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THE GENERALIZED SOLVABILITY OF THE DIRICHLET PROBLEM FOR NON-UNIFORMLY DEGENERATING ELLIPTIC EQUATIONS OF THE SECOND ORDER

Abstract

The Dirichlet problem is considered for non-uniformly degenerating elliptic equations of the second order of divergent structure. The inequalities of Friedrichs type is proved and the conditions are found at which this problem is uniquely generalized solvable in anisotropic Sobolev space.

Introduction. Let E_n be an n dimensional Euclidean space of the points $x=(x_1,...,x_n)$, $n\ge 3, D$ be a bounded domain situated in $E_n, \partial D$ be a boundary of the domain D. Let's consider in D the first boundary value problem

$$Lu=\sum_{i,,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)=f(x)+\sum_{i=1}^{n}\frac{\partial f^i(x)}{\partial x_i}, \tag{1}$$

$$u\big|_{\partial D}=\varphi, \tag{2}$$

where $\|a_{ij}(x)\|$ is a real symmetric matrix with measurable in D elements, moreover for all $x\in D, \zeta\in E_n$ it is fulfilled the condition

$$\gamma\sum_{i=1}^{n}\lambda_i(x)\zeta_i^2\le\sum_{i,j=1}^{n}a_{ij}\zeta_i\zeta_j\le\gamma^{-1}\sum_{i=1}^{n}\lambda_i(x)\zeta_i^2. \tag{3}$$

Here $\gamma\in(0,1]$ is a constant, and the functions $\lambda_i(x)\, i=1,...,n$ almost everywhere in D are finite and positive. The aim of the given paper is to find the conditions on functions $\lambda_i(x), f(x), f^i(x)$ and $\varphi(x)\,(i=1,...,n)$, at which the problem (1), (2) is uniformly generalized solvable in corresponding anisotropic Sobolev weight space. Let's denote that in the case of uniformly elliptic equations we can find the proof of analogous fact in [1-3]. Concerning the equations with uniform degeneration then let's note in this case papers [4-5]. For elliptic equations with weak (logarithmic) non-uniform degeneration the generalized solvability of Dirichlet problem is established in [6]. Let's note also paper [7-8], where the first boundary value problem is investigated for one class of elliptic equations with non-uniform power degeneration at a point.

1^0. The inequality of Friedrichs type. Let's agree in some notations and determinations. Let $W_{2,\lambda}^1(D)$ be a Banach space of the function $u(x)$, given on D with finite norm

$$\|u\|_{W_{2,\lambda}^1(D)}=\left(\int_D\left(u^2(x)+\sum_{i=1}^{n}\lambda_i(x)u_i^2\right)dx\right)^{1/2},$$

where $\lambda=(\lambda_1,...,\lambda_n)$, $u_i=\partial u/\partial x_i$, $(i=1,..,n)$. On the functions $\lambda_i(x)$ $(i=1,...,n)$ we put the next conditions

$$\lambda_i(x)\in L_1(D),\ \lambda_i^{-1}(x)\in L_{n/2}(D), i=1,..,n. \tag{4}$$

Let's denote by $C_0^\infty(D)$ a space of infinite differentiable finite in D functions. Let further $\overset{\circ}{W}{}^1_{2,\lambda}(D)$ be a sub-space $W^1_{2,\lambda}(D)$, compact set in which is the totality of all functions $u(x)\in C_0^\infty$, and $L_{2,\lambda_i^{-1}}(D)$ be the Banach spaces of the functions $u(x)$ given on D, with finite norm

$$\|u\|_{L_2,\lambda_i^{-1}(D)}=\left(\int_D u^2(x)\lambda_i^{-1}(x)dx\right)^{1/2},\ i=1,...,n.$$

We'll understand the boundary condition (2) in the next meaning. Let the function $\Phi(x)\in W^1_{2,\lambda}(D)$ be given. We'll say that " $u|_{\partial D}=\Phi|_{\partial D}$ ", if

$$(u-\Phi)\in \overset{\circ}{W}{}^1_{2,\lambda}(D). \tag{5}$$

The function $u(x)\in W^1_{2,\lambda}(D)$ is the generalized solution of the Dirichlet problem (1), (5), if (5) is fulfilled and for any function $v(x)\in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ the following integral identity is true.

$$\int_D\sum_{i,j=1}^n a_{ij}(x)u_jV_idx=\int_D\left(-fv+\sum_{i=1}^n f^iv_i\right)dx. \tag{6}$$

In further everywhere the notation $C(.,.,.)$ means that the positive constant C depends only on the contents of brackets

Theorem 1. *Let the conditions (4) be fulfilled. Then for any function* $u(x)\in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ *the inequality*

$$\int_D u^2(x)dx\le c_1(\lambda,n,D)\int_D\sum_{i=1}^n\lambda_i(x)u_i^2(x)dx \tag{7}$$

holds.

Proof. It is evident that it is sufficient to prove (7) for the functions $u(x)\in C_0^\infty(D)$. We'll use the following classic embedding theorem (see for ex. [2]); for any function $u(x)\in C_0^\infty(D)$ when $1<p<n$ the following inequality is true

$$\|u\|_{L_{\frac{np}{n-p}}(D)}\le C_2(n,p,D)\|\nabla u\|_{L_p(D)}. \tag{8}$$

Supposing in (8) $p=\frac{2n}{n+2}$, we get

$$\|u\|_{L_2(D)}\le c_2(n,D)\|\nabla u\|_{L_{\frac{2n}{n+2}}(D)}. \tag{9}$$

But on the other hand

$$\|\nabla u\|_{L_{\frac{2n}{n+2}}(D)}=\left(\int_D\sum_{i=1}^n|u_i|^{\frac{2n}{n+2}}dx\right)^{\frac{n+2}{2n}}=$$

$$=\left(\sum_{i=1}^{n}\int_D \lambda_i^{-q}(x)\lambda_i^{q}(x)|u_i|^{\frac{2n}{n+2}}dx\right)^{\frac{n+2}{2n}}\le$$

$$\le\left[\sum_{i=1}^{n}\left(\int_D \lambda_i^{-qs'}(x)dx\right)^{1/s'}\left(\int_D \lambda_i^{qs}(x)|u_i|^{\frac{2ns}{n+2}}dx\right)^{1/s}\right]^{\frac{n+2}{2n}},$$

where $q>0$ and $s>1$ are arbitrary, and $s'=s/(s-1)$. Let's suppose now $s=(n+2)/n$, $q=n/(n+2)$. Then $s'=(n+2)/2$ and therefore

$$\|\nabla u\|_{L_{\frac{2n}{n+2}}(D)}\le C_3^{\frac{n+2}{2n}}\cdot n^{\frac{1}{n}}\left(\int_D\sum_{i=1}^{n}\lambda_i(x)u_i^2dx\right)^{\frac{1}{2}}. \qquad (10)$$

By view of the condition (4)

$$\left(\int_D \lambda_i^{-n/2}(x)dx\right)^{\frac{2}{n+2}}\le c_3(\lambda,n,D);\ i=1,...,n.$$

Thus from (10) we conclude

$$\|\nabla u\|_{L_{\frac{2n}{n+2}}(D)}\le C_3^{\frac{n+2}{2n}}n^{\frac{1}{n}}\left(\int_D\sum_{i=1}^{n}\lambda_i(x)u_i^2dx\right)^{1/2}. \qquad (11)$$

Now it is sufficient to suppose $C_1=C_2^2C_3^{\frac{n+2}{n}}n^{\frac{2}{n}}$. The required estimation (7) follows from (9) and (11).

The theorem is proved.

2^0. The generalized solvability of Dirichlet problem.

Theorem 2. *Let in domain D the coefficient of the operator L satisfying the conditions (3), (4) be determined. Then the first boundary value problem (1), (5) is uniquely generalized solvable at the space $W_{2,\lambda}^1(D)$ for every $\Phi\in W_{2,\lambda}^1(D), f\in L_2(D)$, $f^i\in L_{2,\lambda_i^{-1}}(D);\ i=1,...,n$.*

Proof. Let's consider first of all the case $\Phi\equiv 0$. Let's introduce for $u,v\in W_{2,\lambda}^1(D)$ bilinear form

$$B(u,v)=\int_D\sum_{i,j=1}^{n}\alpha_{ij}(x)u_jv_jdx.$$

Let's show that this form is bounded, i.e.

$$|B(u,v)|\le C_4(\gamma)\|u\|\|v\|, \qquad (12)$$

where $\|\cdot\|=\|\cdot\|_{W_{2,\lambda}^1(D)}$.

We'll use the following inequality

$$\left|\sum_{i,j=1}^{n}b_{ij}\zeta_i\eta_i\right|\le\left(\sum_{i,j=1}^{n}b_{ij}\zeta_i\zeta_j\right)^{1/2}\left(\sum_{i,j=1}^{n}b_{ij}\eta_i\eta_j\right)^{1/2}, \qquad (13)$$

true for all $\zeta,\eta \in E_n$ if only the quadratic form generated by the matrix $\|b_{ij}\|$, is positively determined (see for ex. [1]). We've for $u,v \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ subject to (13) and (3)

$$|B(u,v)| \le \int_D \left| \sum_{i,j=1}^n \alpha_{ij}(x) u_j v_i \right| dx \le$$

$$\le \int_D \left(\sum_{i,j=1}^n \alpha_{ij}(x) u_i u_j \right)^{1/2} \left(\sum_{i,j=1}^n \alpha_{ij}(x) v_i v_j \right)^{1/2} dx \le$$

$$\le \gamma^{-1} \int_D \left(\sum_{i=1}^n \lambda_i(x) u_i^2 \right)^{1/2} \left(\sum_{i=1}^n \lambda_i(x) v_i^2 \right)^{1/2} dx \le$$

$$\le \gamma^{-1} \left(\int_D \sum_{i=1}^n \lambda_i(x) u_i^2 dx \right)^{1/2} \left(\int_D \sum_{i=1}^n \lambda_i(x) \vartheta_i^2 dx \right)^{1/2} \le$$

$$\le \gamma^{-1} \|u\| \cdot \|v\|.$$

By the same token the estimation (12) is proved. Let's show now that the form $B(u,v)$ is coercive, i.e.

$$B(u,u) \ge C_5(\gamma,\lambda,n,D) \|u\|^2 \qquad (14)$$

for any function $u(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$. Subject to (3) and theorem 1 we have

$$B|u,u| \ge \gamma \int_D \sum_{i=1}^n \lambda_i(x) u_i^2 dx \ge$$

$$\ge \frac{\gamma}{2} \int_D \sum_{i=1}^n \lambda_i(x) u_i^2 dx + \frac{\gamma}{2} \int_D \sum_{i=1}^n \lambda_i(x) u_i^2 dx \ge$$

$$\ge \frac{\gamma}{2} \int_D \sum_{i=1}^n \lambda_i(x) u_i^2 dx + \frac{\gamma}{2c_1} \int_D u^2 dx \ge C_5 \|u\|^2,$$

where $c_5 = \min\left\{ \frac{\gamma}{2}, \frac{\gamma}{2c_1} \right\}$. So, the inequality (14) is proved. Let's consider for $v \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ the functional

$$H(v) = \int_D \left[-fv + \sum_{i=1}^n f^i \frac{\partial v}{\partial x_i} \right] dx,$$

where $f \in L_2(D)$, $f^i \in L_{2,\lambda_i^{-1}}(D)$; $i = 1,\dots,n$. Let's show that it is bounded. We have

$$|H(v)| \le \int_D |f| \cdot |v| dx + \sum_{i=1}^n \int_D |f^i| \cdot \left| \frac{\partial v}{\partial x_i} \right| dx \le$$

$$\le \left(\int_D f^2 dx \right)^{1/2} \left(\int_D v^2 dx \right)^{1/2} +$$

$$+\sum_{i=1}^{n}\left(\int_D (f^i)^2\lambda_i^{-1}(x)dx\right)^{1/2}\left(\int_D \lambda_i(x)\vartheta_i^2 dx\right)^{1/2}. \qquad (15)$$

Let's denote $\max\left\{\|f\|_{L_2(D)},\|f^1\|_{L_2,\lambda_1^{-1}(D)},...,\|f^n\|_{L_2,\lambda_n^{-1}(D)}\right\}$ by c_6. Then from (15) it follows that

$$|H(v)|\le 2c_6\|v\|.$$

Now it is sufficient to apply the Lax-Milgram theorem [3] and the statement of the theorem is proved when $\Phi\equiv 0$.

Let now $\Phi\not\equiv 0$. Let's consider the function $w(x)=u(x)-\Phi(x)$. It is clear that $w\in \overset{\circ}{W}{}^1_{2,\lambda}(D)$. At this bilinear form $B(w,v)$ has the following form

$$B(w,v)=\int_D\left[-fv+\sum_{i=1}^{n}f^i v_i\right]dx-\int_D\sum_{i,j=1}^{n}a_{ij}(x)\Phi_j v_i dx,$$

and to complete the proof it is sufficient to show that

$$F_i(x)=\sum_{j=1}^{n}a_{ij}(x)\Phi_j\in L_{2,\lambda_i^{-1}}(D);\ i=1,...,n.$$

In other words we must prove that

$$\frac{F_i(x)}{\sqrt{\lambda_i}}\in L_2(D);\ i=1,...,n.$$

The last is equivalent to the next: if $h_1(x),...,h_n(x)$ are arbitrary functions from $L_2(D)$, then

$$\left|\int_D\sum_{i=1}^{n}\frac{F_i(x)}{\sqrt{\lambda_i(x)}}h_i(x)dx\right|<\infty. \qquad (16)$$

We have

$$\left|\int_D\sum_{i=1}^{n}\frac{F_i(x)}{\lambda_i(x)}h_i(x)dx\right|\le\int_D\left(\sum_{i,j=1}^{n}a_{ij}(x)\Phi_i\Phi_j\right)^{1/2}\times$$

$$\times\left(\sum_{i,j=1}^{n}a_{ij}(x)\frac{h_i}{\sqrt{\lambda_i(x)}}\cdot\frac{h_j}{\sqrt{\lambda_j(x)}}\right)^{1/2}dx\le$$

$$\le\gamma^{-1}\int_D\left(\sum_{i=1}^{n}\lambda_i(x)\Phi_i^2\right)^{1/2}\left(\sum_{i=1}^{n}\lambda_i(x)\frac{h_i^2}{\lambda_i(x)}\right)^{\frac{1}{2}}dx\le$$

$$\le\gamma^{-1}\|\Phi\|_{W^{-1}_{2,\lambda}(D)}(mesD)^{1/2}\sum_{i=1}^{n}\|h_i\|_{L_2(D)}<\infty,$$

and the inequality (16), and together with it the theorem is also proved.

3^0. The estimation of generalized solvability of Dirichlet problem.

Theorem 3. *Let with respect to the coefficients of the operator L in domain D the conditions (3), (4) be fulfilled. Then at any $f\in L_2(D),\Phi\in W^1_{2,\lambda}(D)$, $f^i\in L_{2,\lambda_i^{-1}}(D)$*

$(i=1,...,n)$ *for generalized solution* $u(x)$ *of the problem (1), (5) the following estimation is true*

$$\|u\|_{W^1_{2,\lambda}(D)} \le c_7(\gamma,\lambda,n,D)\left(\|f\|_{L_2(D)} + \|\Phi\|_{W^1_{2,\lambda}(D)} + \sum_{i=1}^{n}\left\|f^i\right\|_{L_{2,\lambda_i^{-1}}(D)}\right). \tag{17}$$

Proof. In accord to the above theorem the generalized solution $u(x)$ of the problem (1), (5) exists. Let's suppose in integral identity (6) $v=u-\Phi$. We'll get

$$\begin{gathered}\int\limits_{D}\sum_{i,j=1}^{n}a_{ij}(x)u_iu_jdx = \int\limits_{D}\sum_{i,j=1}^{n}a_{ij}(x)u_j\Phi_idx - \int\limits_{D}fudx + \int\limits_{D}f\Phi dx + \\ +\sum_{i=1}^{n}\int\limits_{D}f^iu_idx - \sum_{i=1}^{n}\int\limits_{D}f^i\Phi_idx = j_1+j_2+j_3+j_4+j_5.\end{gathered} \tag{18}$$

Further we have

$$\int\limits_{D}\sum_{i,j=1}^{n}a_{ij}(x)u_iu_jdx \ge \gamma\int\limits_{D}\sum_{i=1}^{n}\lambda_i(x)u_i^2(x)dx, \tag{19}$$

$$\begin{gathered}j_1 \le \gamma^{-1}\int\limits_{D}\left(\sum_{i=1}^{n}\lambda_i(x)u_i^2\right)^{1/2}\left(\sum_{i=1}^{n}\lambda_i(x)\Phi_i^2\right)^{1/2}dx \le \\ \le \frac{\gamma^{-1}\varepsilon}{2}\int\limits_{D}\sum_{i=1}^{n}\lambda_i(x)u_i^2dx + \frac{\gamma^{-1}}{2\varepsilon}\int\limits_{D}\sum_{i=1}^{n}\lambda_i(x)\Phi_i^2dx.\end{gathered} \tag{20}$$

Analogously we get

$$j_2 \le \frac{\varepsilon}{2}\int\limits_{D}u^2dx + \frac{1}{2\varepsilon}\int\limits_{D}f^2dx, \tag{21}$$

$$j_3 \le \frac{1}{2}\int\limits_{D}f^2dx + \frac{1}{2}\int\limits_{D}\Phi^2dx, \tag{22}$$

$$j_4 \le \varepsilon_2\int\limits_{D}\sum_{i=1}^{n}\lambda_i(x)u_i^2dx + \frac{1}{2\varepsilon}\int\limits_{D}\sum_{i=1}^{n}\frac{\left(f^i\right)^2}{\lambda_i(x)}dx, \tag{23}$$

$$j_5 \le \frac{1}{2}\int\limits_{D}\sum_{i=1}^{n}\frac{\left(f^i\right)^2}{\lambda_i(x)}dx + \frac{1}{2}\int\limits_{D}\sum_{i=1}^{n}\lambda_i(x)\Phi_i^2dx. \tag{24}$$

Using (19)-(24) in (18) we conclude

$$\begin{gathered}\gamma\int\limits_{D}\sum_{i=1}^{n}\lambda_i(x)u_i^2dx \le \left(\frac{\gamma^{-1}\varepsilon}{2}+\frac{\varepsilon}{2}\right)\int\limits_{D}\sum_{i=1}^{n}\lambda_i(x)u_i^2dx + \\ +\frac{\varepsilon}{2}\int\limits_{D}u^2dx + \left(\frac{1}{2\varepsilon}+\frac{1}{2}\right)\int\limits_{D}f^2dx + \frac{1}{2}\int\limits_{D}\Phi^2dx + \\ +\left(\frac{\gamma^{-1}}{2\varepsilon}+\frac{1}{2}\right)\int\limits_{D}\sum_{i=1}^{n}\lambda_i(x)\Phi_i^2dx + \left(\frac{1}{2\varepsilon}+\frac{1}{2}\right)\int\limits_{D}\sum_{i=1}^{n}\frac{\left(f^i\right)^2}{\lambda_i(x)}dx.\end{gathered} \tag{25}$$

On the other hand

$$\gamma\int_D\sum_{i=1}^n\lambda_i(x)u_i^2dx=\frac{\gamma}{2}\int_D\sum_{i=1}^n\lambda_i(x)u_i^2dx+$$
$$+\frac{\gamma}{2}\int_D\sum_{i=1}^n\lambda_i(x)(u-\Phi)_i^2dx+\frac{\gamma}{2}\int_D\sum_{i=1}^n\lambda_i(x)\Phi_i^2dx+$$
$$+\gamma\int_D\sum_{i=1}^n\lambda_i(x)(u-\Phi)_i\Phi_i dx. \tag{26}$$

Since $u-\Phi\in\overset{\circ}{W}{}^1_{2,\lambda}(D)$, then according to theorem 1

$$\gamma\int_D\sum_{i=1}^n\lambda_i(x)(u-\Phi)_i^2dx\ge\frac{\gamma}{2c_1}\int_D(u-\Phi)^2dx\ge$$
$$\ge\frac{\gamma}{2c_1}\int_D u^2dx+\frac{\gamma}{2c_1}\int_D\Phi^2dx-\frac{\gamma\varepsilon}{2c_1}\int_D u^2dx-\frac{\gamma}{2\varepsilon c_1}\int_D\Phi^2dx. \tag{27}$$

Besides

$$\gamma\int_D\sum_{i=1}^n\lambda_i(x)(u-\Phi)_i\Phi_i dx\ge-\frac{\gamma\varepsilon}{2}\int_D\sum_{i=1}^n\lambda_i(x)(u-\Phi)_i^2dx-$$
$$-\frac{\gamma}{2\varepsilon}\int_D\sum_{i=1}^n\lambda_i(x)\Phi_i^2dx\ge-\gamma\varepsilon\int_D\sum_{i=1}^n\lambda_i(x)u_i^2dx- \tag{28}$$
$$-\left(\frac{\gamma}{2\varepsilon}+\gamma\varepsilon\right)\int_D\sum_{i=1}^n\lambda_i(x)\Phi_i^2dx.$$

Now allowing for (26)-(28) in (25) we get

$$\frac{\gamma}{2}\int_D\sum_{i=1}^n\lambda_i(x)u_i^2dx+\frac{\gamma}{2c_1}\int_D u^2dx\le$$
$$\le\left(\frac{\gamma^{-1}\varepsilon}{2}+\frac{\varepsilon}{2}+\gamma\varepsilon\right)\int_D\sum_{i=1}^n\lambda_i(x)u_i^2dx+$$
$$+\left(\frac{\varepsilon}{2}+\frac{\gamma\varepsilon}{2c_1}\right)\int_D u^2dx+\left(\frac{1}{2\varepsilon}+\frac{1}{2}\right)\int_D f^2dx+ \tag{29}$$
$$+\left(\frac{1}{2}+\frac{\gamma}{2c_1\varepsilon}\right)\int_D\Phi^2dx+\left(\frac{\gamma^{-1}}{2\varepsilon}+\frac{1}{2}+\gamma\varepsilon+\frac{\gamma}{2\varepsilon}\right)\int_D\sum_{i=1}^n\lambda_i(x)\Phi_i^2dx+$$
$$+\left(\frac{1}{2\varepsilon}+\frac{1}{2}\right)\int_D\sum_{i=1}^n\frac{(f^i)^2}{\lambda_i(x)}dx.$$

Let's choose ε_1 and ε_2 from the equalities

$$\frac{\gamma^{-1}\varepsilon_1}{2}+\frac{\varepsilon_1}{2}+\gamma\varepsilon_1=\frac{\gamma}{4},\frac{\varepsilon_2}{2}+\frac{\gamma\varepsilon_2}{2c_1}=\frac{\gamma}{4c_1}$$

correspondingly. Let's fix $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$. Then from (29) it follows

$$\|u\|^2_{W^1_{2,\lambda}(D)}\le c_8(\gamma,\lambda,n,D)\left(\|f\|^2_{L_2(D)}+\|\Phi\|^2_{W^1_{2,\lambda}(d)}+\sum_{i=1}^n\|f\|^2_{L_{2,\lambda_i^{-1}(D)}}\right),$$

Whence the required estimation (17) follows.

The theorem is proved.

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Received February 5, 2001; Revised July 23, 2001.
Translated Mamedova V.I.

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TRACE FORMULA FOR THE STURM-LIOUVILLE OPERATOR WITH SINGULARITY AT $x=0$

Keywords: Trace formula, spectrum.
AMS subject classifications. 35J10, 35J40, 35K35.

Abstract

Let $\mu_1,\mu_2,...,\mu_n,...$ *be the Dirichlet spectrum of the operator* $-d^2/dx^2+q(x)$ *acting on* $L^2(0,\pi)$. *In the special case where* $q(x)\equiv 0$, $\mu_n=n^2$. *In the [1] and others discovered the asymptotic formula*

$$\mu_n=n^2+\frac{1}{\pi}\int_0^\pi q(x)dx+O\left(n^{-2}\right)$$

and the trace formula

$$\sum_n\left[\mu_n-n^2\right]=\frac{q(0)+q(\pi)}{4},$$

provided that $\int_0^\pi q(x)dx=0$, *where* $q(x)\in C^2[0,\pi]$. *These are beautiful formulas with many application for example in solving inverse problems. In this work, the above mentioned problem has been studied for a Sturm-Liouville operator with the potential* $\frac{A}{x}+\frac{\delta}{x^p}+q(x)$ *(*A,δ *is real and* $p\in(1,2)$*) singularity at* $x=0$.

Introduction. Let's take L differential operator which is generated by differential expression $\ell(y)=-y''+\left(\frac{A}{x}+\frac{\delta}{x^p}+q(x)\right)y$ and boundary conditions $y(0)=0$, $y'(\pi)-Hy(\pi)=0$, where $A,\delta,H,p\in(1,2)$ real const., $q(x)\in L_2[0,\pi]$ is real valued function.

The domain of operator L is taken as $D(L)=\{y|y'\in AC[0,\pi],y(0)=0,$ $y'(\pi)-Hy(\pi)=0\}$. Eigenfunctions and eigenvalues of the operator L are given in the [3].

Now, let $A:H\to H$ be an operator. H is a Hilbert space and as it is separable we can use orthonormal systems. $\{e_n\}\in H$ is an orthonormal system and $\|e_n\|_H=1$.

If we take $\{e_n\}$ as the eigenfunctions of $A,(Ae_n=\lambda_n e_n)$, we find

$$\sum_{n=1}^{\infty}<Ae_n,e_n>=\sum_{n=1}^{\infty}<\lambda_n e_n,e_n>=\sum_{n=1}^{\infty}\lambda_n<e_n,e_n>=\sum_{n=1}^{\infty}\lambda_n .$$

If $\sum_{n=1}^{\infty}\lambda_n<+\infty$ then it is the trace of A, shortly $tr(A)=\sum_{n=1}^{\infty}\lambda_n$.

As $L:L_2[0,\pi]\to L_2[0,\pi]$and is a Hilbert space, $\{y_n\}$ can be thought of an orthonormal system. Taking $Ly_n=\lambda_n y_n$, one obtains