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ON TWO APPROXIMATED METHODS FOR SOLUTION OF ONE BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL EQUATION OF THE FOURTH ORDER

Abstract

A boundary value problem is considered for the fourth order differential equation. This equation is reduced to the equivalent Volterra-Fredholm integral equation. The equation is solved by two iteration methods.

The boundary value problem for a differential equation of the fourth order is considered. It is substituted by the integral equation of Volter-Fredholm and the last is solved by two iteration methods.

Let's consider the following boundary value problem for a differential equation of the fourth order

$$\frac{d^4x(t)}{dt^4}+a(t)x(t)=f(t) \quad (0\le t\le 1), \tag{1}$$

$$\left.\begin{aligned} &x(0)=x_0,\ x'(0)=\dot{x}(0),\ x''(0)=\ddot{x}_0,\\ &x(1)=\alpha x(c)+\beta,\ \alpha c^3\neq 1,\ (0<c<1)\end{aligned}\right\} \tag{2}$$

Such problem is met, for example in the sections of construction mechanics – in the problem on equilibrium of beam on elastic base [1-4], in some problems of the theory of cylindrical shells [4].

For approximated solution of the problem (1), (2) the methods given at papers [5], [6] are applied.

Let's suppose that $a(t)$, $f(t)$, $(0\le t\le 1)$ are continuous. It is easily proved that we can substitute the problem (1)-(2) by the equivalent integral equation

$$\begin{aligned} x(t)=f^*(t)+\int_0^1\frac{t^3(1-s)^3}{6(1-\alpha c^3)}a(s)x(s)ds-\\ -\alpha\frac{t^3}{6(1-\alpha c^3)}\int_0^c(c-s)^3a(s)x(s)ds-\int_0^t\frac{(t-s)^3}{6}a(s)x(s)ds,\end{aligned} \tag{3}$$

where

$$\begin{aligned} f^*(t)=\frac{t^3}{1-\alpha c^3}\beta+x_0\left(1-\frac{t^3(1-\alpha)}{1-\alpha c^3}\right)+\dot{x}_0\left(t-\frac{t^3(1-\alpha c)}{1-\alpha c^3}\right)+\ddot{x}_0\left(\frac{t^2}{2}-\frac{t^3(1-\alpha c^2)}{2(1-\alpha c^3)}\right)-\\ -\int_0^1\frac{t^3(1-s)^3}{6(1-\alpha c^3)}f(s)ds+\frac{\alpha t^3}{6(1-\alpha c^3)}\int_0^1(c-s)^3f(s)ds+\frac{1}{6}\int_0^t(t-s)^3f(s)ds,\end{aligned} \tag{4}$$

or

$$x(t)=f^*(t)+\varphi(t)Fx+Vx, \tag{5}$$

where

$$\left.\begin{aligned}
&\varphi(t)=\frac{t^3}{6\left(1-\alpha c^3\right)},\\
&Vx\equiv-\int_0^t\frac{(t-s)^3}{6}a(s)x(s)ds,\\
&Fx\equiv\int_0^1(1-s)^3a(s)x(s)ds-\alpha\int_0^c(c-s)^3a(s)x(s)ds
\end{aligned}\right\}\qquad(6)$$

1. The generalized method of iterations.

For the approximated solution of the problem (5) let's construct the successive approximation by the following form

$$x_n=f^*+(\varphi-\psi)Fx_n+(V+\psi F)x_{n-1},\ n=1,2,\ldots,\qquad(7)$$

where $x_0(t)$ and $\phi(t)$ are any continuous functions on $[0,1]$.

The equations (7) (at the fixed x_n) represents relative Fredholm integral equation of the second kind with the generated kernel which we can solve exactly. We find it's solution by the following form

$$x_n=f^*+(V+\psi F)x_{n-1}+D_n(\varphi-\psi),$$

where D_n is a desired constant.

Supposing that

$$\int_0^1(1-s)^3a(s)(\varphi(s)-\psi(s))ds-\alpha\int_0^c(c-s)^3a(s)(\varphi(s)-\psi(s))ds\neq1,\qquad(8)$$

the constant D_n is determined by the formula

$$D_n=\frac{Ff^*+F(V+\Psi F)x_{n-1}}{1-F(\varphi-\psi)},$$

i.e. the solution of the equation (7) has the form

$$\begin{aligned}x_n=f^*&+\frac{\varphi-\psi}{1-F(\varphi-\psi)}Ff^*+\frac{\varphi-\psi}{1-F(\varphi-\psi)}FVx_{n-1}+\\&+\frac{(\varphi-\psi)F\psi}{1-F(\varphi-\psi)}Fx_{n-1}+(V+\psi F)x_{n-1}.\end{aligned}\qquad(9)$$

Thus the sequence of the function $\{x_n(t)\}$ determined from the problem (7) now are determined by the equalities (9).

Let's consider the auxiliary linear integral equation

$$x_n=f^*+\frac{\varphi-\psi}{1-F(\varphi-\psi)}Ff^*+\frac{\varphi-\psi}{1-F(\varphi-\psi)}FVx+\frac{(\varphi-\psi)F\psi}{1-F(\varphi-\psi)}Fx+(V+\psi F)x.\qquad(10)$$

It is easy to check that the problems (1), (2) and the integral equation (10) are equivalent.

Let's suppose that the condition

$$\begin{aligned}\gamma=&\frac{\|\varphi-\psi\|}{|1-F(\varphi-\psi)|}\cdot\frac{\|a\|^2}{280\cdot24}\left(1+|\alpha|c^8\right)+\\&+\left(\frac{\|\varphi-\psi\|\cdot|F\psi|}{|1-F(\varphi-\psi)|}+\|\psi\|\right)\cdot\frac{\|a\|}{4}\left(1+|\alpha|c^4\right)+\frac{\|a\|}{24}<1,\ \|x\|=\max_{0\le t\le1}|x(t)|,\end{aligned}\qquad(11)$$

is fulfilled.

Then from the principle of oblate mappings follows that the equation (10) has a unique solution and the solution is t he limit of the approximation (9) and the velocity of convergence is determined by the following inequality

$$\|x(t)-x_n(t)\| \le \gamma^n \|x(t)-x_0(t)\|. \tag{12}$$

Thus the following theorem is proved.

Theorem. *Let the functions* $a(t), f(t),\ (0 \le t \le 1)$ *be continuous on* $[0,1]$. *Let*

$$F[\varphi(t)-\psi(t)] \equiv \int_0^1 (1-s)^3 a(s)(\varphi(s)-\psi(s))ds - \alpha\int_0^1 (c-s)^3 a(s)(\varphi(s)-\psi(s))ds \ne 1.$$

Let finally the condition (11) be fulfilled, i.e. $\gamma < 1$. *Then the sequence of functions* $x_n(t)$ *determined from (9) converges to the unique solution of the problem (1), (2) and the velocity is obtained by the formula (12).*

The other estimation of the velocity of convergence of the velocity of convergence of the successive approximations (5) to the solution of the problem (1), (2) is given below.

For this let's subtract from (9) the equality (10). We'll get

$$x_n - x = \frac{\varphi-\psi}{1-F(\varphi-\psi)} FV(x_{n-1}-x) + \frac{(\varphi-\psi)F\psi}{1-F(\varphi-\psi)} F(x_{n-1}-x) + (V+\psi F)(x_{n-1}-x), \tag{13}$$

$$\|x_n(t)-x(t)\| \le \frac{\|\varphi-\psi\|}{|1-F(\varphi-\psi)|}\cdot\frac{\|a\|^2}{24}\left(\int_0^1 |x_{n-1}(s)-x(s)|ds + |\alpha| c^7 \int_0^c |x_{n-1}(s)-x(s)|ds\right) +$$
$$+\|a\|\cdot\left(\frac{\|\varphi-\psi\|\cdot|F\psi|}{|1-F(\varphi-\psi)|} + \|\psi\|\right)\left(\int_0^1 |x_{n-1}(s)-x(s)|ds + |\alpha| c^3 \int_0^c |x_{n-1}(s)-x(s)|ds\right) +$$
$$+\frac{\|a\|}{6}\int_0^t |x_{n-1}(s)-x(s)|ds,$$

$$\|x_n(t)-x(t)\| \le \frac{\|a\|}{6}\int_0^t |x_{n-1}(s)-x(s)|ds + \|a\|\left[\frac{\|\varphi-\psi\|}{|1-F(\varphi-\psi)|}\left(\frac{\|a\|}{24}+|F\psi|\right)+\|\psi\|\right]\times$$
$$\times\int_0^1 |x_{n-1}(s)-x(s)|ds + \alpha c^3\|a\|\left[\frac{\|\varphi-\psi\|}{1-F(\varphi-\psi)}\left(c^4\frac{\|a\|}{24}+|F\psi|+\|\psi\|\right)\right]\int_0^c |x_{n-1}(s)-x(s)|ds,$$

$$|x_n(t)-x(t)| \le L_0\int_0^t |x_{n-1}(s)-x(s)|ds + L_1\int_0^1 |x_{n-1}(s)-x(s)|ds + L_2\int_0^c |x_{n-1}(s)-x(s)|ds, \tag{14}$$

where

$$L_0 = \frac{\|a\|}{6},\ L_1 = \|a\|\left[\frac{\|\varphi-\psi\|}{|1-F(\varphi-\psi)|}\left(\frac{\|a\|}{24}+|F\psi|\right)+\|\psi\|\right],$$
$$L_2 = \|a\| c^3 |\alpha|\cdot\left[\frac{\|\varphi-\psi\|}{|1-F(\varphi-\psi)|}\left(c^4\frac{\|a\|}{24}+|F\psi|\right)+\|\psi\|\right].$$

From (14) we get easily that

$$|x_n(t)-x(t)| \le \|x-x_0\|\frac{L_0^n}{n!} + \sum_{i=0}^{n-1}\frac{L_0^i}{i!}\alpha_{n-1-i},$$

where

$$\alpha_i = \|x - x_0\| \frac{L_0^i\left(L_1 + c^{i+1}L_2\right)}{(i+1)!} + \sum_{j=0}^{i-1} \frac{L_0^{i-1-j}\left(L_1 + c^{i-j}L_2\right)}{(i-j)!}\alpha_j, \tag{15}$$

$$\alpha_0 = (L_1 + cL_2)\|x - x_0\|.$$

Let's consider the two partial cases of the approximations (7). If we suppose $\psi(t) = \varphi(t)$ then the successive approximations (7) coincide with the ordinary successive approximations and if we suppose $\psi\ (t \equiv 0)$ then the successive approximations (7) coincide with the generalized successive approximations [6].

2. The method of successive substitutions.

Now in (6)substituting under the Volterra operator V the right hand side of the equation (5) we'll get on the first step

$$x(t) = f^* + \varphi Fx + V\left(f^* + \varphi Fx + Vx\right) = (I + V)f^* + (I + V)\varphi Fx + V^2x,$$

at repeating

$$x = (I + V)f^* + (I + V)\varphi Fx + V^2\left(f^* + \varphi Fx + Vx\right) = \left(I + V + V^2\right)f^* + \left(I + V + V^2\right)\varphi Fx + V^3x.$$

And finally after the $n-1$ steps

$$x = f^*(t) + \varphi_n(t)Fx + V^n x, \tag{16}$$

where

$$f_0^*(t) = 0,\ f_0(t) = \sum_{i=0}^{n-1} V^i f^* = f^* + Vf_{n-1}^*,\ \ n = 1,2,...,.$$

$$\varphi_0(t) = 0,\ \varphi_n(t) = \sum_{i=0}^{n-1} V^i\varphi = \varphi + V\varphi_{n-1},\ \ n = 1,2,...,$$

$$V^i x = \int_0^t v_i(t,\tau)x(\tau)d\tau,\ \ i = 1,2,...,$$

$$v_i(t,\tau) = \int_\tau^t v_i(t,s)v_{i-1}(s,\tau)ds,\ \ i = 2,3,...,$$

$$v_i(t,\tau) = -\frac{(t-\tau)^3 a(t)}{6}.$$

Let's estimate $V^n x$. From

$$\|v_1(t,\tau)\| \le \frac{\|a(\tau)\|}{6}$$

follows

$$\left\|\int_0^t v_n(t,\tau)d\tau\right\| \le \frac{\|a(\tau)\|^n}{(4n)!}$$

such that for sufficiently big n $\|V^n x\|$ will be sufficiently little. Discarding $V^n x$ from (16) we'll get the approximated equation

$$x_n = f_n^*(t) + \varphi_n(t)Fx_n, \tag{17}$$

which is a Fredgolm integral equation with the degenerated kernel.

We'll find the solution of the equation (17) by the following form

$$x_n(t) = f_n^*(t) + c_n \varphi_n(t).$$

If

$$1 - F\varphi_n \neq 0, \tag{18}$$

then c_n is determined by the formula

$$c_n = \frac{Ff^*}{1 - F\varphi_n}$$

Therefore

$$x_n(t) = f_n^* + \frac{\varphi_n(t)}{1 - F\varphi_n} Ff_n^*. \tag{19}$$

We can assume this solution as the approximated solution of the equation (5), i.e. of the boundary value problem (1), (2).

Now let's estimate the error of the method. Solving the integral equation of error

$$x - x_n = V^n x + \varphi_n F(x - x_n)$$

we'll get that

$$x - x_n = V^n x + d_n \varphi_n,$$

where

$$d_n = \frac{FV^n x}{1 - F\varphi_n}.$$

Thus

$$x - x_n = V^n x + \varphi_n \frac{FV^n x}{1 - F\varphi_n},$$

$$\|x - x_n\| \leq \|V^n x\| \left[1 + \frac{\|\varphi_n\| \|a\|}{4|1 - F\varphi_n|}\left(1 + |\alpha| c^4\right)\right] \tag{20}$$

or

$$\|x - x_n\| \leq \frac{\|a\|^n}{(4n)!} \|x\| \left[1 + \frac{\|\varphi_n\| \|a\|}{4|1 - F\varphi_n|}\left(1 + |\alpha| c^4\right)\right].$$

Thus the theorem is proved.

Theorem 2. *Let the problem (1), (2) have the unique solution* $x(t)$, *determined on* $[0,1]$. *Let the functions* $a(t)$, $f(t)$ *be continuous and fulfilled the condition (8).*

Then the approximated solution (19) converges to the unique solution of the problem (1), (2) and the velocity of convergence is determined by the formula (20).

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Received September 13, 2000; Revised March 16, 2001.
Translated by Mamedova V.I.

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PERFORMANCE ANALYSIS AND OPTIMIZATION OF BUFFER ALLOCATION STRATEGIES: A STATE SPACE MERGING APPROACH

Abstract

Quality-of-Service (QoS) in high speed packet switching networks is largely determined by buffer size and buffer allocation strategies. Performance evaluations of the buffer allocation strategies are computationally difficult problems due to the complexity of the large state space when the number of traffics and/or the buffer size is large. In this paper, we propose the approach based on the state space merging to avoid these difficulties for the systems supporting two traffic flows when buffer size is large enough. The design and optimization problems are discussed for Complete Sharing (CS) strategy more detail and the results of appropriate numerical experiments are carried out. The objective function is to achieve the desirable level of the blocking (loss) probability (PB) under minimal value of the buffer size.

Keywords: packet switching networks, non-push-out strategies, state space merging, analyze and optimization algorithms

1. Introduction.

To evaluate the congestion of traffic at a computer network node in [4], Irland proposed to use the models of multi-stream queuing systems with finite common waiting room and typed channels in which each stream has its own channels. After this work, these models have successfully been used for the analysis of the performance of buffer sharing strategies at a node in a store-and-forward packet switching networks. To obtain optimal system performance of specific sharing strategies, Irland [4] and Latouche [7] developed some heuristic procedures. Performance evaluation of the buffer allocation strategies is computationally a difficult problem due to the complexity of the large state space when the number of traffics and/or the buffer size is large. These problems have intensively been investigated during the recent two decades, especially after the publication of the classical study of Kamoun and Kleinrock [5], where five strategies were proposed. Their showed that for the Poisson arrivals and exponential service times the probability distribution of the buffer occupancy have a well-known product form.

Buffer allocation strategies can be broadly classified into push-out strategies and non-push-out strategies. Strategies, which can accept an arriving packet by dropping another packet from the buffer, are known as push-out strategies. In this paper, we consider the non-push-out type strategies, which do not allow the drop of already accepted packet of any type. Note that the above mentioned five strategies [5] are strategies of non-push-out type. Further references might be found in [3].

There have been a few works in which the finding of the optimal strategy in the class of non-push-out strategies have been addressed. In [1], Foschini and Gopinath proved that in case two output ports for the Markovian systems, the optimal sharing strategy in the sense of minimization of the *PB* (or equivalently of maximization of the throughput) is in the class of