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A PHRAGMEN-LINDELÖF TYPE THEOREM FOR A NON-DIVERGENT STRUCTURE LINEAR ELLIPTIC EQUATION OF THE SECOND ORDER

Abstract

At the paper we consider a non-uniformly degenerating at infinity, non-divergent structure elliptic equation of the second order, containing small members. The condition providing the truth of Phragmen-Lindelöf type theorem for its solutions are found for minor coefficients of the equation.

Let $\mathbf{E}_n$ be an n-dimensional Euclidean space, $n \geq 2$, D be an infinite domain with the boundary ∂D, situated in the cone $K = \{x : \left(\sum_{i=1}^{n-1} x_i^2\right)^{1/2} \leq kx_n, 0 < x_n < \infty\}$, $0 < k < 1/16$.

Let's consider in D the following problem:

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = 0, \qquad (1)$$

$$u\big|_{\partial D} = 0, \qquad (2)$$

where $\|a_{ij}(x)\|$-is a real symmetric matrix.

Let's suppose that for all $x \in D$ and $\xi \in \mathbf{E}_n$ the condition is fulfilled

$$\mu\sum_{i=1}^{n} \lambda_i(x)\xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \mu^{-1}\sum_{i=1}^{n} \lambda_i(x)\xi_i^2, \qquad (3)$$

where $\mu \in (0,1]$ -is a constant, $\lambda_i(x) = \left(1 + |x|_\alpha\right)^{\alpha_i}, |x|_\alpha = \sum_{i=1}^{n} |x_i|^{2/(2+\alpha_i)}, \alpha_i > -2, i = 1,...,n$,

We'll suppose also that

$$-c_0 \leq c(x) \leq 0, \qquad (4)$$

c_0 - is a positive constant.

The aim of the present paper is the obtaining of condition on the minor coefficients $b_i(x), (i = 1,...,n)$ and $c(x)$ at whose fulfillment Phragmen-Lindelöf theorem is valid for solutions of the problem (1)-(2).

Note that for non-divergent second order uniformly elliptic equations not containing minor coefficients analogous results were obtained in papers [1-4]. As for uniformly elliptic equations of divergent structure, then we show in this connection in [5-6]. And we note papers [7-10], in which theorems of Phragmen-Lindelöf type for quasi-linear elliptic and parabolic equations were obtained. In papers [11-12] analogous theorems were proved for degenerating on infinity elliptic equations without minor coefficients. In case when minor coefficients are present in uniform elliptic equation and principal part of the operator $\mathcal{L}$ has divergent form theorem of Phragmen-Lindelöf type was established in [13] provided $divB(x) \leq 0, (B(x), x) \leq 0$ for all $x \in D$, where $B(x) = (b_1(x),...,b_n(x))$.

If in the presence of the minor coefficients the principal part of the operator $\mathcal{L}$ is written in non-divergent form then the analogous results valid under the condition $\left(B(x), x - x^0\right) \leq 0$ for all $x \in D, x^0 \in \bar{l}$, where $\bar{l}$ is some ray, coming from origin and situated outside of the cone K (see [14]). In the present paper the result of paper [15] is generalized in which the case $\alpha_i \in (-2,0]$, $i=1,...,n$ was considered.

For $R>0,\ p>0,\ x^0 \in \mathbf{E}_n$ denote by $E_R^{x^0}(p)$ an ellipsoid

$$\left\{ x : \sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{R^{\alpha_i}} < (pR)^2 \right\}.$$

Further we'll suppose that $R \geq 1$.

Let's consider parallel with the operator $\mathcal{L}$ the "contracted" operator $\mathcal{L}^c = \mathcal{L} - c(x)$.

Lemma 1. *Let* $z \in \partial E_R^0(5) \cap \overline{K},\ \alpha_n = \alpha^- = \min_i\{\alpha_i\},\ x^0(z) \in \partial E_R^z(1) \cap \partial E_R^0(5),$ *moreover* $x_1^0 > 0,\ x_n^0 > 0,\ x_j^0 = 0\ (j=2,...,n-1)$. *Then* $x^0 \notin K$.

Let's denote $E_R^0(1;9) = E_R^0(9) \setminus \overline{E_R^0(1)},\ D_R = D \cap E_R^0(1;9),\ A_R = \bigcup_{z \in D \cap \partial E_R^0(5)} x^0(z)$, $D_R^z = D \cap E_R^z(4)$.

Lemma 2. *Let* $x^0 \in A_R$. *Then for any* $x \in D_R^z$ *there exists a positive constant* β, *depending only on* k, *such that*

$$\sqrt{\sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{R^{\alpha_i}}} \geq \beta R.$$

Let's introduce the notations

$$b_R(x) = \left(\frac{b_1(x)}{R^{\alpha_1}}, ..., \frac{b_n(x)}{R^{\alpha_n}} \right)$$

and

$$G_s^{(R)}(x) = \left(\sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{R^{\alpha_i}} \right)^{-s/2},$$

where $x \in D_R,\ x^0 \in A_R,\ s>0$.

Let's suppose that

$$\left(b_R(x),\ x - x^0\right) \leq 0. \qquad (5)$$

Lemma 3. *Let there conditions (3) and (5) be fulfilled. Then for any fixed* $x^0 \in A_R$ *there exists such* $s = s(\mu,\alpha,n)$ *that for any* $x \in D_R$

$$\mathcal{L}^c G_s^{(R)}(x) \geq 0.$$

Proof. Let's denote $\rho = \sqrt{\sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{R^{\alpha_i}}}$.

We have

$$\mathcal{L}^c G_s^{(R)}(x) = s\rho^{-(s+2)} \left[\frac{(s+2)}{\rho^2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\left(x_i - x_i^0\right)\left(x_j - x_j^0\right)}{R^{\alpha_i + \alpha_j}} - \sum_{i=1}^{n} \frac{a_{ii}(x)}{R^{\alpha_i}} - \sum_{i=1}^{n} b_i(x) \frac{\left(x_i - x_i^0\right)}{R^{\alpha_i}} \right] \geq$$

$$\geq s\rho^{-(s+2)} \left[\frac{(s+2)\mu}{\rho^2} \sum_{i=1}^{n} \frac{\lambda_i(x)}{R^{\alpha_i}} \frac{\left(x_i - x_i^0\right)^2}{R^{\alpha_i}} - \mu^{-1} \sum_{i=1}^{n} \frac{\lambda_i(x)}{R^{\alpha_i}} \right].$$

We can analogously to prove in [16] that there exists a positive constant $C_1(\alpha,n)$ and $C_2(\alpha,n)$, such that $C_1R^{\alpha_i} \le \lambda_i(x) \le C_2R^{\alpha_i}$, $i=1,...,n$.

Consequently

$$\mathcal{L}G_s^{(R)}(x) \ge s\rho^{-(s+2)}\left[(s+2)\mu C_1 - \mu^{-1}nC_2\right].$$

Let's denote $s_0 = \max\left\{3, \dfrac{nC_2}{\mu^2 C_1}\right\}$. it is sufficient to suppose $s = s_0 - 2$ and the lemma is proved.

Corollary. *Let* $z \in D \cap \partial E_R^0(5)$, $x^0 = x^0(z)$, $x \in D \cap E_R^z(4)$, $g_s^{(R)}(x) = = \beta^s R^s G_s^{(R)}(x)$. *Then* $g_s^{(R)}(x) \le 1$.

Lemma 4. *Let* $z \in D \cap \partial E_R^0(5), x^0 = x^0(z)$ *and in* $H = D \cap E_R^z(4)$ *the positive solution* $u(x)$ *of the equation (1) continuous in* $\overline{H}$ *and vanishing in that part* Γ *of the boundary domain* H, *which lies strictly inside* $E_R^z(4)$ *be determined. Then if the conditions (3)-(5) are fulfilled, then there exists a constant* $\eta = \eta(\mu,\alpha,n) > 0$ *such that*

$$\sup_H u \ge (1+\eta) \sup_{H \cap E_R^z(1)} u .$$

Proof. First of all let's show that if $\mathcal{L}u(x) = 0$ then $\mathcal{L}u^2(x) \ge 0$. Really

$$\mathcal{L}u^2(x) = 2u(x)\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + 2\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j} + 2u(x)\sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i} \ge$$

$$\ge 2u(x)\left[\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i}\right] = -2u^2(x)\cdot c(x) \ge 0 .$$

Let $\sup_H u^2 = M$. Let's introduce the auxiliary function

$$U(x) = M\left[1 - g_S^{(R)}(x) + \sup_{x \in \overline{H} \cap \partial E_R^z(4)} g_S^{(R)}(x)\right].$$

It is easy to see that $\mathcal{L}\left(U(x) - u^2(x)\right) \le 0$ in H, $\left(U(x) - u^2(x)\right)\big|_\Gamma \ge 0$, $\left(U(x) - u^2(x)\right)\big|_{\partial H \setminus \Gamma} \ge 0$. Then by the maximum principle $U(x) \ge u^2(x)$ in H and in particular

$$\sup_{H \cap E_R^z(1)} u^2 \le M\left[1 - \left(\inf_{H \cap E_R^z(1)} g_S^{(R)}(x) - \sup_{\overline{H} \cap \partial E_R^z(4)} g_S^{(R)}(x)\right)\right].$$

Let $x \in \overline{H} \cap \partial E_R^z(4)$. Then $\sqrt{\sum_{i=1}^n \frac{\left(x_i - x_i^0\right)^2}{R^{\alpha_i}}} \ge \sqrt{\sum_{i=1}^n \frac{\left(x_i - z_i\right)^2}{R^{\alpha_i}}} - \sqrt{\sum_{i=1}^n \frac{\left(x_i^0 - z_i\right)^2}{R^{\alpha_i}}} = 3R$.

Therefore $\sup_{x \in \overline{H} \cap \partial E_R^z(4)} g_S^{(R)}(x) \le 3^{-S} \beta^S$.

If $x\in H\bigcap E_R^z(1)$, then $\sqrt{\sum_{i=1}^n \frac{\left(x_i-x_i^0\right)^2}{R^{\alpha_i}}}\le\sqrt{\sum_{i=1}^n \frac{\left(x_i-z_i\right)^2}{R^{\alpha_i}}}+$
$+\sqrt{\sum_{i=1}^n \frac{\left(z_i-x_i^0\right)^2}{R^{\alpha_i}}}\le 2R$, therefore $\inf_{x\in H\bigcap E_R^z(1)} g_S^{(R)}(x)\ge 2^{-S}\beta^S$ and we get

$$\sup_{H\bigcap E_R^z(1)} u^2\le M\left[1-\beta^S\left(2^{-S}-3^{-S}\right)\right].$$

Let's suppose that $\eta_1=\beta^S\left(2^{-S}-3^{-S}\right)$, $\eta=\sqrt{1+\eta_1}-1$. Then the statement of lemma follows from the last inequality.

Lemma 5. *Let in* $H_1=D\bigcap E_R^0(1;9)$ *the positive solution* $u(x)$ *of the equation (1), continuous in* $\overline{H}_1$ *and vanishing in that part* Γ *of the boundary domain* H_1 *which lies strictly inside* $E_R^0(1;9)$ *be determined. Then if the condition (3)-(5) are fulfilled then*

$$\sup_{H_1} u\ge(1+\eta)\cdot\sup_{H_1\bigcap\partial E_R^0(5)} u.$$

Proof. Let z that point $\overline{H}_1\bigcap\partial E_R^0(5)$ in which $u(z)=\sup_{H_1\bigcap\partial E_R^0(5)} u$. By lemma 4

$$\sup_{H_1\bigcap E_R^z(4)} u\ge(1+\eta)\cdot\sup_{H_1\bigcap\partial E_R^z(1)} u.$$

On the other hand $\sup_{H_1\bigcap E_R^z(1)} u\ge u(z)$ and $E_R^z(4)\subset E_R^0(1;9)$.

Hence the statement of the lemma follows.

Corollary. *Let in* $H_2=D\bigcap E_R^0(9)$ *the positive solution* $u(x)$ *of the equation (1) continuous in* $\overline{H}_2$ *and vanishing at that part* Γ *of the boundary domain* H_2 *which lies strictly inside* $E_R^0(9)$ *be determined. Then if the conditions (3)-(5) are fulfilled then*

$$\sup_{H_2} u\ge(1+\eta)\cdot\sup_{H_2\bigcap\partial E_R^0(5)} u.$$

Theorem. *Let in domain* D *coefficients of the operator* $\mathcal{L}$ *satisfying the conditions (3)-(5) be determined and* $u(x)$ *be the solution of the problem (1)-(2). Then either* $u(x)\equiv 0$ *in* D, *or* $\varliminf_{r\to\infty}\frac{M(r)}{r^\sigma}>0$, *where* $M(r)=\sup_{D\bigcap\partial E_r^0(1)}|u(x)|$ *and* $\sigma>0$ *depends only on* k,α,μ *and* n.

Proof. Let there exist a point $a\in D$ in which $u(a)=b\ne 0$. We can suppose without losing generality that $b>0$ (otherwise it was sufficient to multiply the solution by minus unit). Let $D^+=\{x: x\in D,\, u(x)>0\}$ and D' be connected component D^+ containing a point a. From the maximum principle it follows that this component is a bounded domain on the boundary of which $u(x)$ turns to zero. Let $\gamma=\left(\frac{5}{9}\right)^{2/(2+\alpha^-)}$. Then for any $R>0$ $E_{\gamma R}^0(1)\subset E_R^0\left(\frac{5}{9}\right)$. Thus for any $R\ge 1$ the corollary to lemma 5 can be formulated as follows:

$$\sup_{D\bigcap E_R^0(1)} u\ge(1+\eta)\cdot\sup_{D\bigcap E_{\gamma R}^0(1)} u.$$

Let m_0 be the smallest natural number for which $a \in E^0_{\gamma-m_0}(1) \cap D'$. Let later $r > 1$ - be arbitrary, moreover $m > m_0$ is a natural number for which

$$\gamma^{-m} \le r < \gamma^{-m-1},$$

$$m \ln \frac{1}{\gamma} \le \ln r < (m+1) \ln \frac{1}{\gamma},$$

i.e.

$$m > \frac{\ln r}{\ln \frac{1}{\gamma}} - 1 .$$

We'll suppose r such big that $\frac{\ln r}{\ln \frac{1}{\gamma}} - 1 \ge \frac{\ln r}{2 \ln \frac{1}{\gamma}}$. Let $N(r) = \sup\limits_{D' \cap E^0_r(1)} u$. Applying successively the corollary to lemma 5 we'll get:

$$N(r) \ge (1+\eta)^{m-m_0} \cdot N\left(\gamma^{-m_0}\right) \ge (1+\eta)^{m-m_0} \cdot b = (1+\eta)^m \cdot \frac{b}{(1+\eta)^{m_0}} = (1+\eta)^m \cdot b_1 \ge$$

$$\ge b_1 \cdot (1+\eta)^{\ln r / 2 \ln \frac{1}{\gamma}} = \eta_2^{\ln r} \cdot b_1 = b_1 \cdot \exp\left(\ln \eta_2^{\ln r}\right) = b_1 \cdot \exp\left(\ln r^{\sigma}\right) = b_1 r^{\sigma},$$

where $\eta_2 = (1+\eta)^{1/2 \ln \frac{1}{\gamma}}$, $\sigma = \ln \eta_2$.

Thus for sufficient big r

$$\frac{N(r)}{r^{\sigma}} \ge b_1 .$$

Applying the maximum principle we get the statement of the theorem.

Author expresses gratitude to his supervisor, corresponding member of NASA, prof. I.T. Mamedov for his statement of the problem and his attention to the work.

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Received March 2, 2001; Revised August 6, 2001.
Translated by Mamedova V.I.

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THE GENERALIZED SOLVABILITY OF THE DIRICHLET PROBLEM FOR NON-UNIFORMLY DEGENERATING ELLIPTIC EQUATIONS OF THE SECOND ORDER

Abstract

The Dirichlet problem is considered for non-uniformly degenerating elliptic equations of the second order of divergent structure. The inequalities of Friedrichs type is proved and the conditions are found at which this problem is uniquely generalized solvable in anisotropic Sobolev space.

Introduction. Let E_n be an n dimensional Euclidean space of the points $x=(x_1,...,x_n)$, $n\geq 3, D$ be a bounded domain situated in $E_n, \partial D$ be a boundary of the domain D. Let's consider in D the first boundary value problem

$$Lu=\sum_{i,,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)=f(x)+\sum_{i=1}^{n}\frac{\partial f^i(x)}{\partial x_i}, \tag{1}$$

$$u|_{\partial D}=\varphi, \tag{2}$$

where $\|a_{ij}(x)\|$ is a real symmetric matrix with measurable in D elements, moreover for all $x\in D, \zeta\in E_n$ it is fulfilled the condition

$$\gamma\sum_{i=1}^{n}\lambda_i(x)\zeta_i^2\leq\sum_{i,j=1}^{n}a_{ij}\zeta_i\zeta_j\leq\gamma^{-1}\sum_{i=1}^{n}\lambda_i(x)\zeta_i^2. \tag{3}$$

Here $\gamma\in(0,1]$ is a constant, and the functions $\lambda_i(x)\, i=1,...,n$ almost everywhere in D are finite and positive. The aim of the given paper is to find the conditions on functions $\lambda_i(x), f(x), f^i(x)$ and $\varphi(x)\,(i=1,...,n)$, at which the problem (1), (2) is uniformly generalized solvable in corresponding anisotropic Sobolev weight space. Let's denote that in the case of uniformly elliptic equations we can find the proof of analogous fact in [1-3]. Concerning the equations with uniform degeneration then let's note in this case papers [4-5]. For elliptic equations with weak (logarithmic) non-uniform degeneration the generalized solvability of Dirichlet problem is established in [6]. Let's note also paper [7-8], where the first boundary value problem is investigated for one class of elliptic equations with non-uniform power degeneration at a point.

1^0. The inequality of Friedrichs type. Let's agree in some notations and determinations. Let $W^1_{2,\lambda}(D)$ be a Banach space of the function $u(x)$, given on D with finite norm

$$\|u\|_{W^1_{2,\lambda}(D)}=\left(\int_D\left(u^2(x)+\sum_{i=1}^{n}\lambda_i(x)u_i^2\right)dx\right)^{1/2},$$

where $\lambda=(\lambda_1,...,\lambda_n)$, $u_i=\partial u/\partial x_i$, $(i=1,..,n)$. On the functions $\lambda_i(x)$ $(i=1,...,n)$ we put the next conditions

$$\lambda_i(x)\in L_1(D),\ \lambda_i^{-1}(x)\in L_{n/2}(D), i=1,..,n. \tag{4}$$