MATHEMATICS

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A THEOREM ON THE OSCILLATION OF SOLUTIONS OF NONUNIFORMLY DEGENERATE PARABOLIC EQUATIONS OF SECOND ORDER

Abstract

A class of second order parabolic equations of non-divergent structure with non-uniform power degeneration is considered in the paper. A theorem on the oscillation of solutions of these equations, and inner a priori estimate of Hölder's norm are proved.

Let $\mathbf{E}_n$ and $\mathbf{R}_{n+1}$ be an Euclidean space of points $x=(x_1,...,x_n)$ and $(x,t)=(x_1,...,x_n,t)$ respectively, $D\subset\mathbf{R}_{n+1}$ be a bounded domain with a parabolic boundary $\Gamma(D)$, $(0,0)\in D$. In D consider a parabolic equation

$$\mathcal{L}u=\sum_{i,j=1}^{n}a_{ij}(x,t)u_{ij}+\sum_{i=1}^{n}b_i(x,t)u_i+c(x,t)u-u_t=0\,, \qquad (1)$$

where $\|a_{ij}(x,t)\|$ is a real symmetric matrix, moreover for all $(x,t)\in D$ and $\xi\in\mathbf{E}_n$

$$\gamma\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le\gamma^{-1}\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2\,. \qquad (2)$$

Here $\gamma\in(0,1]$ is a constant, $\lambda_i(x,t)=\left(|x|_\alpha+\sqrt{|t|}\right)^{\alpha_i}$, $|x|_\alpha=\sum_{i=1}^{n}|x_i|^{\frac{2}{2+\alpha_i}}$, $-2<\alpha_i\le 2$, $u_i=\frac{\partial u}{\partial x_i}$, $u_{ij}=\frac{\partial^2 u}{\partial x_i\partial x_j}$, $i,j=1,...,n$. As to minor coefficients of the equation (1) we shall suppose that they are real and moreover, for all $(x,t)\in D$

$$|b_i(x,t)|\le b_0\,,\quad (i=1,...,n);\ -c_0\le c(x,t)\le 0\,, \qquad (3)$$

where b_0 and c_0 are positive constants.

The aim of the paper is to prove an inner a priori estimate of Holder's norm of solutions of the equation (1). Note that the analogous result for second order parabolic equations of non-divergent structure has been obtained in papers [1-6]. As to non-uniformly degenerating parabolic equations without minor terms with $\alpha_i\in[0,2]$, $i=1,...,n$ we indicate papers [7-8]. A more complete review of results on this theme one can find in monographs [9-11].

At first agree on some denotations and definitions. We shall denote by $\mathcal{E}_R^{x^0}(k)$ an ellipsoid $\left\{x:\sum_{i=1}^{n}\frac{(x_i-x_i^0)^2}{R^{\alpha_i}}<(kR)^2\right\}$, $C_{x^0;k;R}^{t^1,t^2}$ - a cylinder $\mathcal{E}_R^{x^0}(k)\times(t^1,t^2)$. Here $R>0$, $k>0, t^1<t^2, x^0\in\mathbf{E}_n$. The function $u(x,t)\in C^{2,1}(D)$ is called $\mathcal{L}$-subparabolic in D, if $\mathcal{L}u(x,t)\ge 0$, for $(x,t)\in D$. The function $u(x,t)$ is called $\mathcal{L}$ super parabolic in D, if $-u(x,t)$ $\mathcal{L}$-subparabolic in D.

Let $\mathbf{C}^1=C_{0;1;R}^{-bR^2,0}$, $\mathbf{C}^2=C_{0;17;R}^{-4bR^2,0}$, $\mathbf{C}^3=\mathbf{C}^2\setminus\overline{\mathbf{C}}^1$, where the constant $b\in(0,1)$ will be chosen later.

For $s>0$, $\beta>0$ introduce the function

$$G_R^{(s,\beta)}(x,t)=\begin{cases} t^{-s}\exp\left[-\left(\sum_{i=1}^{n}x_i^2/R^{\alpha_i}\right)\Big/4\beta t\right], & \text{if } t>0,\\ 0, & \text{if } t\le 0.\end{cases}$$

The measure μ, determined in B-set $H\subset \mathbf{C}^3$ is called (s,β,R)- admissible, if

$$\int_H G_R^{(s,\beta)}(x-y,t-\tau)\,d\mu(y,\tau)\le 1 \text{ for } (x,t)\notin H.$$

The number $p_R^{(s,\beta)}(H)=\sup\mu(H)$, where the least upper bound is taken on all (s,β,R)- admissible measures is called a parabolic (s,β,R)- capacity of the set H.

Denote $\alpha=(\alpha_1,...,\alpha_n)$, $|\alpha|=\alpha_1+\cdots\alpha_n$, $\alpha^+=\max\{\alpha_1,...,\alpha_n\}$, $\alpha^-=$ $=\min\{\alpha_1,...,\alpha_n\}$. The record $C(\cdots)$ means that a positive constant C depends only on the content of parenthesis. Moreover, $C(\mathcal{L})=C(\gamma,\alpha,b_0,c_0)$.

The following three lemmas are analogies of corresponding statements from [9].

Lemma 1. *If the conditions (2)-(3) are fulfilled with respect to the coefficients of the operator* $\mathcal{L}$, *then there exist constants* $s(b,\mathcal{L},n)$, $\beta(b,\mathcal{L},n)$ *and* $R_0(b,\mathcal{L},n)$ *such that for* $R\le R_0$, $(y,\tau)\in\mathbf{C}^3$

$$\mathcal{L}_{(x,t)}G_R^{(s,\beta)}(x-y,t-\tau)\ge \textit{for } (x,t)\in\mathbf{C}^3\setminus\{(y,\tau)\}. \tag{4}$$

Proof. We have for $t>\tau$ allowing for the conditions (2)-(3)

$$\mathcal{L}G_R^{(s,\beta)}=G_R^{(s,\beta)}\left\{\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{(x_i-y_i)(x_j-y_j)}{4\beta^2R^{\alpha_i+\alpha_j}(t-\tau)^2}-\frac{1}{2\beta(t-\tau)}\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{R^{\alpha_i}}-\right.$$

$$\left.-\frac{1}{2\beta(t-\tau)}\sum_{i=1}^{n}b_i(x,t)\frac{(x_i-y_i)}{R^{\alpha_i}}+c(x,t)+\frac{s}{t-\tau}-\frac{\rho^2}{4\beta(t-\tau)^2}\right\}\ge$$

$$\ge G_R^{(s,\beta)}\left\{\frac{\gamma}{4\beta^2(t-\tau)^2}\sum_{i=1}^{n}\frac{\lambda_i(x,t)}{R^{\alpha_i}}\frac{(x_i-y_i)^2}{R^{\alpha_i}}-\frac{\gamma^{-1}}{2\beta(t-\tau)}\sum_{i=1}^{n}\frac{\lambda_i(x,t)}{R^{\alpha_i}}-\right.$$

$$\left.-\frac{b_0}{2\beta(t-\tau)}\sum_{i=1}^{n}\frac{|x_i-y_i|}{R^{\alpha_i}}-C_0+\frac{s}{t-\tau}-\frac{\rho^2}{4\beta(t-\tau)^2}\right\}, \tag{5}$$

where $\rho^2=\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}$ on the other hand for $(x,t)\in\mathbf{C}^3$ $|x_i|\le 17R^{1+\frac{\alpha_i}{2}}$, $i=1,...,n$; $\sqrt{|t|}\le 2\sqrt{b}R\le 2R$ therefore

$$|x|_\alpha+\sqrt{|t|}\le\left(n17^{\frac{2}{2+\alpha^-}}+2\right)R=C_1R. \tag{6}$$

Analogously, if $(x,t)\in\mathbf{C}^3$, then either $\sum_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}\ge R^2$ or $\sqrt{|t|}\ge\sqrt{b}R$. In the first case there will be found such i_0, $1\le i_0\le n$ that

$$|x_{i_0}|\ge\frac{R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}},$$

i.e. $|x|_\alpha n^{\frac{1}{2+\alpha^-}} R$.

Thus,

$$|x|_\alpha + \sqrt{|t|} \ge \min\left\{ n^{-\frac{1}{2+\alpha^-}}, \sqrt{b} \right\} R = C_2 R . \tag{7}$$

We conclude from (6) and (7) that

$$C_3 R^{\alpha_i} \le \lambda_i(x,t) \le C_4 R^{\alpha_i} \ ; \ i = 1,\dots,n , \tag{8}$$

where $C_3 = C_5(\alpha,n) b^{\frac{|\alpha^-|}{2}}$, $C_4 = C_6(\alpha,n) b^{-\frac{|\alpha^-|}{2}}$. (Without losing generality we assume that there exists $j, 1 \le j \le n$, for which $\alpha_j < \alpha^+$).

Allowing for (8) in (5) we get

$$\measuredangle G_R^{(s,\beta)} \ge G_R^{(s,\beta)} \left\{ \left(\frac{\gamma C_3}{\beta} - 1 \right) \frac{\rho^2}{4\beta(t-\tau)^2} - \frac{\gamma^{-1} n C_4}{2\beta(t-\tau)} - \frac{b_0}{2\beta(t-\tau)} \times \right.$$
$$\left. \times \sum_{i=1}^{n} \frac{|x_i - y_i|}{R^{\alpha_i}} - C_0 + \frac{s}{t-\tau} \right\}. \tag{9}$$

Since $|x_i - y_i| \le 34 R^{1+\frac{\alpha_i}{2}}$, then for $R \le 1$

$$\sum_{i=1}^{n} \frac{|x_i - y_i|}{R^{\alpha_i}} \le 34 \sum_{i=1}^{n} R^{1-\frac{\alpha_i}{2}} \le 34n . \tag{10}$$

Besides

$$C_0 \le \frac{C_0}{t-\tau} \cdot 4bR^2 \le \frac{s}{2(t-\tau)}, \tag{11}$$

if only $R \le R_1 = \left(\frac{s}{8C_0} \right)^{\frac{1}{2}}$. Fix $R_0 = \min\{1, R_1\}$. Then it follows from (9)-(11) that

$$\measuredangle G_R^{(s,\beta)} \ge G_R^{(s,\beta)} \left\{ \left(\frac{\gamma C_3}{\beta} - 1 \right) \frac{\rho^2}{4\beta(t-\tau)^2} + \frac{1}{t-\tau} \left(-\frac{C_7}{\beta} + \frac{s}{2} \right) \right\}, \tag{12}$$

where $C_7 = C_8(\alpha,n) b^{-\frac{|\alpha^-|}{2}}$. Now it suffices assume

$$\beta = \gamma C_3 , \quad s = \frac{2C_7}{\beta} \tag{13}$$

and the required inequality (4) follows from (12)-(13). The lemma is proved.

Everywhere later not specifying this we shall consider that the constants s and β are chosen in correspondence with the equalities (13). For the brevity of the record we shall denote the function $G_R^{(s,\beta)}$ and capacity $p_R^{(s,\beta)}$ by G_R and p_R respectively.

Lemma 2. *Let* $\mathbf{C}_\rho = C_{x';\rho;R}^{t'-\rho^2R^2,t'} \subset \mathbf{C}^3$. *Then*

$$p_R(\mathbf{C}_\rho) \ge C_9(n,\alpha,\rho,b) R^{2s} . \tag{14}$$

Proof. Consider the measure μ concentrated at the center of the lower foundation of a cylinder $\mathbf{C}_\rho$ with density $(4\beta s)^{-s}(\rho R)^{2s}$. Let the point (x,t) be arranged on the lateral surface $\mathbf{C}_\rho$ we have:

$$J(x,t)=\int\limits_{\{(x',t'-\rho^2R^2)\}} G_R\left(x-x',t-t'+\rho^2R^2\right)d\mu\left(x',t'-\rho^2R^2\right)=$$

$$=\int\limits_{\{(x',t'-\rho^2R^2)\}} \left(t-t'+\rho^2R^2\right)^{-s}\exp\left[-\frac{\rho^2R^2}{4\beta\left(t-t'+\rho^2R^2\right)}\right]d\mu\left(x',t'-\rho^2R^2\right). \qquad (15)$$

For $z>0$ consider the function $h(z)=z^{-s}\exp\left[-\frac{\rho^2R^2}{4\beta z}\right]$, and find the value z at which $h(z)$ attains its maximum. We find from the equation $h'(z)$

$$z=\frac{\rho^2R^2}{4\beta s}.$$

Thus, we get from (15)

$$J(x,t)\le(4\beta s)^{-s}(\rho R)^{-2s}e^{-s}\mu\left\{\left(x',t'-\rho^2R^2\right)\right\}\le 1.$$

It a point (x,t) is on the upper foundation of $\mathbf{C}_\rho$, then

$$J(x,t)\le(\rho R)^{-2s}\mu\left\{\left(x',t'-\rho^2R^2\right)\right\}\le 1.$$

Thus, $J\big|_{\Gamma(\mathbf{R}_{n+1}\setminus\mathbf{C}_\rho)}\le 1$.

Allowing for $\lim\limits_{(x,t)\to\infty} J(x,t)=0$, by lemma 1 and a maximum principle we deduce

$$J(x,t)\le 1 \quad \text{for} \quad (x,t)\in\mathbf{R}_{n+1}\setminus\mathbf{C}_\rho.$$

Consequently

$$p_R\left(\mathbf{C}_\rho\right)\ge\mu\left\{\left(x',t'-\rho^2R^2\right)\right\}=(4\beta s)^{-s}(\rho R)^{2s}$$

and the required estimate (14) is proved

Let $\mathbf{C}^4=C_{0;9;R}^{-2bR^2,0}$, $\left(x^0,t^0\right)\in\Gamma\left(\mathbf{C}^4\right)$, $\mathbf{C}^5=C_{x^0;8;R}^{t^0-bR^2,t^0}$, $\mathbf{C}^6=C_{x^0;1;R}^{t^0-\frac{bR^2}{4},t^0}$, $\mathbf{C}^7=C_{x^0;1;R}^{t^0-bR^2,t^0-\frac{bR^2}{2}}$.

Choose and fix b so that the condition $b^{1-\frac{|\alpha^-|}{2}}\le\frac{49}{8C_8}$ be fulfilled.

Lemma 3. *Let in the cylinder $\mathbf{C}^5$ be arranged a domain P having limit points on $\Gamma\left(\mathbf{C}^5\right)$ and intersecting $\mathbf{C}^6$. Then let continuous in $\overline{P}$, and vanishing in $\Gamma=\Gamma(P)\cap\mathbf{C}^5$ positive $\mathcal{L}$- subparabolic function $u(x,t)$ be determined in P. Then if as to the coefficients of the operator $\mathcal{L}$ the conditions (2)-(3) are fulfilled, there exist such $\eta_1(\mathcal{L},n)$ that for $R\le R_0$*

$$\sup_P u\ge\left(1+\eta_1R^{-2s}p_R(H_1)\right)\sup_{P\cap\mathbf{C}^6}u, \qquad (16)$$

where $H_1=\mathbf{C}^7\setminus P$.

Proof. We can consider that $p_R(H_1)>0$. Fix an arbitrary $\varepsilon\in(0,p_R(H_1))$, and let the measure μ on H_1, be such that

the measure μ on H_1, be such that

$$V(x,t)=\int_{H_1} G_R(x-y,t-\tau)d\mu(y,\tau)\le 1 \text{ for } (x,t)\notin H_1, \tag{17}$$

$$\mu(H_1)\ge p_R(H_1)-\varepsilon. \tag{18}$$

Let (y,τ) be an arbitrary fixed point from H_1, S be a lateral surface of $\mathbf{C}^5$. Now estimate the quantity $\sup\limits_{(x,t)\in S} G_R(x-y,t-\tau)$. To this end we fix $x\in \mathcal{E}_R^{x^0}(8)$ and find that value of $t>\tau$, at which the function $\vartheta(t)=G_R(x-y,t-\tau)$ attains its maximum. Equaling $\vartheta'(t)$ to zero, we get

$$t-\tau=\frac{\sum\limits_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}}{4\beta s}. \tag{19}$$

But by Minkowsky inequality

$$\sqrt{\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}}\ge\sqrt{\sum_{i=}^{n}\frac{(x_i-x_i^0)}{R^{\alpha_i}}}-\sqrt{\sum_{i=1}^{n}\frac{(y_i-x_i^0)^2}{R^{\alpha_i}}}\ge 7R.$$

Besides $t-\tau\le bR^2$, $4\beta s=8C_8 b^{-\frac{|\alpha^-|}{2}}$. Since

$$b^{1-\frac{|\alpha^-|}{2}}\le\frac{49}{8C_8},$$

then from (19) and monotonicity of $\vartheta(t)$ up to the first maximum we deduce

$$\sup_{(x,t)\in S} G_R(x-y,t-\tau)\le (bR^2)^{-s}\exp\left[-\frac{49}{4\beta b}\right]. \tag{20}$$

Now let's estimate $\inf\limits_{(x,t)\in \mathbf{C}^6} G_R(x-y,t-\tau)$. We have

$$\inf_{(x,t)\in\mathbf{C}^6} G_R(x-y,t-\tau)\ge (bR^2)^{-s}\exp\left[-\frac{\left(2\sum\limits_{i=1}^{n}\frac{(x_i-x_i^0)^2}{R^{\alpha_i}}+2\sum\limits_{i=1}^{n}\frac{(y_i-x_i^0)^2}{R^{\alpha_i}}\right)}{4\beta\frac{bR^2}{4}}\right]\ge$$

$$\ge (bR^2)^{-s}\exp\left[-\frac{4}{\beta b}\right]. \tag{21}$$

Let's introduce an auxilary function

$$W(x,t)=M\left[1-V(x,t)+(bR^2)^{-s}\exp\left[-\frac{49}{4\beta b}\right]\right]-u(x,t),$$

where $M=\sup\limits_{P} u$. By lemma 1, the function $W(x,t)-\mathcal{L}$ is superparabolic in P. It follows from (20) that $W|_{\Gamma(P)\cap S}\ge 0$.

The analogously inequality holds on Γ (by virtue of (7)) and that part of $\Gamma(P)$, which is arranged on the lower foundation of $\mathbf{C}^5$ (as that $V=0$). Thus, $W|_{\Gamma(P)}\ge 0$, and by a maximum principle $W(x,t)\geq 0$ for $(x,t)\in P$. In particular, allowing for (18) and (21) we have

$$\sup_{P\cap \mathbf{C}^6} u \leq M\left[1-\inf_{P\cap \mathbf{C}^6} V(x,t)+\left(bR^2\right)^{-s}\exp\left[-\frac{49}{4\beta b}\right]\right]\leq$$

$$\leq M\left[1-\left(bR^2\right)^{-s}\left(\exp\left[-\frac{4}{\beta b}\right]-\exp\left[-\frac{49}{4\beta b}\right]\right)p_R(H_1)+\varepsilon\exp\left[-\frac{4}{\beta b}\right]\right]. \quad (22)$$

Now allowing for that arbitrariness of ε, we arrive at the required inequality (16) from (22). The lemma is proved.

Corollary. *If the conditions of the lemma is fulfilled, and H_1 contains a cylinder $\mathbf{C}_\rho$, then*

$$\sup_P u \geq \left(1+\eta_2(\mathcal{L},n,\rho)\right)\sup_{P\cap \mathbf{C}^6} u.$$

Lemma 4. *Let the conditions of the previous lemma be fulfilled. Then there exists such $\delta(\mathcal{L},n)$ that if $mesP\leq \delta_1 mes\mathbf{C}^5$ and $R\leq R_0$, then*

$$\sup_P u \geq 2\sup_{P\cap \mathbf{C}^6}. \quad (23)$$

Proof. Let's consider an auxiliary function

$$Y(x,t)=M\left[\frac{1}{64R^2}\sum_{i=1}^{n}\frac{\left(x_i-x_i^0\right)^2}{R^{\alpha_i}}+\frac{t^0-t}{bR^2}\right]-u(x,t).$$

If is easy to see that

$$\mathcal{L}Y\leq M\left[\frac{1}{32R^2}\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{R^{\alpha_i}}+\frac{1}{32R^2}\sum_{i=1}^{n}\frac{b_i(x,t)\left(x_i-x_i^0\right)}{R^{\alpha_i}}+\frac{1}{bR^2}\right]\leq$$

$$\leq\frac{M}{R^2}\left[\frac{\gamma^{-1}}{32}\sum_{i=1}^{n}\frac{\lambda_i(x,t)}{R^{\alpha_i}}+\frac{b_0}{4}\sum_{i=1}^{n}R^{1-\frac{\alpha_i}{2}}+\frac{1}{b}\right]\leq\frac{MC_{10}(\mathcal{L},n)}{R^2}. \quad (24)$$

On the other hand $Y|_{\Gamma(P)}\geq 0$. Let's consider the domain $P'\subset\mathbf{C}^5$, $P'\supset P$, $mesP'\leq 2mesP$ and the function $F(x,t)$ such that $F(x,t)=-\frac{C_{10}}{R^2}$ for $(x,t)\in P$, $F(x,t)=0$ for $(x,t)\notin P'$, $-\frac{C_{10}}{R^2}\leq F(x,t)\leq 0$. Let $w(x,t)$ be a solution of the following first boundary value problem

$$\mathcal{L}w(x,t)=F(x,t),\ (x,t)\in\mathbf{C}^5;\quad w|_{\Gamma(\mathbf{C}^5)}=0.$$

By a maximum principle $w(x,t)\geq 0$ for $(x,t)\in\mathbf{C}^5$. Besides, by A.D. Aleksandrov-N.V. Krylov inequality [11]

$$\sup_{\mathbf{C}^5} w\leq\frac{C_{11}(\mathcal{L},n)\left(\prod_{i=1}^{n}R^{1+\frac{\alpha_i}{2}}\right)^{\frac{1}{n+1}}}{\inf_{\mathbf{C}^5}\left(\prod_{i=1}^{n}\lambda_i(x,t)\right)^{\frac{1}{n+1}}}\|F\|_{L_{n+1}}(P')\leq$$

$$\leq C_{12}(\mathcal{L},n)\frac{R^{\frac{n}{n+1}+\frac{|\alpha|}{2(n+1)}}}{R^{\frac{|\alpha|}{n+1}}}2^{\frac{1}{n+1}}\delta_1^{\frac{1}{n+1}}\cdot R^{-2}R^{\frac{n+2}{n+1}}R^{\frac{|\alpha|}{2(n+1)}}=C_{13}(\mathcal{L},n)\delta_1^{\frac{1}{n+1}}. \qquad (25)$$

Now if we put $Y'(x,t)=Y(x,t)+M\,w(x,t)$, then by virtue of (24) the function Y' is $\mathcal{L}$-superparabolic in P, $Y'|_{\Gamma(P)}\geq 0$ and by a maximum principle allowing for (25)

$$\sup_{P\cap\mathbf{C}^6}u\leq M\left[\frac{1}{64}+\frac{1}{4}+C_{13}\delta_1^{\frac{1}{n+1}}\right]. \qquad (26)$$

Choose δ_1 such that $C_{13}\delta^{\frac{1}{n+1}}=\frac{15}{64}$. Then the inequality (23) follows from (26). The lemma is proved.

Let $\mathbf{C}^8=C_{0;8,5;R}^{-\frac{5b}{2}R^2,-\frac{bR^2}{2}}$, $\mathbf{C}^9=C_{0;9,5;R}^{-3bR^2,-\frac{bR^2}{2}}$, $\mathbf{C}^{10}=\mathbf{C}^9\setminus\mathbf{C}^8$.

Lemma 5. *Let a domain P having limit points on $\Gamma(\mathbf{C}^3)$ and intersecting $\mathbf{C}^4$ be arranged on $\mathbf{C}^3$. Then let a continuous in $\overline{P}$ and vanishing on $\Gamma_1=\Gamma(P)\cap\mathbf{C}^3$ positive $\mathcal{L}$-superparabolic function $u(x,t)$ be determined on P. Then if the conditions (2)-(3) are fulfilled with respect to the coefficients of an operator $\mathcal{L}$, then there exists such $\eta_3(\mathcal{L},n)$ that for $R\leq R_0$*

$$\sup_P u\geq\left(1+\eta_3R^{-2s}p_R(H_2)\right)\sup_{P\cap\Gamma(\mathbf{C}^4)}u, \qquad (27)$$

where $H_2=\mathbf{C}^{10}\setminus P$.

Proof. On $\Gamma(\mathbf{C}^4)$ choose a minimal number of points $(x^1,t^1),\ldots,(x^m,t^m)$, so that if $\mathbf{C}_{R,1}^i=C_{x^i;1;R}^{t^i-\frac{bR^2}{4},t^i}$, $\mathbf{C}_{R,2}^i=C_{x^i;1;R}^{t^i-bR^2,t^i-\frac{bR^2}{2}}$ $(i=1,\ldots,m)$, then:

a) $\bigcup_{i=1}^m\overline{\mathbf{C}}_{R,1}^i\supset\Gamma(\mathbf{C}^4)$;

b) $\bigcup_{i=1}^m\mathbf{C}_{R,2}^i\supset\mathbf{C}^{10}$

c) for any i_0 and any point $(x^0,t^0)\in\Gamma(\mathbf{C}^4)$ there will be found such a chain $(x^{i_1},t^{i_1}),\ldots,(x^{i_k},t^{i_k})$, that $(x^0,t^0)\in\mathbf{C}_{R,1}^{i_k}$, and in the intersection $\mathbf{C}_{R,2}^{i_l}\cap\overline{\mathbf{C}}_{R,2}^{i_{l+1}}$ a cylinder $\mathbf{C}_{R,3}^l$ of the height $\frac{bR^2}{16}$ whose foundation is an ellipsoid with semiaxes $\frac{1}{4}R^{1+\frac{\alpha_i}{2}}$; $l=0,\ldots,k-1;\ i=1,\ldots,n$ is contained.

It is clear that $m=m(\alpha,n)$. Without losing generality we shall assume that $\sup_{P\cap\Gamma(\mathbf{C}^4)}u=1$. Let the point $(x^0,t^0)\in\Gamma(\mathbf{C}^4)$ be such that $u(x^0,t^0)=1$. There exists such i_0, $1\leq i_0\leq n$, that

$$P_R\left(\mathbf{C}_{R,2}^{i_0}\cap H_2\right)\geq\frac{P_R(H_2)}{m}. \qquad (28)$$

Let's assume that

$$\sup_{\mathbf{C}^{i_0}_{R,1}\cap P} u \geq 1-\delta, \tag{29}$$

where $\delta = \dfrac{\eta_1 R^{-2s} p_R(H^0)}{2(1+\eta_1 R^{-2s} p_R(\mathbf{C}^{10}))}$, $H^0 = \mathbf{C}^{i_0}_{R,2} \cap H_2$.

Applying lemma 3 and allowing for (28)-(29) we get:

$$\sup_P u \geq \sup_{P\cap C^{t^{i_0}-\delta R^2, t^{i_0}}_{x^{i_0},9,R}} u \geq \left(1+\eta_1 R^{-2s} p_R(H^0)\right)(1-\delta) \geq$$

$$\geq 1+\frac{\eta_1}{2} R^{-2s} p_R(H^0) \geq 1+\frac{\eta_1}{2m} R^{-2s} p_R(H_2)$$

and in this case the lemma is proved.

Now let $u(x,t)<1-\delta$ for $(x,t)\notin \mathbf{C}^{i_0}_{R,1}\cap P$. Assume $\vartheta_1(x,t)=u(x,t)-1+\delta$ and the set $P_1=\{(x,t):(x,t)\in P, \vartheta_1(x,t)>0\}$. The cylinder $\mathbf{C}^{i_0}_{R,1}$ must be a complement to P_1, moreover a cylinder $\overline{\mathbf{C}}^{i_0}_{R,2} \cap \overline{\mathbf{C}}^{i_1}_{R,2}$ is contained in the intersection $\mathbf{C}^{1}_{R,3}$. By the corollary from lemma 3

$$\sup_P \vartheta_1 \geq \sup_{P_1} \vartheta_1 \geq \left(1+\eta_2(\mathcal{L},n)\right) \sup_{P_1\cap \mathbf{C}^{i_1}_{R,1}} \vartheta_1. \tag{30}$$

Assume $\sigma = \dfrac{\eta_2}{2(1+\eta_2)}$. If

$$\sup_{P_1\cap \mathbf{C}^{i_1}_{R,1}} \vartheta_1 \geq \delta(1-\sigma), \tag{31}$$

i.e.

$$\sup_{P_1\cap \mathbf{C}^{i_1}_{R,1}} u \geq 1-\delta\sigma,$$

then it follows from (30)-(31)

$$\sup_P u \geq 1-\delta+(1+\eta_2)\delta(1-\sigma) = 1+\frac{\eta}{2}\delta$$

and in this case the lemma is proved. But if $u(x,t)<1-\delta\sigma$ for $(x,t)\in \mathbf{C}^{i_1}_{R,1}\cap P_1$ then we consider the function $\vartheta_2(x,t)=u(x,t)-1+\delta\sigma$ and the set $P_2=\{(x,t):(x,t)\in P_1, \vartheta_2(x,t)>0\}$. Continue the process in an analogous way. No later than in the k-th step we arrive at the alternation

$$\sup_{P_k\cap \mathbf{C}^{i_k}_{R,1}} u \geq 1-\delta\sigma^k,$$

i.e. $(x^0,t^0)\in \mathbf{C}^{i_k}_{R,1}$ and $u(x^0,t^0)=1$. By the same token the lemma is proved.

The following lemma is proved analogously.

Lemma 6. *Let the conditions of lemma 5 be fulfilled. Then there exists such $\delta_1(\mathcal{L},n)$ that if $mesP \leq \delta_1 mes\mathbf{C}^3$, then*

$$\sup_P u \geq 2 \sup_{P\cap\Gamma(C^4)} u.$$

Now using the method of paper [6] we arrive at the two statements.

Theorem 1. *Let a domain P having limit points on $\Gamma(\mathbf{C}^2)$ and intersecting $\mathbf{C}^1$ be arranged on $\mathbf{C}^2$. Then let a continuous in $\overline{P}$ and vanishing on $\Gamma(P)\cap\mathbf{C}^2$ positive $\mathcal{L}$-subpararbolic function $u(x,t)$ be determined in P. Then if $mesH_2 \ge ames\mathbf{C}^{10}$ $(a>0)$, then*

$$\sup_P u \ge (1+\eta_4(\mathcal{L},n,a))\sup_{P\cap\mathbf{C}^1} u .$$

Theorem 2. *Let in domain $D\subset\mathbf{R}_{n+1}$ a solution $u(x,t)$ of the equation (1) be determined, moreover as to the coefficients of the operator $\mathcal{L}$, the conditions (2)-(3) be fulfilled. Then if $R\le R_0$ is such that $\overline{\mathbf{C}}^2\subset D$. Then*

$$\operatorname*{osc}_{\mathbf{C}^2} u \ge (1+\eta_5(\mathcal{L},n))\operatorname*{osc}_{\mathbf{C}^1} u .$$

In conclusion we cite one more important corollary of theorem 2. It is proved by the same sketch that a corresponding fact for uniform parabolic equations (see for ex. [9]).

Let $C_\lambda(D)$, $(0<\lambda<1)$ be a Banach space of functions $u(x,t)$, given on D with a finite norm

$$\|u\|_{C_\lambda(D)} = \|u\|_{C(D)} + \sup_{\substack{(x,t),(y,\tau)\in D\\ (x,t)\ne(y,\tau)}} \frac{|u(x,t)-u(y,\tau)|}{|x-y|^\lambda + |t-\tau|^{\lambda/2}},$$

D_ρ is a totality of all points $(x,t)\in D$, for which $\inf_{(y,\tau)\in\Gamma(D)}\left(|x-y|+|t-\tau|^{\frac{1}{2}}\right)>\rho>0$.

Theorem 3. *If the conditions of the previous theorem are fulfilled, then for any $\rho>0$*

$$\|u\|_{C_\lambda(D_\rho)} \le H\|u\|_{C(D)}.$$

The constant λ here depends only on the coefficients of the operator $\mathcal{L}$ and n, and H also on ρ.

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ON ONE INVERSE PROBLEM FOR A SEMI-LINEAR EQUATION OF PARABOLIC TYPE

Abstract

In the paper the inverse problem on defining the right hand side of the semi-linear equation of parabolic type is considered. The theorems of existence, uniqueness and stability of solution are proved. For approximate solving the considered inverse problem the method of successive approximations was suggested and its convergence rate was estimated.

In the paper the inverse problem on defining the right hand side of a semi-linear equation of parabolic type is considered. The existence, uniqueness and solution stability theorems are proved.

For approximate solving of the considered problem the method of successive approximations was suggested and its convergence rate was estimated.

The inverse problems on defining the unknown source (in applications the right hand member usually is the sense of the source) of the linear parabolic equation was considered in papers [1-6].

We'll accept the following denotations: D is a bounded domain from R^n with the boundary ∂D, $Q_T = D \times (0,T], S_T = \partial D \times [0,T], 0 < T = const, \|\cdot\|_{C^l} = \|\cdot\|_l$, the spaces $C^l(\cdot), C^{(l+\alpha)/2}(\cdot),\ C^{l+\alpha,(l+\alpha)/2}(\cdot), 0 < \alpha < 1,\ l = 0,1,2,$ and corresponding norms are defined, for example, in [9, p.12].

Consider the problem on defining $\{f(x), u(x,t)\}$ from the conditions

$$u_t - Lu = f(x)g(u), \quad (x,t) \in Q_T, \tag{1}$$

$$u(x,0) = \varphi(x), x \in \overline{D};\ u(x,t) = \psi(x,t),\ (x,t) \in S_T, \tag{2}$$

$$\int_0^T u(x,t)dt = h(x),\ x \in \overline{D}, \tag{3}$$

where

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} = \sum_{i=1}^{n} b_i(x)u_{x_i} + C(x)u.$$

The functions $a_{ij}(x), b_i(x),\ i,j = \overline{1,n}, C(x), g(\cdot), \varphi(x), \psi(x,t), h(x)$ are given. Everywhere below we'll suppose that for arbitrary real vector $\eta = (\eta_1,...,\eta_n)$ for any $(x,t) \in \overline{Q}_T$

$$m_0 \sum_{i=1}^{n} \eta_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\eta_i\eta_j \le m_1 \sum_{i=1}^{n} \eta_i^2; \quad 0 < m_0 < m_1.$$

If the function $f(x)$ in the equation (1) is given, then naturally, the condition (3) isn't given. The problem on defining $u(x,t)$ from (1)-(2) in more general statement is considered, for example in paper [7].

The problems (1)-(3) relate to the class of incorrect by Hadamard problems. Examples show that the solution of this problem doesn't always exist and even if exists, then it may not be unique and stable.