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ON ONE INVERSE PROBLEM FOR A SEMI-LINEAR EQUATION OF PARABOLIC TYPE

Abstract

In the paper the inverse problem on defining the right hand side of the semi-linear equation of parabolic type is considered. The theorems of existence, uniqueness and stability of solution are proved. For approximate solving the considered inverse problem the method of successive approximations was suggested and its convergence rate was estimated.

In the paper the inverse problem on defining the right hand side of a semi-linear equation of parabolic type is considered. The existence, uniqueness and solution stability theorems are proved.

For approximate solving of the considered problem the method of successive approximations was suggested and its convergence rate was estimated.

The inverse problems on defining the unknown source (in applications the right hand member usually is the sense of the source) of the linear parabolic equation was considered in papers [1-6].

We'll accept the following denotations: D is a bounded domain from R^n with the boundary ∂D, $Q_T = D\times(0,T]$, $S_T = \partial D\times[0,T]$, $0 < T = const$, $\|\cdot\|_{C^l} = \|\cdot\|_l$, the spaces $C^l(\cdot), C^{(l+\alpha)/2}(\cdot)$, $C^{l+\alpha,(l+\alpha)/2}(\cdot)$, $0<\alpha<1$, $l=0,1,2$, and corresponding norms are defined, for example, in [9, p.12].

Consider the problem on defining $\{f(x), u(x,t)\}$ from the conditions

$$u_t - Lu = f(x)g(u), \quad (x,t)\in Q_T, \tag{1}$$

$$u(x,0)=\varphi(x),\ x\in\overline{D};\ u(x,t)=\psi(x,t),\ (x,t)\in S_T, \tag{2}$$

$$\int_0^T u(x,t)dt = h(x),\ \ x\in\overline{D}, \tag{3}$$

where

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} = \sum_{i=1}^{n} b_i(x)u_{x_i} + C(x)u .$$

The functions $a_{ij}(x), b_i(x),\ i,j=\overline{1,n}, C(x), g(\cdot), \varphi(x), \psi(x,t), h(x)$ are given. Everywhere below we'll suppose that for arbitrary real vector $\eta=(\eta_1,...,\eta_n)$ for any $(x,t)\in\overline{Q}_T$

$$m_0\sum_{i=1}^{n}\eta_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\eta_i\eta_j \le m_1\sum_{i=1}^{n}\eta_i^2; \quad 0<m_0<m_1 .$$

If the function $f(x)$ in the equation (1) is given, then naturally, the condition (3) isn't given. The problem on defining $u(x,t)$ from (1)-(2) in more general statement is considered, for example in paper [7].

The problems (1)-(3) relate to the class of incorrect by Hadamard problems. Examples show that the solution of this problem doesn't always exist and even if exists, then it may not be unique and stable.

Therefore we have to regard the problem (1)-(3) proceeding from general conceptions of the theory of incorrect problems [8].

Definition 1. *Functions* $\{f(x), u(x,t)\}$ *are called the solution of the problem (1)-(3), if* $f(x) \in C(\overline{D}), u(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ *the relations (1)-(3) are satisfied.*

Definition 2. *We'll say that solution of the problem (1)-(3) belongs to the set* K*, if:*

$$f(x) \in C^{\alpha}(\overline{D}),\ |f(x)| \le m_2, |u(x,t)| \le m_3,\ \ (x,t) \in \overline{Q}_T$$

Theorem 1. *Let* 1^0. $a_{ij}(x), b_i(x),\ C(x) \in C^{\alpha}(\overline{D}),\ i,j = \overline{1,n},$

$g(u) \in Lip(-\infty,+\infty),\ \varphi(x), h(x) \in C^{2+\alpha}(\overline{D}), \Psi(x,t) \in C^{2+\alpha,1+\alpha/2}(S_T), \partial D \in C^{2+\alpha}$;

2^0. $\left|\int_0^T g(u)dt\right| \ge m_4 > 0,\ x \in \overline{D}$, *the conditions of agreement:*

1) *of o order* $\varphi(x) = \psi(x,0),\ \int_0^T \psi(x,t)dt = h(x),\ x \in \partial D,$

2) *of* 1^{st} *order* $[\psi_t(x,0) - L\varphi(x)] \cdot g(\psi(x,T)) = [\psi_t(x,T)] \cdot g(\varphi(x)),\ \ x \in \partial D$ *be satisfied.*

Then there exists such $0 < T_1 \le T$ *that for all* $(x,t) \in \overline{Q}_T$ *the solution of the problem (1)-(3) on the set* K *is unique and the estimation of stability:*

$$\|u - \overline{u}\| + \|f - \bar{f}\| \le M_1 \left[\|g - \overline{g}\| + \|\varphi - \overline{\varphi}\|_2 + \|\psi - \overline{\psi}\|_{2,1} + \|h - \overline{h}\|_2\right] \qquad (4)$$

is true, where $M_1 > 0$ *depends on the problem data and the set* K *(further everywhere we'll denote positive constants by* M_i*, which depend on the problem data and the set* K *and by* N_i *which depend only on the problem data)* $\{\bar{f}(x), \overline{u}(x,t)\}$ *is the solution of the problem (1)-(3) with data* $\overline{g}(\overline{u}), \overline{\varphi}(x), \overline{\psi}(x), \overline{h}(x)$*, which satisfy the conditions* 1^0*,* 2^0 *respectively.*

Proof. We'll define the function

$$F(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T),\ F(x,0) = \varphi(x), x \in \overline{D}, F(x,t) = \psi(x,t), (x,t) \in S_T. \quad (5)$$

Let $z(x,t) = u(x,t) - \overline{u}(x,t), \lambda(x) = f(x) - \tilde{f}(x), \delta_1(u) = g(u) - g(\overline{u}),\ \delta_2(x,t) = F(x,t) - \overline{F}(x,t), \delta_3(x) = \varphi(x) - \overline{\varphi}(x), \delta_4(x) = h(x) - \overline{h}(x)$.

It's easy to check that the function $\{\lambda(x), v(x,t) = z(x,t) - \delta_2(x,t)\}$ satisfy the system

$$v_t - Lv = \lambda(x)g(u) + \tilde{f}(x)[\delta_1(u) + (\overline{g}(u) - \overline{g}(\overline{u}))] - \delta_{2t}(x,t) + L\delta_2(x,t), (x,t) \in Q_T, \quad (6)$$

$$v(x,0) = 0, x \in \overline{D}; v(x,t) = 0, (x,t) \in S_T, \quad (7)$$

$$\lambda(x) = [z(x,T) - \delta_3(x) - L\delta_4(x)] / \int_0^T g(u)dt + \left\{[\overline{\varphi}(x) + L\overline{h}(x) - \overline{u}(x,T)] \times \right.$$
$$\left. \times \int_0^T [\delta_1(u) + \overline{g}(u) - \overline{g}(\overline{u})]dt\right\} / \left(\int_0^T g(u)dt \cdot \int_0^T \overline{g}(\overline{u})dt\right),\ x \in \overline{D}. \quad (8)$$

On the propositions of the theorem and from the definition of the set K if follows that the coefficients and the right side hand member of the equation (6) satisfy the Hölder condition. It means that there exists a classical solution of the problem on defining $v(x,t)$ from the conditions (6), (7) and it can be represented in the form *[9, p.468]*:

$$v(x,t)=\int_0^t\int_D G(x,t;\xi,\tau)\{\lambda(\xi)g(u)+\bar{f}(\xi)[\delta_1(u)+(\bar{g}(u)-\bar{g}(\bar{u}))]-\delta_{2\tau}(\xi,\tau)+ + L\delta_2(\xi,\tau)\}d\xi d\tau, \tag{9}$$

where $G(x,t;\xi,\tau)$ is a Green function of the problem (6), (7) for which the following estimation is true [9, p.469]:

$$|G(x,t;\xi,\tau)|\le N_1(t-\tau)^{-\frac{n}{2}}\exp\left(-N_2\frac{|x-\xi|^2}{t-\tau}\right). \tag{10}$$

Allowing for $v(x,t)=z(x,t)-\delta_2(x,t)$ from (9) we'll obtain

$$|z(x,t)|\le|\delta_2(x,t)|+\int_0^t\int_D|G(x,t;\xi,\tau)|\left\{|\lambda(\xi)g(u)+\bar{f}(\xi)(\bar{g}(u)-\bar{g}(\bar{u}))|+ +|\delta_1(u)\bar{f}(\xi)-\delta_{2\tau}(\xi,\tau)+L\delta_2(\xi,\delta)|\right\}d\xi d\tau.$$

Assume

$$æ=\|u-\bar{u}\|+\|f-\bar{f}\|.$$

Under conditions of the theorem and from the definition of the set K, subject to estimation (10) we'll obtain

$$|z(x,t)|\le M_2\left[\|\delta_1\|+\|\delta_2\|_{2,1}\right]+M_3tæ,\quad (x,t)\in\overline{Q}_T. \tag{11}$$

Again under conditions of the theorem and from the definition of the set K, allowing for the estimation (11) from (8) we'll obtain for $\lambda(x)$

$$|\lambda(x)|\le M_4\left[\|\delta_1\|+\|\delta_3\|+\|\delta_4\|_2\right]+M_5tæ,\quad x\in\overline{D}. \tag{12}$$

The inequalities (11), (12) are satisfied for any values of $(x,t)\in\overline{Q}_T$. Therefore they must be satisfied also for maximal values of the left hand members. Consequently

$$æ\le M_6\left[\|\delta_1\|+\|\delta_2\|_{2,1}+\|\delta_4\|_2\right]+M_7tæ.$$

Let $0<T_1\le T$ be such a number that $M_7T_1<1$. Then from the last inequality we'll obtain the estimation of stability (4) for solution of the problem (1)-(3). The uniqueness of the solution of the problem (1)-(3) follows from the estimation (4) at $g(u)=\bar{g}(\bar{u}),\varphi(x)=\bar{\varphi}(x),\psi(x,t)=\bar{\psi}(x,t),h(x)=\bar{h}(x)$.

The theorem is proved.

The method of successive estimations as applied to the problem (1)-(3) is as following: let $\{f^{(s)}(x),u^{(s)}(x,t)\}$ be already found. Consider the problem on defining $u^{(s+1)}(x,t)$ from the conditions:

$$u_t^{(s+1)}-Lu^{(s+1)}=f^{(s)}(x)g(u^{(s)}),\ (x,t)\in Q_T, \tag{13}$$

$$u^{(s+1)}(x,0)=\varphi(x),x\in\overline{D};\ u^{(s+1)}(x,t)=\psi(x,t),\quad (x,t)\in S_T \tag{14}$$

this problem has a unique classical solution (if input data satisfy conditions 1^0, 2^0 of theorem 1) belonging to $C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ [9, p.364]. Later by the function $u^{(s+1)}(x,t)$ from the condition

$$f^{(s+1)}(x)=\left[u^{(s+1)}(x,T)-\varphi(x)-Lh(x)\right]/\int_0^T g\left(u^{(s+1)}\right)dt,\quad x\in\overline{D}. \tag{15}$$

$f^{(s+1)}(x) \in C^{\alpha}(\overline{D})$ is defined and this function is used for performing the next step of iteration. So, at $s = 0,1,2...$ the sequence of functions $\{f^{(s)}(x), u^{(s)}(x,t)\}$ is found.

Theorem 2. *Let 1) the conditions 1^0, 2^0 of theorem 1 be satisfied. 2) there exist a unique solution of the problem (1)-(3) from the set K. Then there exists such $0 < T_2 \le T$ that at $(x,t) \in \overline{Q}_{T_2}$ functions $\{f^{(s)}(x), u^{(s)}(x,t)\}$, obtained from (13)-(15) uniformly tend to the solution of the problem (1)-(3) with the rate of geometric progression.*

Proof. At first we'll show that the sequences $\{f^{(s)}(x)\}$, $\{u^{(s)}(x,t)\}$ are uniformly bounded in $\overline{Q}_{T_3}$ (T_3 is defined below) for any $s = 0,1,2,...$.

Really, as was said above, if we choose $\{f^{(0)}(x), u^{(0)}(x,t)\} \in K$, then under conditions of theorem 2 and by virtue of statement of the theorem, proved [9, p.364] it follows that for any $s = 1,2,... \{f^{(s)}(x), u^{(s)}(x,t)\} \in K$. Further with the help of Green's function [9, p.468] we'll find the expressions for solution of the problem on defining $u^{(s+1)}(x,t)$ from (13)-(14).

$$u^{(s+1)}(x,t) = F(x,t) + \int_0^t \int_D G(x,t;\xi,\tau)\left[f^{(s)}(\xi)g(u^{(s)}) - F_\tau(\xi,\tau) + LF(\xi,\tau)\right]d\xi d\tau. \quad (16)$$

Acting in the same way as in the proof of theorem 1, subject to the estimation (10) and conditions of theorem 2 from the east equality and from (15) we'll obtain:

$$\left|u^{(s+1)}(x,t)\right| \le N_3\|F\|_{2,1} + N_4 t\left|f^{(s)}(x)\right|, \ (x,t) \in \overline{Q}_T,$$

$$\left|f^{(s+1)}(x)\right| \le N_5\left[\|\varphi\| + \|h\|_2\right] + N_6 T\left|u^{(s+1)}(x,T)\right|, \ x \in \overline{D},$$

or

$$\gamma^{(s+1)} \le N_7\left[\|F\|_{2,1} + \|h\|_2\right] + N_8 t\gamma^{(s)},$$

where

$$\gamma^{(s)} = \left\|u^{(s)}\right\| + \left\|f^{(s)}\right\|.$$

From the last inequality we have:

$$\gamma^{(s+1)} \le N_7\left[\|F\|_{2,1} + \|h\|_2\right](1 - q^s)/(1-q) + q^s\gamma^{(0)}.$$

Let $0 < T_3 \le T$ be such a number that $q = N_8 T_3 < 1$. Then we'll obtain that the sequences $\{f^{(s)}(x)\}, \{u^{(s)}(x,t)\}$ are uniformly bounded at $(x,t) \in \overline{Q}_{T_3}$ for any $s = 0,1,2,...$.

Now we'll show that the functions $\{f^{(s)}(x), u^{(s)}(x,t)\}$ obtained from (13)-(15) uniformly converge to the solution of the problem (1)-(3).

Subtracting from the correlations of the system (1)-(3) corresponding correlations of the system (13)-(15) we'll obtain that the functions $z^{(s)}(x,t) = u(x,t) - u^{(s)}(x,t)$, $\lambda^{(s)}(x) = f(x) - f^{(s)}(x)$ satisfy the conditions of the system:

$$z_t^{(s+1)} - Lz^{(s+1)} = \lambda^{(s)}(x)g(u^{(s)}) + f(x)\left[g(u) - g(u^{(s)})\right], \ (x,t) \in Q_T, \quad (17)$$

$$z^{(s+1)}(x,0) = 0, \ x \in \overline{D}; \ z^{(s+1)}(x,t) = 0, \ (x,t) \in S_T, \quad (18)$$

$$\lambda^{(s+1)}(x)=z^{(s+1)}(x,T)\Big/\int_0^T g(u)dt+\left\{\left[\varphi(x)+Lh(x)-u^{(s+1)}(x,T)\right]\times\right.$$
$$\left.\times\int_0^T\left[g(u)-g\left(u^{(s+1)}\right)\right]dt\right\}\Big/\left(\int_0^T g(u)dt\cdot\int_0^T g\left(u^{(s+1)}\right)dt\right),\quad x\in\overline{D}. \tag{19}$$

Under conditions of theorem 2 the coefficients and the right hand member of (17) are Hölderian and therefore [9, p.468]:

$$z^{(s+1)}(x,t)=\int_0^t\int_D G(x,t;\xi,\tau)\left[\lambda^{(s)}(\xi)g\left(u^{(s)}\right)+f(\xi)\left(g(u)-g\left(u^{(s)}\right)\right)\right]d\xi d\tau, \tag{20}$$

where $G(x,t;\xi,\tau)$ is a Green function of the problem (17), (18), for which the estimation (10) is true.

Acting in the same way as obtaining the inequality (11), from (2) we have

$$\left|z^{(s+1)}(x,t)\right|\le M_8 t\text{æ}^{(s)},\quad (x,t)\in\overline{Q}_T, \tag{21}$$

where

$$\text{æ}^{(s)}=\left\|u-u^{(s)}\right\|+\left\|f-f^{(s)}\right\|.$$

Allowing to the inequality (21) and the conditions of theorem 2 from (19) we'll obtain

$$\left|\lambda^{(s+1)}(x)\right|\le M_9 t\text{æ}^{(s)},\quad x\in\overline{D}. \tag{22}$$

From (21), (22) follows that

$$\text{æ}^{(s+1)}\le M_{10}t\text{æ}^{(s)}$$

or

$$\text{æ}^{(s+1)}\le\left(M_{10}t\right)^s\text{æ}^{(0)}.$$

Let $0<T_4\le T$ be such a number that $M_{10}T_4<1$. Then from the last inequality we'll obtain that at $(x,t)\in\overline{Q}_{T_2}$, $T_2=\min\{T_3,T_4\}$ the statement of theorem 2 is true.

Theorem 3. *Let the conditions 1^0, 2^0 of theorem 1 be satisfied. Then it can be found such $0<T^*\le T$ that at $(x,t)\in\overline{Q}_T$ a unique solution of the problem (1)-(3) in the sense of definition 1.*

The proof is led by the method of successive approximations applied on the scheme (13)-(15).

It was shown above that the sequences $\left\{f^{(s)}(x)\right\},\left\{u^{(s)}(x,t)\right\}$, obtained from (13)-(15) are uniformly bounded (by the norm C) at $(x,t)\in\overline{Q}_{T_3}$.

Equicontinuity of these sequence follows from the inequalities:

$$\left|u^{(s+1)}(x,t)-u^{(s+1)}(x',t')\right|\le\left|u^{(s+1)}(x,t)-u^{(s+1)}(x',t)\right|+\left|u^{(s+1)}(x',t)-u^{(s+1)}(x',t')\right|\le$$
$$\le\left|F(x,t)-F(x',t)\right|+\left|F(x',t)-F(x',t')\right|+\int_0^t\int_D\left|G(x,t;\xi,\tau)-G(x',t;\xi,\tau)\right|\left|f^{(s)}(\xi)g\left(u^{(s)}\right)-\right.$$
$$\left.-F_\tau(\xi,\tau)+LF(\xi,\tau)\right|d\xi d\tau+\int_0^{t'}\int_D\left|G(x',t;\xi,\tau)-G(x',t';\xi,\tau)\right|\left|f^{(s)}(\xi)g\left(u^{(s)}\right)-\right.$$
$$\left.-F_\tau+LF\right|d\xi d\tau+\int_{t'}^t\int_D\left|G(x',t;\xi,\tau)\right|\left|f^{(s)}(\xi)g\left(u^{(s)}\right)-F_\tau+LF\right|d\xi d\tau,$$

$$\left|f^{(s)}(x)-f^{(s)}(x')\right|\le\left|u^{(s)}(x,T)-u^{(s)}(x',T)\right|\Big/\left|\int_0^T g\left(u^{(s)}(x,T)\right)dt\right|+\left\{\left|\varphi(x)+Lh(x)-u^{(s)}(x',T)\right|\times\right.$$

$$\times\left|\int_0^T\left[g\left(u^{(s)}(x,T)\right)-g\left(u^{(s)}(x',T)\right)\right]dt\right|+\left[\left|\varphi(x)-\varphi(x')\right|+\left[L(h(x)-h(x'))\right]\right]\left|\int_0^T g\left(u^{(s)}(x,T)\right)dt\right|\Bigg\}\Big/$$

$$\Big/\left|\int_0^T g\left(u^{(s)}(x,T)\right)dt\int_0^T g\left(u^{(s)}(x',T)\right)dt\right|,$$

subject to the uniform boundedness of $\{f^{(s)}(x)\}$ and $\{u^{(s)}(x,t)\}$, continuity and boundedness of the input data, the estimation (10) for the Green function. According to Arzeli theorem from the sequences $\{f^{(s)}(x)\}$, $\{u^{(s)}(x,t)\}$ subsequences convergent to some functions $f^*(x),u^*(x,t)$ respectively can be chosen and $f^*(x)\in C(\overline{D}),\ u^*(x,t)\in C(\overline{Q}_{T_3})$.

Passing to the limit at $s\to\infty$ in the correlations (15), (16) we'll obtain

$$u^*(x,t)=F(x,t)+\int_0^t\int_D G(x,t;\xi,\tau)\left[f^*(\xi)g(u^*)-F_\tau+LF\right]d\xi d\tau, \qquad (23)$$

$$f^*(x)=\left[u^*(x,T_3)-\varphi(x)-Lh(x)\right]\Big/\int_0^{T_3}g(u^*)dt. \qquad (24)$$

The functions $\{f^*(x),u^*(x,t)\}$ satisfy the conditions of the system (1)-(3). Moreover, the estimations of derivatives of the Green function [9, p.469] allow to conclude that $C^{2,1}(Q_{T_3})\cap C(\overline{Q}_{T_3})$, i.e. functions $\{f^*(x),u^*(x,t)\}$ are the solution of the problem (1)-(3) in the sense of definition 1. So, the existence of at least one solution of the problem (1)-(3) is proved.

Now we'll show that the found solution is unique. It's easy to check that under conditions of the theorem the functions $\{f^*(x),u^*(x,t)\}$ belong to the class K. Therefore, from the statement of theorem 1 follows that the functions $\{f^*(x),u^*(x,t)\}$ will be a unique solution of (1)-(3) at $(x,t)\in\overline{Q}_{T^*}$, $T^*=\min\{T_1,T_3\}$. The theorem is proved.

Remark. The inverse problems on defining the unknown coefficient and right hand member of the system of semi-linear parabolic equations of the second order are considered analogously to the problem (1)-(3).

The analogs of the proved below theorems hold for these problems.

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GLOBAL SOLUTIONS OF NON-LINEAR FOURTH ORDER HYPERBOLIC INEQUALITIES

Abstract

In this paper considered variation inequality for the fourth order non-linear hyperbolic operators and proved one-valued solvability of this problem.

Let $\Omega \subset R^n$ be a bounded domain with the smooth boundary Γ. Let's consider the fourth order non-linear hyperbolic operator in the cylinder $Q = (0,T) \times \Omega$

$$L(u) = u'' + \Delta\big(a(t,x,u)\Delta u\big), \tag{1}$$

with the boundary conditions of Dirichlet

$$u\big|_\Gamma = 0, \quad \frac{\partial u}{\partial \nu}\bigg|_\Gamma = 0 \tag{2}$$

and with the initial conditions

$$u(0,x) = u_0(x), \quad u'(0,x) = u_1(x), \tag{3}$$

where $\frac{\partial}{\partial \nu}$ - is the derivative in the direction of external normal ν, $u' = u_t$, $u'' = u_{tt}$.

$$a(\cdot) \in C^2[0,T] \times \overline{\Omega} \times R, \quad a(t,x,u) \ge a_0 > 0. \tag{4}$$

Let

$$\overset{\circ}{W}{}_2^{2k} = \left\{ u:\ u \in W_2^{2k}(\Omega);\ \frac{\partial^i u}{\partial \nu^i}\bigg|_\Gamma = 0,\ i = 0,1,\ldots,k-1 \right\},$$

where $W_2^{2k}(\Omega)$ - is a Sobolev space.

Let's determine the space

$$H_T(2k,2m) \equiv H_T\left(\overset{\circ}{W}{}_2^{2k}, \overset{\circ}{W}{}_2^{2m}, L_2(\Omega)\right) = \left\{ u : u \in L_\infty\left(0,T; \overset{\circ}{W}{}_2^{2k}\right),\right.$$

$$\left. u' \in L_\infty\left(0,T; \overset{\circ}{W}{}_2^{2m}\right), u'' \in L_\infty\big(0,T; L_2(\Omega)\big) \right\}.$$

Let us denote by K_0 and K_λ the corresponding convex closed sets in the spaces $\overset{\circ}{W}{}_2^2$ and $\overset{\circ}{W}{}_2^4$:

$$K_0 = \left\{ u : u \in \overset{\circ}{W}{}_2^2, |\Delta u(x)| \le 1, \text{ almost everywhere on } \Omega \right\},$$

$$K_\lambda = \left\{ u : u \in \overset{\circ}{W}{}_2^4,\ |\Delta u(x)| \le 1,\ \left|\Delta^2 u(x)\right| \le \frac{1}{\lambda} \quad \text{almost everywhere on } \Omega \right\}.$$

Let's determine the set

$$H_T(2m, 2k, K_\lambda) \equiv H_T\left(\overset{\circ}{W}{}_2^{2m}, \overset{\circ}{W}{}_2^{2k}, K_\lambda\right) = \left\{ u : u \in H_T\left(\overset{\circ}{W}{}_2^{2m}, \overset{\circ}{W}{}_2^{2k}, L_2(\Omega)\right),\right.$$

$$\left. u'(t,\cdot) \in K_\lambda \text{ almost everywhere on } (0,\mathrm{T}) \right\}.$$

Let's introduce notations: