac{(n+1/2)^{p-1}}{\mu+(n+1/2)^2}-\frac{A}{\pi}\frac{\ln(n+1/2)}{\mu+(n+1/2)^2}\right)^j+\frac{\lambda_0-1/4}{\mu}+O\left(\frac{1}{\mu^2}\right).$$

Therefore

$$\ln\Psi(\mu)=\sum_{n=1}^{\infty}\left(\frac{\lambda_n-(n+1/2)^2-2c_1(n+1/2)^{p-1}-\frac{A}{\pi}\ln(n+1/2)}{\mu+(n+1/2)^2}\right)+\frac{c_1\pi}{\sin p\pi/2}\frac{1}{\mu^{2-p/2}}+$$

$$+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+\left[\frac{3A}{2\pi}(1+\ln(3/2))-\frac{3^p c_1}{2^{p-1}p}+\lambda_0-\frac{1}{4}\right]\frac{1}{\mu}+O\left(\frac{1}{\mu^2}\right).$$

Making suitable addings and subtractions to this expression

$$\ln\Psi(\mu)=\sum_{n=1}^{\infty}\left(\lambda_n-(n+1/2)^2-2c_1(n+1/2)^{p-1}-\frac{A}{\pi}\ln(n+1/2)-2c_2\frac{2c_3}{(n+1/2)^{3-2p}}-\right.$$

$$-2c_4\frac{\ln(n+1/2)}{(n+1/2)^{2-p}}-\frac{2c_5}{(n+1/2)^{2-p}}-2c_9\frac{\ln(n+1/2)}{(n+1/2)}-\frac{2c_{10}}{(n+1/2)}-\frac{2c_6}{(n+1/2)^{5-3p}}-$$

$$-2c_7\frac{\ln(n+1/2)}{(n+1/2)^{4-2p}}-\frac{\gamma_1}{(n+1/2)^{4-2p}}+\frac{A^2\delta c_p}{2\pi}\frac{\ln^2(n+1/2)}{(n+1/2)^{3-p}}-\gamma_2\frac{\ln(n+1/2)}{(n+1/2)^{3-p}}-$$

$$\left.-\frac{\gamma_3}{(n+1/2)^{3-p}}+\frac{A^3}{12\pi}\frac{\ln^3(n+1/2)}{(n+1/2)^2}-\gamma_4\frac{\ln^2(n+1/2)}{(n+1/2)^2}-\gamma_5\frac{\ln(n+1/2)}{(n+1/2)^2}\right)\frac{1}{\mu+(n+1/2)^2}+$$

$$+2c_2\sum_{n=1}^{\infty}\frac{1}{\mu+(n+1/2)^2}+2c_3\sum_{n=1}^{\infty}\frac{1}{(n+1/2)^{3-p}\left(\mu+(n+1/2)^2\right)}+$$
$$+2c_4\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^{2-p}\left(\mu+(n+1/2)^2\right)}+2c_5\sum_{n=1}^{\infty}\frac{1}{(n+1/2)^{2-p}\left(\mu+(n+1/2)^2\right)}+$$
$$+2c_9\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)\left(\mu+(n+1/2)^2\right)}+2c_{10}\sum_{n=1}^{\infty}\frac{1}{(n+1/2)\left(\mu+(n+1/2)^2\right)}+$$
$$+2c_6\sum_{n=1}^{\infty}\frac{1}{(n+1/2)^{5-3p}\left(\mu+(n+1/2)^2\right)}+2c_7\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^{4-2p}\left(\mu+(n+1/2)^2\right)}+$$
$$+\gamma_1\sum_{n=1}^{\infty}\frac{1}{(n+1/2)^{4-2p}\left(\mu+(n+1/2)^2\right)}-\frac{A^2\delta c_p}{2\pi}\sum_{n=1}^{\infty}\frac{\ln^2(n+1/2)}{(n+1/2)^{3-p}\left(\mu+(n+1/2)^2\right)}+$$
$$+\gamma_2\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^{3-p}\left(\mu+(n+1/2)^2\right)}+\gamma_3\sum_{n=1}^{\infty}\frac{1}{(n+1/2)^{3-p}\left(\mu+(n+1/2)^2\right)}-$$
$$-\frac{A^3}{12\pi}\sum_{n=1}^{\infty}\frac{\ln^3(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+\gamma_4\sum_{n=1}^{\infty}\frac{\ln^2(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+$$
$$+\gamma_5\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+\frac{c_1\pi}{\sin(p\pi/2)}\frac{1}{\mu^{2-p/2}}+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+$$
$$+\left[\frac{3A}{2\pi}(1+\ln(3/2))-\frac{3^p c_1}{2^{p-1}p}+\lambda_0-\frac{1}{4}\right]\frac{1}{\mu}+O\left(\frac{1}{\mu^2}\right).$$

Lemma. *Let's get* $F(\alpha)=\int_0^{\infty}\frac{(x+1/2)^{\alpha}}{\mu+(x+1/2)^{\alpha}}dx$, *where* $\alpha\in(0,1)$, $\mu>0$ *for* $k=1,2,3$ *integral of* $F^{(k)}(\alpha)$ *is uniform convergent.*

Proof. We shall write $F(\alpha)=\int_0^{\infty}\frac{(x+1/2)^{\alpha}}{\mu+(x+1/2)^2}dx$ as

$$\int_0^{\infty}\frac{(x+1/2)^{\alpha}}{\mu+(x+1/2)^2}dx=\int_0^{1}\frac{(x+1/2)^{\alpha}}{\mu+(x+1/2)^2}dx+\int_1^{\infty}\frac{(x+1/2)^{\alpha}}{\mu+(x+1/2)^2}dx=F_1(\alpha)+F_2(\alpha).$$

The integral which is given by function of $F_1(\alpha)$ is uniform convergent with respect to comparison theorem.

Moreover, due to differentiable theorem of integrals corresponding to parameter, integrals which express with function of $F_1^{(k)}(\alpha)$, $(k=1,2,3)$ are also uniform convergent.

Now, we shall show that uniform convergent α parameter of by $F_2(\alpha)$, differentiable to this parameter and integrals expressed by functions of $F_2^{(k)}(\alpha)$, $(k=1,2,3)$ are uniform convergent.

Since for $\forall\alpha\in(0,1)$ and $\forall x\in(0,+\infty)$ $f(x,\alpha)=\frac{(x+1/2)^{\alpha}}{\mu+(x+1/2)^2}>0$ and function of $F_2(\alpha)$ is continuous, integral given by $F_2(\alpha)$ is uniform convergent corresponding to α parameter.

Moreover, for $k = 1,2,3$ since integrals expressed by functions of $F_2^{(k)}(\alpha)$ also satisfy same conditions, these integrals are also uniform convergent.

If above lemma is used and necessary processes are done,

$$\ln\Psi(\mu) = \frac{1}{\mu}\sum_{n=0}^{\infty}\left(\lambda_n - (n+1/2)^2 - 2c_1(n+1/2)^{p-1} - \frac{A}{\pi}\ln(n+1/2) - 2c_2\frac{2c_3}{(n+1/2)^{3-2p}} - \right.$$

$$-2c_4\frac{\ln(n+1/2)}{(n+1/2)^{2-p}} - \frac{2c_5}{(n+1/2)^{2-p}} - 2c_9\frac{\ln(n+1/2)}{(n+1/2)} - \frac{2c_{10}}{(n+1/2)} - \frac{2c_6}{(n+1/2)^{5-3p}} -$$

$$-2c_7\frac{\ln(n+1/2)}{(n+1/2)^{4-2p}} - \frac{\gamma_1}{(n+1/2)^{4-2p}} + \frac{A^2\delta c_p}{2\pi}\frac{\ln_2(n+1/2)}{(n+1/2)^{3-p}} - \gamma_2\frac{\ln(n+1/2)}{(n+1/2)^{3-p}} -$$

$$\left. -\frac{\gamma_3}{(n+1/2)^{3-p}} + \frac{A^3}{12\pi}\frac{\ln_3(n+1/2)}{(n+1/2)^2} - \gamma_4\frac{\ln^2(n+1/2)}{(n+1/2)^2} - \gamma_5\frac{\ln(n+1/2)}{(n+1/2)^2}\right) -$$

$$-\frac{1}{\mu}\left(\lambda_0 - \frac{1}{4} - c_1 2^{2-p} + \frac{A}{\pi}\ln 2 - 2c_2 - 2^{4-2p}c_3 + 2^{3-p}c_4\ln 2 - 2^{3-p}c_5 + 4c_9\ln 2 - \right.$$

$$-4c_{10} - 2^{6-3p}c_6 + 2^{5-2p}c_7\ln 2 - 2^{4-2p}\gamma_1 + \frac{A^2\delta c_p}{\pi}2^{2-p}\ln^2 2 + 2^{3-p}\gamma_2\ln 2 - 2^{3-p}\gamma_3 -$$

$$\left. -\frac{A^3}{3\pi}\ln^3 2 - 4\gamma_4\ln^2 2 + 4\gamma_5\ln 2\right) - \frac{1}{\mu}\sum_{n=1}^{\infty}\left(\lambda_n - (n+1/2)^2 - 2c_1(n+1/2)^{p-1} - \frac{A}{\pi}\ln(n+1/2) - \right.$$

$$-2c_2\frac{2c_3}{(n+1/2)^{3-2p}} - 2c_4\frac{\ln(n+1/2)}{(n+1/2)^{2-p}} - \frac{2c_5}{(n+1/2)^{2-p}} - 2c_9\frac{\ln(n+1/2)}{(n+1/2)} - \frac{2c_{10}}{(n+1/2)} -$$

$$-\frac{2c_6}{(n+1/2)^{5-3p}} - 2c_7\frac{\ln(n+1/2)}{(n+1/2)^{4-2p}} - \frac{\gamma_1}{(n+1/2)^{4-2p}} + \frac{A^2\delta c_p}{2\pi}\frac{\ln^2(n+1/2)}{(n+1/2)^{3-p}} - \gamma_2\frac{\ln(n+1/2)}{(n+1/2)^{3-p}} -$$

$$\left. -\frac{\gamma_3}{(n+1/2)^{3-p}} + \frac{A^3}{12\pi}\frac{\ln^3(n+1/2)}{(n+1/2)^2} - \gamma_4\frac{\ln^2(n+1/2)}{(n+1/2)^2} - \gamma_5\frac{\ln(n+1/2)}{(n+1/2)^2}\right)\frac{(n+1/2)^2}{\mu + (n+1/2)^2} +$$

$$+2c_2\sum_{n=1}^{\infty}\frac{1}{\mu + (n+1/2)^2} + 2c_3\sum_{n=1}^{\infty}\frac{1}{(n+1/2)^{3-p}\left(\mu + (n+1/2)^2\right)} +$$

$$+2c_4\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^{2-p}\left(\mu + (n+1/2)^2\right)} + 2c_5\sum_{n=1}^{\infty}\frac{1}{(n+1/2)^{2-p}\left(\mu + (n+1/2)^2\right)} +$$

$$+2c_9\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)\left(\mu + (n+1/2)^2\right)} + 2c_{10}\sum_{n=1}^{\infty}\frac{1}{(n+1/2)\left(\mu + (n+1/2)^2\right)} +$$

$$+2c_6\sum_{n=1}^{\infty}\frac{1}{(n+1/2)^{5-3p}\left(\mu + (n+1/2)^2\right)} + 2c_7\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^{4-2p}\left(\mu + (n+1/2)^2\right)} +$$

$$+\gamma_1\sum_{n=1}^{\infty}\frac{1}{(n+1/2)^{4-2p}\left(\mu + (n+1/2)^2\right)} - \frac{A^2\delta c_p}{2\pi}\sum_{n=1}^{\infty}\frac{\ln^2(n+1/2)}{(n+1/2)^{3-p}\left(\mu + (n+1/2)^2\right)} +$$

$$+\gamma_2\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^{3-p}\left(\mu + (n+1/2)^2\right)} + \gamma_3\sum_{n=1}^{\infty}\frac{1}{(n+1/2)^{3-p}\left(\mu + (n+1/2)^2\right)} -$$

$$-\frac{A^3}{12\pi}\sum_{n=1}^{\infty}\frac{\ln^3(n+1/2)}{(n+1/2)^2\left(\mu + (n+1/2)^2\right)} + \gamma_4\sum_{n=1}^{\infty}\frac{\ln^2(n+1/2)}{(n+1/2)^2\left(\mu + (n+1/2)^2\right)} +$$

$$+\gamma_5\sum_{n=1}^{\infty}\frac{\ln(n+1/2)}{(n+1/2)^2\left(\mu+(n+1/2)^2\right)}+\frac{c_1\pi}{\sin p\pi/2}\frac{1}{\mu^{2-p/2}}+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+$$

$$+\left[\frac{3A}{2\pi}(1+\ln(3/2))-\frac{3^p c_1}{2^{p-1}p}+\lambda_0-\frac{1}{4}\right]\frac{1}{\mu}+O\left(\frac{1}{\mu^2}\right).$$

We also obtain the asymptotic expression of $\varphi'(\pi,-\mu)-H\varphi(\pi,-\mu)$

$$\varphi'(\pi,-\mu)-H\varphi(\pi,-\mu)=$$

$$=\frac{e^{\sqrt{\mu}\pi}}{2}\left\{1+\frac{\delta}{2}(\alpha_2-2\alpha_4)\frac{1}{\mu^{\frac{2-p}{2}}}+\frac{A}{2}\frac{\ln\sqrt{\mu}}{\sqrt{\mu}}+\left[\frac{1}{2}\int_0^{\pi}q(t)dt-A\alpha_3+\frac{\delta}{2(1-p)\pi^{p-1}}+\frac{A\ln\pi}{2}+\right.\right.$$

$$\left.+\frac{A\alpha_1}{2}-H\right]\frac{1}{\sqrt{\mu}}-\frac{\delta H}{2}(\alpha_2-2\alpha_4)\frac{1}{\mu^{\frac{3-p}{2}}}-\frac{AH\ln\sqrt{\mu}}{2\mu}+\left[\frac{q(\pi)-q(0)}{4}+\frac{A}{4\pi}+\frac{\delta}{4\pi^p}-\right.$$

$$\left.\left.-\frac{H}{2}\int_0^{\pi}q(t)dt-\frac{H\delta}{2(1-p)\pi^{p-1}}+AH\alpha_3-\frac{AH\ln\pi}{2}-\frac{AH\alpha_1}{2}\right]\frac{1}{\mu}\right\}+O\left(\frac{1}{\mu^{3/2}}\right)$$

the trace is obtained from the equation $\varphi'(\pi,-\mu)-H\varphi(\pi,-\mu)=A\Phi(\lambda)$.

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PARTIAL OPERATOR-DIFFERENTIAL EQUATIONS IN SPACES OF TYPE S

Abstract

In the paper partial differential equations with unbounded operator coefficients in space of type S. Theorem on existence and asymptotic representation of solution of the given equation are proved. In conclusion examples of application of obtained abstract results to solution of Newmann problem for elliptic equations in unbounded cylindrical domains are give.

Differential equations with operator coefficients in Banach spaces were studied in many papers [1-3]. Results of these papers have a series of applications in theory of boundary-value problems. The main applications concern with the question of behaviour of solutions in infinitive cylinder [4] or in the neighborhood of conical point of he boundary [5].

In comparison with ordinary operator-differential equations few papers are devoted to investigation of solvability of partial operator-differential equations in Hilbert spaces. We'll note papers [6,7], where solvability of boundary-value problems for some classes of partial operators-differential equations in functional spaces are studied.

Partial differential equations with operator coefficients were investigated in a number of papers [8-16]. Theorems on single valued, normal and Fredholm solvability, on asymptotic behavior and smoothness of solution in Hilbert spaces such equations also have applications in theory of boundary-value problems.

1. Let H be a Hilbert space. Consider differential operator

$$L(D)u = \sum_{|\alpha|\le m} A_\alpha D^\alpha u ,$$

where A_α are bounded operators: $H \to H$.

Denote by $R(\lambda)=\left[\sum_{|\alpha|\le m}(i\lambda)^\alpha A_\alpha\right]^{-1}$ the operator $H \to H$. At first we'll define the space $S(R^n;H)$. Let $u(x)\in H$ at any $x\in R^n$ and $D^\alpha u$ exists at all α . Suppose that $(1+|x|)^p \|D^\alpha u\|_H \le C_{\alpha,p}$ take place at any p, α .

Then we'll say that $u(x)\in S(R^n;H)$. If it's clear what H we are talking about then we'll use denotation $u(x)\in S(R^n)$. Let continuous mapping $T: u(x)\to H$ $u(x)\leftarrow S(R^n)$ be given. The set of all such T we'll denote by S^i .

Theorem 1. *If $R(\lambda)$ exists at all real λ, infinity differentiable function λ is such that*

$$\|D^\alpha R(\lambda)\|_H \in C_\alpha (1+|\lambda|)^s ,$$

where s depends on α , then $L(D)$ maps λ on

$$\|D^\alpha R(\lambda)\|_H \le C_\alpha (1+|\lambda|)^s .$$

Proof. If $u(x)\in S(R^n)$, then it's clear that $Lu\in S(R^n)$. Let $f\in S(R^n)$. We have to prove that there exists such $u(x)\in S(R^n)$, that $Lu=f$. Consider $\tilde{u}(\lambda)=R(\lambda)\tilde{f}(\lambda)$,