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THE APPROXIMATION OF SOLUTION OF THE STATIONARY PROBLEM OF SALINE FINGERS AT HIGH SUPERCRITICALITY

Abstract

The problem of stationary binary finger convention has been considered. The article proved that the conception of paralysis layer is quite just. The solution of this problem is described by means of simple and the algebraic equation. In the layers attaching the ends of saline fingers the problem leads off to the solution of non-linear regional sum for ordinary equation with the parameter depending on the square function.

1. Saline fingers very often appear in ocean at warm up and salinization of sea-water from above and create very intensive mechanism of heat and salt transport in top-layer of the ocean. The basic experimental and theoretical facts about saline fingers stated in the monograph [1]; point to the lag of the theoretical investigations of stabilization of saline fingers was clarified quite recently [2], and direct numerical analysis of the problem [3] was carried out at not do high supercriticaly in the parameter region far from the real oceanic conditions. In this connection in papers [4, 5] the mean field method for description of double convective diffusion (of heat and salt) in sea-water quite analogous to the mean field theory for the ordinary convection [6] was developed.

The single-mode theory of saline fingers allowing to reveal the qualitative regularities of the phenomenon was suggested in paper [7]. Papers [8, 9] are divoted to defining the limits of applicability of the single-mode approximation.

In particular, in these papers it was clarified that single-mode approximation is suitable for not high supercriticality and description of saline fingers was advanced to mild supercriticalities [8, 9].

It's important to note that the investigation of stationary situation for the free layers was carried out in papers [7-9], since in the all pointed out earlier papers the solution of non-stationary equations for saline fingers reach to stable stationary mode independent on initial data, and boundary conditions for velocity don't exert considerable influence on the solution of problem [1].

Papers [8, 9] showed that with increasing the supercriticality the number of generated modes sharply increases with connection of which at high supercriticality it's rational to refuse from the mode analysis of the problem which use the Fourier expansions of dynamic variables on the vertical coordinate.

2. It's easy to obtain dimensionless stationary correlations of non-linear theory of saline fingers in the approximation of mean field at large Luise numbers $\tau = {}^{\nu_\theta}\!/_{\nu_s} >> 1$ (ν_θ and ν_s are the coefficients of thermal diffusivity and salt diffusion) from the equations given in paper [7].

These correlations have the form

$$\left(\delta^2 D^2 - 1\right)^3 W = Q\left[W - \left(\delta^2 D^2 - 1\right)S\right], \tag{1}$$

$$W\left(F + WS\right) = \left(\delta^2 D^2 - 1\right)S, \tag{2}$$

$$D\bar{S} = F + WS - r, \tag{3}$$

where

$$Q = Ra\left({}^{\delta}\!/_{\pi}\right)^4, \quad D = {}^{d}\!/_{dz_1}, \tag{4}$$

and the boundary conditions for the system (1)-(3) are

$$W(0)=W''(0)=W^{IV}(0)=0\,,\ \ S(0)=\overline{S}(0)=0\,,$$
$$W(\pi)=W''(\pi)=W^{IV}(\pi)=0\,,\ \ S(\pi)=\overline{S}(\pi)=0\,. \tag{5}$$

Here in addition to (1)-(5) there is one more boundary value problem

$$\left(\delta^2 D^2-1\right)T=W\,, \tag{6}$$
$$D\overline{T}=WT-F_\theta\,, \tag{7}$$
$$T(0)=\overline{T}(0)=T(\pi)=\overline{T}(\pi)=0\,. \tag{8}$$

Here W,S,T are dimensionless vertical distribution of the vertical and also the fluctuations of salinity and temperature, $\overline{S}$ and $\overline{T}$ are dimensionless mean magnitudes of salinity and temperature, F is dimensionless diffusion flux, F_θ is some part of the heat flow, z_1 is dimensionless vertical variable, Ra are Rayleigh numbers. The dimensionless variables are

$$k_\perp=\frac{\pi}{H},\ \ \delta=\frac{k_\perp}{k},\ \ \ z_1=k_\perp z,\ \ \ W=\Phi(x,y)\nu_\theta k_\perp W(z_1)/\delta\tau\,,$$
$$S=\Phi(x,y)\frac{B}{k_\perp}\frac{\delta}{r}S(z_1)\,,\ \theta=\Phi(x,y)\frac{A}{k_\perp}\frac{\delta}{\tau}T(z_1),$$
$$\overline{S}=\frac{B}{k_\perp}\frac{1}{r}\overline{S}(z_1)\,,\ \ \ \overline{\theta}=\frac{A}{k_\perp}\frac{1}{\tau^2}\overline{T}(z_1),\ F_\theta=\nu_\theta A\left(-1+F_\theta/\tau^2\right),\ F_s=-\nu_S B\frac{F}{r}\,,$$
$$\left(r={}^{\tau}\!/_{R}\equiv\frac{\beta B}{\alpha A}\tau\right). \tag{9}$$

Here H is the thickness of the layer, $k_\perp$ and k are vertical and horizontal wave numbers, x, y are horizontal coordinates and z is vertical coordinate, B and A are the given vertical salinity and temperature gradients, R is floatability number, α and β are density coefficients of the temperature and salinity, $\Phi(x,y)$ is the function satisfying the Helmholtz inequality

$$\Delta_H\Phi+k^2\Phi=0,\ \ \ \overline{\Phi}^2=1\ \ \ \left(\Delta_H=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{y^2}\right) \tag{10}$$

and describing the horizontal structure of saline fingers.

It is easy to see [4, 5] that the pulsating variables W,S,T are even, but mean magnitudes $\overline{S},\overline{T}$ are odd functions with respect to the middle of the layer, i.e. points $z_1={}^{\pi}\!/_{2}$. Hence it follows that in addition to the condition (5), (8)

$$\overline{S}\left({}^{\pi}\!/_{2}\right)=\overline{T}\left({}^{\pi}\!/_{2}\right)=0\,. \tag{11}$$

Then integrating the correlations (3) and (7) over z_1 from 0 to ${}^{\pi}\!/_{2}$ we'll obtain

$$F=r-\frac{2}{\pi}\int_0^{\pi/2}WS\,dz_1\,, \tag{12}$$

$$F_\theta=\frac{2}{\pi}\int_0^{\pi/2}WT\,dz_1\,. \tag{13}$$

It's easy to show that the integrals in (12), (13) are negative, i.e. $F>0, F_\theta<0$, which corresponds to the evident physical meaning on the directions of flows of heat and salt.

The equations (1), (2) with the conditions (4) and with the definition of F from (12), Q and D from (4) completely define the nonlinear boundary value problem for the variables W and S when the external parameters Ra and r are given if the dimensionless value $\delta = {k_\perp}/{k}$ defining the width of the connective bolt nest is known. The problem (6), (7) is easily solved in exactly the same way if the function $W(z_1)$ is known and the value F_θ is defined from (13). In order to define the known value δ we'll use the extremal principle of maximal power of weighting [7], generalizing the well-known Molcus principle on maximum of the heat flow [10]. The weighting power averaged on layer in the dimensionless variables (9) will take the form

$$<f_v>=g v_\theta \alpha \frac{2}{\pi}\int_0^{\pi/2} W(T-S)dz_1 .$$

Then maximal power of weighting principle will take the form

$$\int_0^{\pi/2} W(T-S)dz_1 = \max_\delta \qquad (14)$$

or allowing for (12) and (13)

$$F+F_\theta = \max_\delta . \qquad (14^a)$$

3. As is well known [1] at high supercriticality, i.e. sufficiently large values Ra and r saline fingers are very thin, i.e. $\delta<<1$. In this case we can use ideals of boundary layer [11] solving the given system and as "external" solutions we can use equations with theorem addents of the order δ and higher. Obviously this system is reduced to algebraic one and we write its solution as:

$$S_\ell=\frac{Q+1}{Q}\sqrt{F\frac{Q}{Q+1}-1}\,,\ T_\ell=-W_\ell=\sqrt{F\frac{Q}{Q+1}-1}\,,$$

$$D\bar{S}_\ell=\frac{Q+1}{Q}-r\,,\ \ D\bar{T}_\ell=-\frac{Q}{Q+1}F-F_\theta . \qquad (15)$$

The formulas (15) show that external solutions for all pulsatinal vaiables are constants and for mean fields they are linear functions from vertical variables satisfying the condition (11). Further the condition of existence of nontrivial external solution has the form:

$$F>\frac{Q+1}{Q}, \qquad (16)$$

and the physical condition for the mean salinity fields $\bar{S}$ to indemnify the given external salinity field BZ has the form

$$r>\frac{Q+1}{Q}. \qquad (17)$$

This condition leads to the fact that the complete salinity gradient in the external region turns out to be less than given external salinity gradient B. Such a "rinsing out" of the salinity gradient was repeatedly justified by the experiment.

It's also clear that the external solution is true outside of small neighborhoods of the points 0 and π with the length of the order δ and as δ less as the condition (15) becomes more "representative".

4. It's absolutely obvious that all the structure of solutions is defined exactly in narrow band of the boundary layers with the width of the order δ. In this case the problem on solving of the boundary value problem (1)-(5) appears which evidently can be realized only by numerical methods. However, the approximate analytical solution of the problem to which we'll pass now is of interest.

Consider the problem (1)-(5) where at first the parameter δ will be supposed known. We shall formulate the following procedure of the method of successive approximations

$$\left(\delta^2 D^2 - 1\right)^3 W_{n+1} = Q\left[W_{n+1} - \left(\delta^2 D^2 - 1\right)S_{n+1}\right], \tag{18}$$

$$\left(\delta^2 D^2 - 1\right)S_{n+1} = W_n\left(F_n + W_n S_n\right), \tag{19}$$

$$F_n = r - \frac{2}{\pi}\int_0^{\pi/2} W_{n+1} S_{n+1} dz_1 , \tag{20}$$

$$(n = 0,1,2,...)$$

where W_n and S_n satisfy the boundary conditions (5). We'll choose the external solution (15) as zero approximation. Then the system of the first approximation will take the form

$$\left(\delta^2 D^2 - 1\right)^3 W_1 = Q\left(W_1 - W_\ell \frac{Q+1}{Q}\right), \tag{21}$$

$$\left(\delta^2 D^2 - 1\right)S_1 = \frac{Q+1}{Q} W_\ell . \tag{22}$$

Defining F^0 through W_ℓ with the help of the formula for W_ℓ in (15) we'll obtain that the formula (12) will take the form

$$F^0 \equiv \frac{Q+1}{Q}\left(W_\ell^2 + 1\right) = r - \frac{2}{\pi}\int_0^{\pi/2} W_1 S_1 dz_1 . \tag{23}$$

For dimensionless temperature from (6) we'll obtain

$$\left(\delta^2 D^2 - 1\right)T_1 = W_1 . \tag{24}$$

the boundary value problems (21), (22) with the conditions (5), (24) and (8) are linear and their solution is easily found by the Fourier method and has the form:

$$\{W_1, S_1, T_1\} = \sum_{k=1}^{\infty} \{W_k, S_k, T_k\} \sin(2k-1)z ,$$

$$W_k = \frac{4}{\pi}\frac{(Q+1)W_\ell}{(2k-1)\left[Q + \left(1+\delta_k^2\right)^3\right]}, \quad S_k = -\frac{4}{\pi}\frac{(Q+1)W_\ell}{Q(2k-1)\left(1+\delta_k^2\right)},$$

$$T_k = -\frac{W_k}{1+\delta_k^2}, \ \left(\delta_k = (2k-1)\delta\right). \tag{25}$$

The expressions for the flows F^0 and F_θ^0 can be represented in the form

$$F^0 = \frac{Q+1}{Q} + \frac{1}{\varepsilon_W}\left(r - \frac{Q+1}{Q}\right), \tag{26}$$

$$F_\theta^0 = -\left(\frac{Q_r}{Q+1} - 1\right)\frac{\varepsilon_T}{\varepsilon_W}, \tag{27}$$

where

$$\varepsilon_W = \varepsilon_W(\delta, Q) = \frac{8\delta^2}{\pi^2}\left\{\sum_{k=1}^{\infty}\frac{1}{1+\delta_k^2} + \sum_{k=1}^{\infty}\frac{3(1+\delta_k^2)+\delta_k^4}{(1+\delta_k^2)\left[Q+(1+\delta_k^2)^3\right]}\right\}, \tag{28}$$

$$\varepsilon_T = \varepsilon_T(\delta, Q) = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{(Q+1)^2}{(2k-1)^2(1+\delta_k^2)\left[(1+\delta_k^2)^3+Q\right]^2}. \tag{29}$$

5. Now we'll estimate the asymptotic behaviour of the values $\varepsilon_W(\delta, Q)$ and $\varepsilon_T(\delta, Q)$ at $\delta \to 0$ formally assuming that Q is a parameter independent on δ. Replacing summation in (28), (29) by integration on trapezoidal formula

$$f \equiv \sum_{k=1}^{\infty} f_k \approx \frac{f_1}{2} + \int_1^{\infty} f_k dk$$

we'll represent the series (28) and (29) in the from

$$\varepsilon_W \approx \frac{2\delta}{\pi}[1 + f(Q)] + \varphi(\delta, Q), \tag{30}$$

where

$$\varphi(\delta, Q) = \frac{4\delta^2}{\pi^2}\frac{Q+(1+\delta^2)^3+3(1+\delta^2)+\delta^4}{(1+\delta^2)\left[Q+(1+\delta^2)^3\right]} -$$

$$-\frac{4\delta}{\pi^2}\left\{arctg\delta + \int_0^{\pi}\frac{(3+3x^2+x^4)}{(1+x^2)\left[Q+(1+x^2)^3\right]}dx\right\}, \tag{31}$$

$$f(Q) = \frac{2}{\pi}\int_0^{\infty}\frac{(3+3x^2+x^4)}{(1+x^3)\left[Q+(1+x^2)^3\right]}dx, \tag{32}$$

$$\varepsilon_T = 1 + \psi(\delta, Q), \tag{33}$$

$$\varphi(\delta, Q) = \frac{4\delta^2}{\pi^2}\frac{Q+(1+\delta^2)^3+3(1+\delta^2)+\delta^4}{(1+\delta^2)\left[Q+(1+\delta^2)^3\right]} -$$

$$-\frac{4\delta}{\pi^2}\left\{arctg\delta + \int_0^{\pi}\frac{(3+3x^2+x^4)}{(1+x^2)\left[Q+(1+x^2)^3\right]}dx\right\}. \tag{34}$$

From (31) and (34) follows that at $0 < \delta << 1$

$$\varphi(\delta, Q) = \varphi_0(Q)0(\delta_3), \qquad \psi(\delta, Q) = \psi_0(Q)0(\delta) \tag{35}$$

holds.

Then from (30) and (33) we have

$$\varepsilon_W \approx \frac{2\delta}{\pi}(1 + f(Q)), \quad \varepsilon_T \approx 1 \tag{36}$$

and formula for the flows F^0 and F_θ^0 take the form

$$F^0 \approx \frac{Q+1}{Q} + \frac{\pi[(r-1)Q-1]}{2\delta Q[f(Q)+1]}, \qquad (37)$$

$$F_\theta^0 \approx -\frac{[(r-1)Q-1]}{(Q+1)[f(Q)+1]} \cdot \frac{\pi}{2\delta Q}, \qquad (38)$$

$$\left(Q = Ra\left(\frac{\delta}{\pi}\right)^4\right).$$

Further the function $f(Q)$ is easily found in explicit form if the integral (32) would be calculated with the help of the theorem on residues, the function $f(Q)$ defined in such a way has the form

$$f(Q) = \frac{1}{3Q}\left[3 - \frac{1+Q}{\left(1+Q^{1/3}\right)^{3/2}} - \frac{1+Q}{\rho^3}\left(2-\rho-Q^{1/3}\right)\sqrt{2+2\rho-Q^{1/3}}\right], \qquad (39)$$

$$\rho = \left(Q^{2/3} - Q^{1/3} + 1\right)^{1/2}.$$

at the same time it's easy to show that at $Q \to 0$ the formula (39) have a sense at $Q \to 0$ we

$$f(Q) \approx 1{,}0833 - 0{,}7451Q + 0\left(Q^2\right). \qquad (40)$$

6. Using formulas (37)-(39) from the principle (14^a) the value of δ for each given value Ra an r can be found and then the connective dimensionless salinity flows $Nu_S - 1$ depending on the effective Rayleigh numbers $Ra_{eff} = Ra_S - Ra = (r-1)Ra$ can be defined. This procedure was carried out numerically.

The results are in table 1.

The results of numerical computation of the first approximation

Table 1

Ra	r	Q^0	δ^0	F^0	$(Nu_s-1)Ra_s$
10^2	10	0.1499	0.6182	1.059*10^2	0.9531*10^4
	100	0.02733	0.4039	1.528*10^3	1.5127*10^5
10^3	10	0.2457	0.3933	1.503*10^2	1.3527*10^5
	100	0.03545	0.2424	2.476*10^3	2.4512*10^6
10^4	10	0.3111	0.2346	2.414*10^2	2.1726*10^6
	100	0.04039	0.1408	2.207*10^3	2.1849*10^7
10^5	10	0.3504	0.1359	4.073*10^2	3.6657*10^7
	100	0.04344	0.08065	7.305*10^3	7.2319*10^8
10^6	10	0.3741	0.07769	7.045*10^2	6.3405*10^8
	100	0.04514	0.04759	1.282*10^4	1.2692*10^{10}

The dependence of $Ra_s(Nu_s-1)$ on $Ra_{eff}=Ra_s-Ra$ according to data of table 1 in double logarithmically coordinates is represented in fig. 1. The data for $r=10^3, 10^4, 10^5$, being absent in the table are brought in the same place. With the exception of some first values archived at comparatively small Ra_{eff} all other values are sufficiently good placed on the straight line, whose equation has the form

$$Ra_s(Nu_s-1)\approx 0{,}384(Ra_s-Ra)^{1{,}310}\,. \tag{41}$$

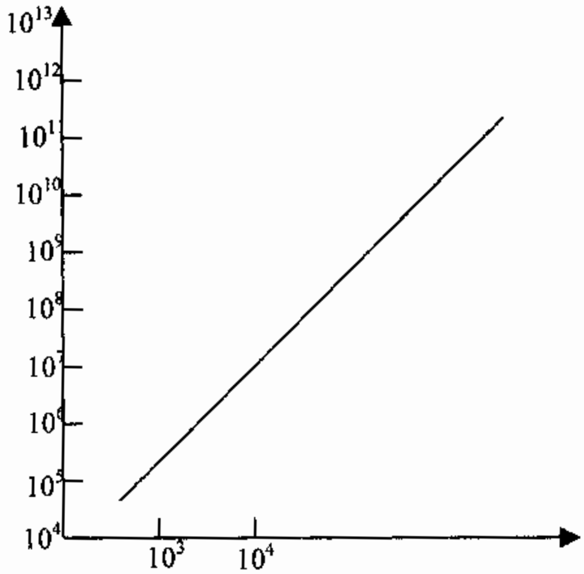

Fig. 1. Independence $Ra_s(Nu_s-1)$ **of** $Ra_{eff}=Ra_s-Ra$ **for the first approximation.**

The dependence (41) practically coincides with the dependence given in numerical calculations of Straus [3] though it rather deviates from "the law $4/3$". Nevertheless the formula (41) gives the values of the flow of salinity which are less almost for an order than those, which were observed in the experiment of Terner [1]. Such variance can be connected first of all with insufficiency of the first approximation. For clarification of circumstances of this variance the complete numerical analysis of the nonlinear boundary value problem (1)-(12) is necessary to which the special paper will be devoted.

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ON NON-LINEAR LONGITUDINAL VIBRATION OF RECTILINEAR PILE

Abstract

The paper is devoted to the investigation of longitudinal vibrations of rectilinear pile. The effect of frequencies to the critical loading is studied. The field of application of linear theory is shown.

The given paper is devoted to the investigation of longitudinal vibration of rectilinear pile subject to the geometrical non-linearity. As is known [1], there exist such values of frequencies at which the permutations increase infinitely, i.e. the linear theory isn't applicable. Therefore for these cases it is necessary to take into account the geometrical non-linearity.

In frames of geometrical non-linear theory at Lagrangian approach the vibration has the following form [2]:

$$\left[\sigma^{ij}\left(\delta_i^k + u,_j^k\right)\right]_{,j} = \rho\frac{\partial^2 u^k}{\partial t^2}, \tag{1}$$

where δ_i^k is the Kronecker symbol, the comma means the covariant differentiating, ρ - is density of non-deformed body. In addition the tensor's components of deformation ε_{ij} are determined by the components of the vector of permutations u_i by the following form [2]:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u,_i^k\, u_{k,j}\right). \tag{2}$$

In case of longitudinal vibrations it is accepted

$$u_x = u(x), u_y = u_z = 0\,.$$

Then in Descartes coordinate system we've:

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2;\ \frac{\partial}{\partial x}\left[\sigma\left(1 + \frac{\partial u}{\partial x}\right)\right] = \rho\frac{\partial^2 u}{\partial t^2}.$$

Supposing the material of the pile to be elastic, finally we'll get:

$$c^2\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\left(1 + \frac{1}{2}\frac{\partial u}{\partial x}\right)\left(1 + \frac{\partial u}{\partial x}\right)\right] = \frac{\partial^2 u}{\partial t^2}, \tag{3}$$

where $c = \sqrt{E/\rho}$.

The boundary conditions we'll take in the following form:

when $x = 0\ \ u = 0$,

when $x = L\ \ \frac{1}{E}\sigma\left(1 + \frac{\partial u}{\partial x}\right) \equiv \frac{\partial u}{\partial x}\left(1 + \frac{1}{2}\frac{\partial u}{\partial x}\right)\left(1 + \frac{\partial u}{\partial x}\right) = \tau_0 \sin\omega_0 t$. (4)

So, the equation (3) at the boundary conditions (4) allows to investigate the longitudinal vibrations of points of the pile. Let's note that in common case to find the analytical solution of the equation (3) is impossible. Therefore the necessity of applying the approximate methods appears. One of the effective methods is the variational one. In our case let's apply the Reysner's variation principle. As for the considered equations the Reysner functional has the following form [2]: