ALIEV A.B., AHMEDOVA Zh.B.

GLOBAL SOLUTIONS OF NON-LINEAR FOURTH ORDER HYPERBOLIC INEQUALITIES

Abstract

In this paper considered variation inequality for the fourth order non-linear hyperbolic operators and proved one-valued solvability of this problem.

Let $\Omega \subset R^n$ be a bounded domain with the smooth boundary Γ. Let's consider the fourth order non-linear hyperbolic operator in the cylinder $Q = (0,T) \times \Omega$

$$L(u) = u'' + \Delta\big(a(t,x,u)\Delta u\big), \tag{1}$$

with the boundary conditions of Dirichlet

$$u\big|_\Gamma = 0, \quad \frac{\partial u}{\partial \nu}\bigg|_\Gamma = 0 \tag{2}$$

and with the initial conditions

$$u(0,x) = u_0(x), \quad u'(0,x) = u_1(x), \tag{3}$$

where $\frac{\partial}{\partial \nu}$ - is the derivative in the direction of external normal ν, $u' = u_t$, $u'' = u_{tt}$.

$$a(\cdot) \in C^2[0,T] \times \overline{\Omega} \times R, \quad a(t,x,u) \ge a_0 > 0. \tag{4}$$

Let

$$\overset{\circ}{W}{}_2^{2k} = \left\{ u:\ u \in W_2^{2k}(\Omega); \frac{\partial^i u}{\partial \nu^i}\bigg|_\Gamma = 0,\ i = 0,1,\dots,k-1 \right\},$$

where $W_2^{2k}(\Omega)$ - is a Sobolev space.

Let's determine the space

$$H_T(2k,2m) \equiv H_T\left(\overset{\circ}{W}{}_2^{2k}, \overset{\circ}{W}{}_2^{2m}, L_2(\Omega)\right) = \left\{ u : u \in L_\infty\left(0,T;\overset{\circ}{W}{}_2^{2k}\right),\right.$$

$$\left. u' \in L_\infty\left(0,T;\overset{\circ}{W}{}_2^{2m}\right), u'' \in L_\infty\big(0,T;L_2(\Omega)\big) \right\}.$$

Let us denote by K_0 and K_λ the corresponding convex closed sets in the spaces $\overset{\circ}{W}{}_2^{2}$ and $\overset{\circ}{W}{}_2^{4}$:

$$K_0 = \left\{ u : u \in \overset{\circ}{W}{}_2^{2}, |\Delta u(x)| \le 1, \text{ almost everywhere on } \Omega \right\},$$

$$K_\lambda = \left\{ u : u \in \overset{\circ}{W}{}_2^{4},\ |\Delta u(x)| \le 1,\ \left|\Delta^2 u(x)\right| \le \frac{1}{\lambda} \quad \text{almost everywhere on } \Omega \right\}.$$

Let's determine the set

$$H_T(2m,2k,K_\lambda) \equiv H_T\left(\overset{\circ}{W}{}_2^{2m}, \overset{\circ}{W}{}_2^{2k}, K_\lambda\right) = \left\{ u : u \in H_T\left(\overset{\circ}{W}{}_2^{2m}, \overset{\circ}{W}{}_2^{2k}, L_2(\Omega)\right),\right.$$

$$\left. u'(t,\cdot) \in K_\lambda \text{ almost everywhere on } (0,T) \right..$$

Let's introduce notations:

$$<u,v>(t)=\int_\Omega u(t,x)v(t,x)dx.$$

The next result is true.

The main theorem: *Let the condition (4) is satisfied. Then for any* $f\in W_2^1[0,T;L_2(\Omega)]$,

$$u_0\in \overset{\circ}{W}{}_2^4\cap C^4(\overline{\Omega}),\ u_1\in K_0 \tag{5}$$

exists the unique function $u(\cdot)\in H_T(2,2,K_0)$, *which satisfies the conditions (2), (3) and the inequality.*

$$<u'',v-u'>(t)+<a(t,x,u)\Delta u,\Delta v-\Delta u'>(t)\geq <f,v-u'>(t), \tag{6}$$

almost everywhere on $(0,T)$,

where $v\in K_0$.

We denote that the analogous problem for the operator (1) with the Rike's boundary conditions.

$$u|_\Gamma=0,\quad \Delta u|_\Gamma=0 \tag{7}$$

was investigated in paper [1]. The method of receiving some a priori estimates, which used in [1] are not applied. The reason of this is that the resolvent of Laplacian operator with the Dirichlet boundary condition doesn't transfer the set K_λ into itself. It is impossible to pas to the limit in standard scheme without these a priori estimates. In this paper we use some different methods.

For simplicity we will use $a(u)=a(t,x,u)$ for proff.

At first we consider the variation inequality as in [1].

$$<L(u_\lambda)-f,v-u_\lambda>(t)\geq 0,\ v\in K_\lambda \text{ almost everywhere on } (0,T), \tag{8}$$

$$u_\lambda(0,\lambda)=u_0(x),\quad u_\lambda'(0,\lambda)=u_{1\lambda}(x),\ x\in\Omega, \tag{9}$$

where

$$u_{1\lambda}\in K_\lambda \quad \text{and} \quad u_{1\lambda}\to u_1 \quad \text{in} \quad \overset{\circ}{W}{}_2^2. \tag{10}$$

Using the proof method of the corresponding theorem (with the boundary conditions Rike (7) given in [1]) it is proved that at any $\lambda>0$ the problem (8)-(9) has a unique solution

$$u_\lambda\in H_T(4,4,K_\lambda). \tag{11}$$

In view of (11) we have

$$|\Delta u_\lambda'(t,x)|\leq 1 \text{ almost everywhere on Q}. \tag{12}$$

Hence using (5) we get the next a priori estimation

$$|\Delta u_\lambda(t,x)|\leq C,\quad \text{almost everywhere on Q}, \tag{13}$$

where $C_1>0$ doesn't depend on λ. Using the shift method in future we will get the priori estimation

$$\int_\Omega |u_\lambda''(t,x)|^2dx\leq C_2, \tag{14}$$

where $C_2>0$ doesn't depend on $\lambda>0$.

By virtue of (12)-(14) we can choose such sub-sequence denoted by $\{u_{\lambda_n}\}$, that

$$u_{\lambda_n}\to u \quad * - \text{weakly in } L_\infty\left(0,T;\overset{\circ}{W}{}_2^2\right), \tag{15}$$

$$u'_{\lambda_n} \to u' \quad * - \text{weakly in } L_\infty\left(0,T;\overset{\circ}{W}{}_2^2\right), \tag{16}$$

$$\Delta u_{\lambda_n} \to \Delta u \quad * - \text{weakly in } L_\infty(Q), \tag{17}$$

$$\Delta u'_{\lambda_n} \to \Delta u' \quad * - \text{weakly in } L_\infty(Q), \tag{18}$$

$$u''_{\lambda_n} \to u'' \quad * - \text{weakly in } L_\infty(o,T;L_2(\Omega)), \tag{19}$$

$$a(u_{\lambda_n})\Delta u_{\lambda_n} \cdot \Delta u'_{\lambda_n} \to A(t,x) \quad * - \text{weakly in } L_\infty(Q), \tag{20}$$

where $\lambda_n \to 0$ when $n \to \infty$.

From (15), (16) and (19) follows that

$$u_{\lambda_n} \to u \text{ in } C([0,T]L_2(\Omega)), \tag{21}$$

$$u'_{\lambda_n} \to u' \text{ in } C([0,T], L_2(\Omega)). \tag{22}$$

From (4), (13) and (21) follows that

$$a(u_{\lambda_n}) \to a(u) \text{ in } C([0,T], L_2(\Omega)). \tag{23}$$

Statement: *The sequence* $\{u_{\lambda_n}\}$ *is fundamental in* $C\left([0,T];\overset{\circ}{W}{}_2^2(\Omega)\right)$.

Proof: Let's consider the expression

$$R_t(u,v) = \int_0^t\int_\Omega (a(u)\Delta u - a(v)\Delta v)(\Delta u' - \Delta v')dxds$$

and let's prove that

$$\overline{\lim_{n\to\infty}}\,\overline{\lim_{m\to\infty}} R_t(u_{\lambda_n}, u_{\lambda_m}) = \overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}} 2\int_0^t < L(u_{\lambda_m}),\ u'_{\lambda_n} - u'_{\lambda_m} > ds. \tag{24}$$

Indeed we have the next equality using (15)-(20).

$$\overline{\lim_{m\to\infty}} R_t(u_{\lambda_n}, u_{\lambda_m}) = \int_0^t\int_\Omega a(u_{\lambda_n})\Delta u_{\lambda_n}\Delta u'_{\lambda_n} dxds - \int_0^t\int_\Omega a(u_{\lambda_n})\Delta u_{\lambda_n}\Delta u' dxds -$$

$$- \int_0^t\int_\Omega a(u)\Delta u \Delta u'_{\lambda_n} dxds + \int_0^t\int_\Omega A(s,x)dxds.$$

Then passing to the limit at $\lambda_n \to 0$ and taking into account (15)-(20) we have

$$\overline{\lim_{n\to\infty}}\,\overline{\lim_{m\to\infty}} R_t(u_{\lambda_n}, u_{\lambda_m}) = 2\int_0^t\int_R A(s,x) - 2\int_0^t\int_\Omega a(u)\Delta u \Delta u' dxds. \tag{25}$$

On the other hand

$$\overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}} 2\int_0^t\int_\Omega < L(u_{\lambda_n}),\ u'_{\lambda_n} - u'_{\lambda_m} > dxds = \overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}} 2\int_0^t\int_\Omega < u''_{\lambda_n}, u'_{\lambda_n} - u'_{\lambda_m} > dxds -$$

$$- \overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}} 2\int_0^t\int_\Omega \Delta(a(u_{\lambda_m})\Delta u_{\lambda_n})(u'_{\lambda_n} - u'_{\lambda_m})dxds. \tag{26}$$

From (16),(19) it follows that the first sum is equal to zero and from (16),(20) it follows that the second sum is equal to

$$2\int_0^t\int_\Omega A(s,x)dxds - 2\int_0^t\int_\Omega a(u)\Delta u \Delta u' dxds. \tag{27}$$

The equality (24) is the corollary of (25)-(27).

We consider the inequality

$$< L(u_{\lambda_n}), v - u'_{\lambda_n} > (t) \geq 0,\ v \in K_n \quad \text{almost everywhere on } (0,T). \tag{28}$$

Since $K_{\lambda_m} \subset K_{\lambda_n}$ when $n < m$ than $u'_{\lambda_m}(t,\cdot) \in K_{\lambda_m}$ almost everywhere on (0,T).

Therefore we can take $v = u_{\lambda_m}(t,\cdot)$ in (28). Then we will get the next inequality

$$< L(u_{\lambda_n}), u'_{\lambda_m} - u'_{\lambda_n} > (t) \geq 0. \tag{29}$$

From (24) and (29) it follows that

$$\overline{\lim_{n\to\infty}}\ \overline{\lim_{m\to\infty}} R_t(u_{\lambda_n}, u_{\lambda_m}) \leq 0. \tag{30}$$

We consider the expression

$$E(u,v) = \int_\Omega (a(u)\Delta u - a(v)\Delta v)(\Delta u - \Delta v) dx. \tag{31}$$

Using the priori estimations (12)-(14) and also applying the Lagrangian formula and the Hölder inequality we will get that

$$E(u_{\lambda_n}, u_{\lambda_m}) \geq c_3 \left\| u_{\lambda_n} - u_{\lambda_m} \right\|^2_{\overset{\circ}{W}{}_2^2} - c_4 \left\| u_{\lambda_n} - u_{\lambda_m} \right\|^2_{L_2(\Omega)}, \tag{32}$$

where $c_3 > 0$, $c_4 > 0$ doesn't depend on n and m.

On the other hand

$$E(u(t,\cdot), v(t,\cdot)) = E(u(0,\cdot), v(0,\cdot)) + \int_0^t \frac{d}{ds} E(u(s,\cdot), v(s,\cdot)) ds = E(u(0,\cdot), v(0,\cdot)) +$$

$$+ \int_0^t \int_\Omega (a(u), \Delta u - a(v)\Delta v)(\Delta u' - \Delta v') dx ds + \int_0^t \int_\Omega (a(u), \Delta u' - a(v)\Delta v')(\Delta u - \Delta v) dx ds +$$

$$+ \int_0^t \int_\Omega (a'(u), u'\Delta u - a'(v) v' \Delta v)(\Delta u - \Delta v) dx ds = I_0 + I_1 + I_2 + I_3,$$

where

$$I_0 = E(u(0,\cdot), v(0,\cdot)),$$

$$I_1 = R_t(u,v),$$

$$I_2 = R_t(u,v) + I'_2 = \int_0^t \int_\Omega (\Delta u' \Delta v + \Delta u \Delta v')(a(u) - a(v)) dx ds$$

$$I_3 = \int_0^t \int_\Omega \left(\int_0^1 a''(v + \tau(u - v)) d\tau \right) u' \Delta u\, (u - v)(\Delta u - \Delta v) dx ds +$$

$$+ \int_0^t \int_\Omega a'(v) \Delta u (u' - v')(\Delta u - \Delta v) dx ds + \int_0^t \int_\Omega a'(v) v' (\Delta u - \Delta v)^2 dx ds.$$

Thus

$$E(u,v) = E(u_0, v_0) + 2R_t(u,v) + G(u,v), \tag{33}$$

where

$$G(u,v) = I'_2 + I_3.$$

Taking into account the priori estimations (12)-(13) and using the inequality we'll get the following estimation:

$$\left| G(u_{\lambda_n}, u_{\lambda_m}) \right| \leq c_5 \int_0^t \left[\left\| u_{\lambda_n} - u_{\lambda_m} \right\|_{L_2(\Omega)} + \left\| u_{\lambda_n} - u_{\lambda_m} \right\|^2_{\overset{\circ}{W}{}_2^2} + \left\| u'_{\lambda_n} - u'_{\lambda_m} \right\|^2_{L_2(\Omega)} \right] ds. \tag{34}$$

From (33) and (34) we'll get that

$$\overline{\lim_{n\to\infty}}\,\overline{\lim_{m\to\infty}} E\left(u_{\lambda_n},u_{\lambda_m}\right)\leq 2\,\overline{\lim_{n\to\infty}}\,\overline{\lim_{m\to\infty}} R_t\left(u_{\lambda_n},u_{\lambda_m}\right)+c_5\,\overline{\lim_{n\to\infty}}\,\overline{\lim_{m\to\infty}}\int_0^t\Big[\left\|u_{\lambda_n}-u_{\lambda_m}\right\|_{L_2(\Omega)}+$$

$$+\left\|u_{\lambda_n}-u_{\lambda_m}\right\|^2_{\overset{\circ}{W}{}_2^2}+\left\|u'_{\lambda_n}-u'_{\lambda_m}\right\|^2_{L_2(\Omega)}\Big]ds\,.$$

Using (21), (22), (30), (32) and (34) we have the next estimation from the last inequality

$$\overline{\lim_{n\to\infty}}\,\overline{\lim_{m\to\infty}}\left\|u_{\lambda_n}-u_{\lambda_m}\right\|^2_{\overset{\circ}{W}{}_2^2}\leq\overline{\lim_{n\to\infty}}\,\overline{\lim_{m\to\infty}}\int_0^t\left\|u_{\lambda_n}-u_{\lambda_m}\right\|^2_{\overset{\circ}{W}{}_2^2}ds\,.$$

From here we have

$$\overline{\lim_{n\to\infty}}\,\overline{\lim_{m\to\infty}}\left\|u_{\lambda_n}-u_{\lambda_m}\right\|^2_{\overset{\circ}{W}{}_2^2}=0\,.$$

Hence it follows that

$$\lim_{\substack{n\to\infty\\ m\to\infty}}\left\|u_{\lambda_n}-u_{\lambda_m}\right\|_{\overset{\circ}{W}{}_2^2}=0\,, \tag{35}$$

i.e. at any $t\in[0,T]$ the sequence $\left\{u_{\lambda_n}(t,\cdot)\right\}$ is fundamental in $\overset{\circ}{W}{}_2^2$.

Now let us prove that the sequence $\left\{u_{\lambda_n}\right\}$ is fundamental in $C\left([0,T],\overset{\circ}{W}{}_2^2\right)$.

We estimate the $R_t\left(u_{\lambda_n},u_{\lambda_m}\right)$ from above using (12)-(13). Taking into account the given estimation in (33) we will get that

$$E\left(u_{\lambda_n},u_{\lambda_m}\right)\leq c_7\int_0^t\left\|u_{\lambda_n}-u_{\lambda_m}\right\|_{\overset{\circ}{W}{}_2^2}ds+\psi\left(u_{\lambda_n},u_{\lambda_m}\right)(t)\,, \tag{36}$$

where

$$\psi\left(u_{\lambda_n},u_{\lambda_m}\right)(t)=c_8\int_0^t\left(\left\|u_{\lambda_n}-u_{\lambda_m}\right\|_{L_2(\Omega)}+\left\|u'_{\lambda_n}-u'_{\lambda_m}\right\|^2_{L_2(\Omega)}=\left\|u_{\lambda_n}-u_{\lambda_m}\right\|^2_{L_2(\Omega)}\right)ds\,.$$

From (21) and (22) follows that

$$\psi\left(u_{\lambda_n},u_{\lambda_m}\right)(t)\to 0 \text{ in } C[0,T] \text{ when } n,m\to\infty\,. \tag{37}$$

In view of (13) and (35) by the Lebesque theorem on limit passage under the integral we will get:

$$\int_0^T\left\|u_{\lambda_n}-u_{\lambda_m}\right\|_{\overset{\circ}{W}{}_2^2}dt\to 0 \text{ when } n,m\to\infty\,. \tag{38}$$

Taking into account the (37) and (38) in (36) we have

$$E\left(u_{\lambda_n},u_{\lambda_m}\right)\to 0 \text{ in } C[0,T] \text{ when } n,m\to\infty\,.$$

Then allowing for this in (32) we will get that

$$\left\|u_{\lambda_n}-u_{\lambda_m}\right\|_{C\left([0,T],\overset{\circ}{W}{}_2^2\right)}\to 0 \text{ when } n,m\to\infty\,, \tag{39}$$

i.e. $\left\{u_{\lambda_n}\right\}$ is fundamental in $C\left([0,T];\overset{\circ}{W}{}_2^2\right)$.

Now passing to the limit in the inequality (8)-(9) we will get the statement of basic theorem on the existence of the solution.

The uniqueness of the solution is proved by a standard scheme (s.[2]).

References

[1]. Aliev A.B., Ahmedova J.B. *Variational inequalities for fourth order quasilinear hyperbolic operator*. Proceedings of IMM Azerbaijan AS, vol.XII(XX), 2000, 10-17.
[2]. Lions J.L. *Some methods of solution of non-linear boundary problems.* M., Mir, 1972.

Akbar B. Aliev, Zhala B. Ahmedova
Azerbaijan Technical University.
25, H.Javid av., 370073, Baku, Azerbaijan.

Received December 18, 2000; Revised May 14, 2001.
Translated Mamedova V.I.

ALIGULIYEV R.M.

A PHRAGMEN-LINDELÖF TYPE THEOREM FOR A NON-DIVERGENT STRUCTURE LINEAR ELLIPTIC EQUATION OF THE SECOND ORDER

Abstract

At the paper we consider a non-uniformly degenerating at infinity, non-divergent structure elliptic equation of the second order, containing small members. The condition providing the truth of Phragmen-Lindelöf type theorem for its solutions are found for minor coefficients of the equation.

Let $\mathbf{E}_n$ be an n-dimensional Euclidean space, $n \ge 2$, D be an infinite domain with the boundary ∂D, situated in the cone $K = \{x : \left(\sum_{i=1}^{n-1} x_i^2\right)^{1/2} \le kx_n, 0 < x_n < \infty\}$, $0 < k < 1/16$.

Let's consider in D the following problem:

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x) = 0, \tag{1}$$

$$u\big|_{\partial D} = 0, \tag{2}$$

where $\|a_{ij}(x)\|$-is a real symmetric matrix.

Let's suppose that for all $x \in D$ and $\xi \in \mathbf{E}_n$ the condition is fulfilled

$$\mu\sum_{i=1}^{n}\lambda_i(x)\xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \mu^{-1}\sum_{i=1}^{n}\lambda_i(x)\xi_i^2, \tag{3}$$

where $\mu \in (0,1]$ -is a constant, $\lambda_i(x) = \left(1 + |x|_\alpha\right)^{\alpha_i}, |x|_\alpha = \sum_{i=1}^{n}|x_i|^{2/(2+\alpha_i)}, \alpha_i > -2, i = 1,...,n$,

We'll suppose also that

$$-c_0 \le c(x) \le 0, \tag{4}$$

c_0 - is a positive constant.

The aim of the present paper is the obtaining of condition on the minor coefficients $b_i(x), (i = 1,...,n)$ and $c(x)$ at whose fulfillment Phragmen-Lindelöf theorem is valid for solutions of the problem (1)-(2).

Note that for non-divergent second order uniformly elliptic equations not containing minor coefficients analogous results were obtained in papers [1-4]. As for uniformly elliptic equations of divergent structure, then we show in this connection in [5-6]. And we note papers [7-10], in which theorems of Phragmen-Lindelöf type for quasi-linear elliptic and parabolic equations were obtained. In papers [11-12] analogous theorems were proved for degenerating on infinity elliptic equations without minor coefficients. In case when minor coefficients are present in uniform elliptic equation and principal part of the operator $\mathcal{L}$ has divergent form theorem of Phragmen-Lindelöf type was established in [13] provided $divB(x) \le 0, (B(x), x) \le 0$ for all $x \in D$, where $B(x) = (b_1(x),...,b_n(x))$.

If in the presence of the minor coefficients the principal part of the operator $\mathcal{L}$ is written in non-divergent form then the analogous results valid under the condition $\left(B(x), x - x^0\right) \le 0$ for all $x \in D, x^0 \in \bar{l}$, where $\bar{l}$ is some ray, coming from origin and