SULEYMANOV S.E., MAMEDOVA Sh.D.

LIMITING ABSORPTION PRINCIPLE FOR THE HELMHOLTZ EQUATION IN A MANY-DIMENSIONAL LAYER WITH A GENERAL BOUNDARY CONDITION

Abstract

In the paper the Green function of the boundary value problem is constructed and for this problem the limiting absorption principle is proved.

Introduction. Wave propagation in a homogeneous layer bounded from two sides by plane-parallel boundaries leads to different boundary value problems in a layer for the Helmholtz equation. The limiting absorption principle for the Helmholtz equation in a two-dimensional layer with the Dirichlet or Neumann boundary conditions is considered in L.M. Brekhovskikh's book [1], in a three-dimensional layer the limiting amplitude and partial conditions of radiation for this problem are considered in A.G.Sveshnikov's article [2]. In this paper A.G. Sveshnikov introduced also new conditions which ensure the uniqueness of solutions of a boundary value problem for the Helmholtz equation in a three-dimensional layer. Now these conditions are called A.G. Sveshnikov's partial conditions. The radiation principle in a many dimensional layer for the Helmholtz equation with Dirichlet and Neumann boundary conditions were studied in [3]. The radiation principles in a three-dimensional cylindrical domain are studied in [4], and in a many-dimensional cylindrical domain - in [5,6]. The radiation principles for the higher order elliptic equations with constant coefficients in a many-dimensional cylinder are studied in [7,8]. In [3,5-8] for the first time the resonance phenomenon was studied and the rate of increase of solutions of non-stationary problem is mentioned when $t \to \infty$. In [9] the radiation principles for the Holmholtz equation are studied in a many-dimensional layer with impedance boundary conditions.

In the present paper the Green function for the Helmholtz equation is constructed in a many-dimensional layer with a general boundary condition and the limiting absorption principle is studied. The results by limiting amplitude principle for this problem will be published later.

§ 1. Construction of the Green function.

Let

$$\Pi = \{x : (x', x_{n+1}), x' = (x_1, x_2, \dots, x_n), -\infty < x_j < +\infty, j = 1,2,\dots,n; -h < x_{n+1} < +h\}$$

be a layer in the $n+1$ dimensional Euclidean space R_{n+1}. Consider the following boundary value problem in Π

$$\left(\Delta + k^2\right) u(k,x) = f(x), \tag{1.1}$$

$$\left(\frac{\partial}{\partial x_{n+1}} + p(k)\right) u(k,x)\Big|_{x_{n+1} = \pm h} = 0, \tag{1.2}$$

where Δ is a Laplacian operator, $f(x)$ is a finite infinitely differentiable function with support in Π, k is a complex parameter with $\operatorname{Im} k > 0$, $p(k) = ak + b$, a and b are real numbers.

Definition 1. *Under the solution of the problem (1.1)-(1.2) we'll understand the decreasing on infinity function $u(k,x)$ satisfying the equation (1.1) and the boundary conditions (1.2) in sense of generalized functions ([10], p.40-187).*

Along with the problem (1.1)-(1.2) we can consider also the problem

$$\left(\Delta + k^2\right)G(k,x,y) = \delta\left(x' - y', x_{n+1}, y_{n+1}\right), \qquad (1.3)$$

$$\left(\frac{\partial}{\partial x_{n+1}} + p(k)\right)G(k,x,y)\Big|_{x_{n+1}=\pm h} = 0, \qquad (1.4)$$

where $\delta(x)$ is a Dirac function.

Definition 2. *Let* $\operatorname{Im} k^2 > 0$. *The decreasing in infinity solution of the problem (1.3)-(1.4) we'll call a Green function of the problem (1.3)-(1.4).*

Now we pass to the construction of the Green function of the problem (1.1)-(1.2). The following theorem is correct.

Theorem 1. *When* $p(k) \neq \pm i\frac{\pi\nu}{h}$ $(\nu = 1,2,3,...)$ *the Green function of the problem (1.1)-(1.2) in the many-dimensional band* Π *is an analytical function of* k *excluding denumerable number of the points* $k = \pm ip(k)$, $k = \pm\frac{\pi\nu}{2h}$ $(\nu = 1,2,3,...)$ *being branching points and for it the following representation holds.*

$$G(k,x,y) = -\frac{i}{2}(2\pi)^{-\frac{n}{2}}\left|x' - y'\right|^{1-\frac{n}{2}}\left\{g_0(k,x_{n+1})g_0(k,y_{n+1})K_0^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}(k_0 r) + \right.$$

$$\left. + \sum_{\nu=1}^{\infty} g_\nu(k,x_{n+1})g_\nu(k,y_{n+1})k_\nu^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}(rk_\nu)\right\},$$

where

$$k_0 = \sqrt{p^2(k) + k^2}, \quad k_\nu = \pm\sqrt{k^2 - \frac{\pi^2\nu^2}{4h}},$$

$$g_0(k,x_{n+1}) = \left[\frac{p(k)}{2sh2hp(k)}\right]^{1/2} e^{-p(x)x_{n+1}}$$

$$g_\nu(k,x_{n+1}) = \frac{\frac{\pi\nu}{2h}\cos(h - x_{n+1})\frac{\pi\nu}{2h} + p(k)\sin(h - x_{n+1})\frac{\pi\nu}{2h}}{\frac{1}{2h}\left(p^2(k) + \frac{\pi^2\nu^2}{4h^2}\right)^{1/2}}, \quad \nu = 1,2,3,.... \qquad (1.5)$$

For $r > 0, p(k) \neq i\frac{\pi\nu}{2h}$ *for any* n *and* $k \neq \frac{\pi\nu}{2h}$ *for* $n = 1,2$, *the series in (1.5) uniformly convergence to* k *in every compact.*

Proof. Assuming $G(k,x,y)$ as a generalized function we accomplish in (1.1)-(1.2) the Fourier transformation by x'. Then we obtain

$$\left(\frac{d^2}{dx_{n+1}^2} + k^2 - \rho^2\right)\hat{G}(k,\rho,x_{n+1},y_{n+1}) = \delta\left(x_{n+1},y_{n+1}\right)e^{i(\xi',y')} \qquad (1.6)$$

with the boundary condition

$$\left(\frac{d}{dx_{n+1}} + p(k)\right)\hat{G}(k,\rho,x_{n+1},y_{n+1})\Big|_{x_{n+1}=\pm h} = 0, \qquad (1.7)$$

where $\xi' = (\xi_1, \xi_2, ..., \xi_n)$ is a dual to x' variable relative to Fourier transformation, $\rho = |\xi| = (\xi_1^2 + \xi_2^2 + ... + \xi_n^2)^{1/2}$, $\hat{G}(k, \rho, x_{n+1}, y_{n+1})$ is a Fourier transformation of $G(k,x,y)$ by $x' - y'$. The decreasing when $\rho \to \infty$ solution of the equation (1.6) will be

$$E(k, \rho, x_{n+1}, y_{n+1}) = \frac{\exp\left[-|x_{n+1} - y_{n+1}|\sqrt{\rho^2 - k^2} + i(\xi', y')\right]}{2\sqrt{\rho^2 - k^2}}.$$

We'll search the solution of the problem (1.6)-(1.7) in the form of

$$\hat{G}(k, \rho, x_{n+1}, y_{n+1}) = E(k, \rho, x_{n+1}, y_{n+1}) + V(k, \rho, x_{n+1}, y_{n+1}),$$

where $V(k, \rho, x_{n+1}, y_{n+1})$ is a solution of the problem

$$\left(\frac{d^2}{dx_{n+1}^2} + k^2 - \rho^2\right) V(k, \rho, x_{n+1}, y_{n+1}) = 0 \qquad (1.8)$$

with the boundary condition

$$\left(\frac{d}{dx_{n+1}} + p(k)\right) V(k, \rho, x_{n+1}, y_{n+1})\Big|_{x_{n+1}=\pm h} = \frac{1}{2}\left(\frac{p(k)}{\sqrt{\rho^2 - k^2}} \mp 1\right) e^{(-h \pm y_{n+1})\sqrt{\rho^2 - k^2} + i(\xi', y')}. \qquad (1.9)$$

Solving the problem (1.8)-(1.9) that is adjoint by complicated calculations we obtain

$$\hat{G}(k, \rho, x_{n+1}, y_{n+1}) = \frac{e^{i(\xi', y')}}{2\sqrt{\rho^2 - k^2}} \left\{ \frac{\left(p^2(k) + \rho^2 - k^2\right) ch(x_{n+1} + y_{n+1})\sqrt{\rho^2 - k^2}}{\left(p^2(k) - \rho^2 + k^2\right) sh 2h\sqrt{\rho^2 - k^2}} - \right.$$
$$\left. - \frac{2p(k)\sqrt{\rho^2 - k^2}\, sh(x_{n+1} + y_{n+1})\sqrt{\rho^2 - k^2}}{\left(p^2(k) - \rho^2 + k^2\right) sh 2h\sqrt{\rho^2 - k^2}} - \frac{ch\left(2h - |x_{n+1} - y_{n+1}|\sqrt{\rho^2 - k^2}\right)}{sh 2h\sqrt{\rho^2 - k^2}} \right\}. \qquad (1.10)$$

Thus we obtain that the function $\hat{G}(k, \rho, x_{n+1}, y_{n+1})$ is an even analytical function by ρ with simple poles at the points

$$\rho_{1,2}^* = \pm k,\ \rho_{3,4}^* = \pm\sqrt{p^2(k) + k^2} \ \text{ and } \ \rho_\nu^\pm = \pm i\sqrt{\frac{\pi^2\nu^2}{4h^2} - k^2},\ \ \nu = 1,2,3,\ldots . \qquad (1.11)$$

Now we calculate the inverse Fourier transformation of the function $\hat{G}(k, \rho, x_{n+1}, y_{n+1})$ by ρ. Then for the Green function $G(k,x,y)$ of the problem (1.1)-(1.2) we obtain

$$G(k,x,y) = \frac{1}{(2\pi)^n} \int_{R_n} \hat{G}(k, \rho, x_{n+1}, y_{n+1}) e^{-i(x', y')} d\xi'. \qquad (1.12)$$

Since the integrand in (1.12) is spherically symmetric, i.e. depends only on ρ, then passing to spherical coordinates we obtain

$$G(k,x,y) = \frac{1}{(2\pi)^n} \int_0^\infty \left\{ \int_{\Omega_\rho} e^{-i\rho r \cos\theta} d\omega \right\} \hat{G}(k, \rho, x_{n+1}, y_{n+1}) \rho^{n-1} d\rho,$$

where θ is an angle between the directions $x' - y'$ and ξ', $r = |x' - y'|, \omega$ is a point of unit sphere, Ω_ρ is a sphere of the radius ρ with center in the origin of coordinate. Allowing for

$$\int_{\Omega_\rho} e^{-i\rho r\cos\theta} d\omega = (2\pi)^{\frac{n}{2}-1}(\rho r)^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\rho r),$$

([11], p.377), where $J_\nu(z)$ is Bessel's function of the order ν, we obtain

$$G(k,x,y) = (2\pi)^{-\left(\frac{n}{2}+1\right)} |x' - y'|^{1-\frac{n}{2}} \int_0^\infty \hat{G}(k,\rho,x_{n+1},y_{n+1}) \rho^{\frac{n}{2}} J_{\frac{n}{2}-1}(\rho r) dr. \qquad (1.13)$$

For calculation of the integral in (1.13) we behave in the following way.

Let n be an odd number. Then $z^{\frac{n}{2}} J_{\frac{n}{2}-1}(z)$ is an even function. The even function by ρ is also the function $\hat{G}(k,\rho,x_{n+1},y_{n+1})$. Therefore if we continue the integrand in (1.13) by ρ even way for negative ρ we obtain

$$G(k,x,y) = \frac{1}{\pi}(2\pi)^{-\left(\frac{n}{2}+1\right)} |x' - y'|^{1-\frac{n}{2}} \int_{-\infty}^{\infty} \hat{G}(k,\rho,x_{n+1},y_{n+1}) \rho^{\frac{n}{2}} J_{\frac{n}{2}-1}(\rho r) dr.$$

Expressing the Bessel function by Hankel function ([12], p.175)

$$J_\nu(z) = \frac{1}{2}\left(H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right) \qquad (1.14)$$

we obtain

$$G(k,x,y) = \frac{1}{2}(2\pi)^{-\left(\frac{n}{2}+1\right)} |x' - y'|^{1-\frac{n}{2}} \left\{ \int_{-\infty}^{\infty} \hat{G}(k,\rho,x_{n+1},y_{n+1}) \rho^{\frac{n}{2}} H_{\frac{n}{2}-1}^{(1)}(\rho r) d\rho + \right.$$
$$\left. + \int_{-\infty}^{\infty} \hat{G}(k,\rho,x_{n+1},y_{n+1}) \rho^{\frac{n}{2}} H_{\frac{n}{2}-1}^{(2)}(\rho r) d\rho \right\}. \qquad (1.15)$$

We denote

$$I^{(1,2)}(k,x,y) = \int_{-\infty}^{\infty} \hat{G}(k,\rho,x_{n+1},y_{n+1}) \rho^{\frac{n}{2}} H_{\frac{n}{2}-1}^{(1,2)}(\rho r) d\rho,$$

$$I_N^{(1,2)}(k,x,y) = \int_{-N}^{N} \hat{G}(k,\rho,x_{n+1},y_{n+1}) \rho^{\frac{n}{2}} H_{\frac{n}{2}-1}^{(1,2)}(\rho r) d\rho.$$

Then

$$I^{(1,2)}(k,x,y) = \lim_{N\to\infty} I_N^{(1,2)}(k,x,y).$$

Denote by C_N^+, C_N^- the semicircles corresponding to upper and lower half-planes with centers in the origin of coordinates and the radiuses N. We put (1.10) in (1.14) and calculate the obtained integral by the method of residues

$$I_N^{(1)}(k,x,y) = 2\pi i \sum_{\nu=1}^{\nu(N)} \underset{\rho=\rho_1^*,\rho_2^*,\rho_\nu^+}{\mathrm{Res}} \hat{G}(k,\rho,x_{n+1},y_{n+1}) \rho^{\frac{n}{2}} H_{\frac{n}{2}}^{(1)}(\rho r) -$$

$$-\int_{C_N^+}\hat{G}(k,\rho,x_{n+1},y_{n+1})\rho^{\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(\rho r)d\rho, \tag{1.16}$$

where $\nu(N)$ is a number of the poles $\hat{G}(k,\rho,x_{n+1},y_{n+1})$ being in $C_N^+U(-N,N)$. By virtue of asymptotics of Hankel's functions for large values of modulus of the argument

$$H^{(1,2)}_{\mu}(z)=\sqrt{\frac{2}{\pi z}}\exp\left[\pm i\left(z-\frac{\pi\mu}{2}-\frac{\pi}{4}\right)\right]\left(1+O\left(\frac{1}{z}\right)\right) \tag{1.17}$$

([12], p.169) and decrease of the integrand in (1.6) we obtain that the integral by C_N^+ tends to zero when $N\to+\infty$. Then from (1.16) we obtain

$$I^{(1)}(k,x,y)=2\pi i\left\{-\frac{p(x)e^{-(x_{n+1}+y_{n+1})p(k)}}{2sh2hp(k)}\left(\sqrt{p^2(k)+k^2}\right)^{\frac{n}{2}-1}\times\right.$$

$$\times H^{(1)}_{\frac{n}{2}}\left(r\sqrt{p^2(k)+k^2}\right)+\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}}{4h}\left[-\cos\left(\left(2h-\left|x_{n+1}-y_{n+1}\right|\frac{\pi\nu}{2h}\right)+\frac{p^2(k)-\frac{\pi^2\nu^2}{4h^2}}{p^2(k)+\frac{\pi^2\nu^2}{4h^2}}\right)\times\right.$$

$$\left.\times\cos(x_{n+1}+y_{n+1})\frac{\pi\nu}{2h}+\frac{2p(k)\frac{\pi\nu}{2h}}{p^2(k)+\frac{\pi^2\nu^2}{2h^2}}\sin(x_{n+1}+y_{n+1})\frac{\pi\nu}{2h}\right]\times,$$

$$\left.\times\left(\sqrt{k^2-\frac{\pi^2\nu^2}{4h^2}}\right)^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}\left(r\sqrt{k^2-\frac{\pi^2\nu^2}{4h^2}}\right)\right\}. \tag{1.18}$$

We can show that the expression in big parenthesis in (1.18) which we denote by B_ν is representable in the form of

$$B_\nu=(-1)^{\nu+1}2\left[\frac{\pi\nu}{2h}\cos(h-y_{n+1})\frac{\pi\nu}{2h}+p(k)\sin(h-y_{n+1})\frac{\pi\nu}{2h}\right]\times$$

$$\times\left[\frac{\pi\nu}{2h}\cos(h-x_{n+1})\frac{\pi\nu}{2h}+p(k)\sin(h-x_{n+1})\frac{\pi\nu}{2h}\right].$$

By substituting this expression of B_ν in (1.18) we obtain

$$I^{(1)}(k,x,y)=-2\pi i\left\{\frac{p(k)e^{-p(k)(x_{n+1}+y_{n+1})}}{2sh2hp(k)}\left(\sqrt{p^2(k)+k^2}\right)^{\frac{n}{2}-1}\times\right.$$

$$\times H^{(1)}_{\frac{n}{2}}\left(r\sqrt{p^2(k)+k^2}\right)+\sum_{\nu=1}^{\infty}\frac{1}{2h}\left[\frac{\pi\nu}{2h}\cos(h-y_{n+1})\frac{\pi\nu}{2h}+p(k)\sin(h-y_{n+1})\frac{\pi\nu}{2h}\right]\times$$

$$\times\left[\frac{\frac{\pi\nu}{2h}\cos(h-x_{n+1})\frac{\pi\nu}{2h}+p(k)\sin(h-x_{n+1})\frac{\pi\nu}{2h}}{\left(p^2(k)+\frac{\pi^2\nu^2}{4h^2}\right)}\right]\times$$

$$\times\left(\sqrt{k^2-\frac{\pi^2\nu^2}{4h^2}}\right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}\left(r\sqrt{k^2-\frac{\pi^2\nu^2}{4h^2}}\right)\Bigg\}. \qquad (1.19)$$

Now consider $I^{(2)}(k,x,y)$. As above we obtain that

$$I_N^{(2)}(k,x,y)=-2\pi i\sum_{\nu=1}^{\nu(N)}\operatorname{Res}\hat{G}(k,\rho,x_{n+1},y_{n+1})\rho^{\frac{n}{2}}H^{(2)}_{\frac{n}{2}}(\rho r)-$$
$$-\int_{C_N^-}\hat{G}(k,\rho,x_{n+1},y_{n+1})\rho^{\frac{n}{2}}H^{(2)}_{\frac{n}{2}-1}(\rho r)d\rho, \qquad (1.20)$$

where $\nu(N)$ is a number of the poles $\hat{G}(k,\rho,x_{n+1},y_{n+1})$ being in $C_N^- U(N,-N)$. By virtue of the asymptotics (1.17) of Hankel's functions and decrease of the integrand $\hat{G}(k,\rho,x_{n+1},y_{n+1})$ in (1.20) when $\rho\to+\infty$ we obtain that the integral by C_N^- tends to zero when $N\to\infty$. Therefore from (1.20) we obtain

$$I^{(2)}(k,x,y)=-2\pi i\Bigg\{-\frac{p(k)ch(x_{n+1}+y_{n+1})p(k)}{2sh2hp(k)}\left(-\sqrt{p^2(k)+k^2}\right)^{\frac{n}{2}-1}\times$$
$$\times H^{(2)}_{\frac{n}{2}}\left(-\sqrt{p^2(k)+k^2r}\right)+\frac{p(k)sh(x_{n+1}+y_{n+1})p(k)}{2sh2hp(k)}\left(-\sqrt{p^2(k)+k^2}\right)^{\frac{n}{2}-1}\times$$
$$\times H^{(2)}_{\frac{n}{2}}\left(-\sqrt{p^2(k)+k^2r}\right)+\sum_{\nu=1}^{\infty}\left[(-1)^{\nu}\frac{\left(p^2(k)+\frac{\pi^2\nu^2}{4h^2}\right)ch(x_{n+1}+y_{n+1})i\frac{\pi\nu}{2h}}{4h\left(p^2(k)+\frac{\pi^2\nu^2}{4h^2}\right)}+\right.$$
$$\left.+(-1)^{\nu+1}\frac{i\frac{\pi\nu}{2h}p(k)sh(x_{n+1}+y_{n+1})i\frac{\pi\nu}{2h}}{2h\left(p^2(k)+\frac{\pi^2\nu^2}{4h^2}\right)}+(-1)^{\nu+1}\frac{ch(2h-|x_{n+1}+y_{n+1}|)i\frac{\pi\nu}{2h}}{4h}\right]\times$$
$$\times\left(\sqrt{k^2-\frac{\pi^2\nu^2}{4h^2}}\right)^{\frac{n}{2}-1}H^{(2)}_{\frac{n}{2}-1}\left(-r\sqrt{k^2-\frac{\pi^2\nu^2}{4h^2}}\right)\Bigg\}. \qquad (1.21)$$

Allowing for

$$H^{(2)}_{\frac{n}{2}-1}(-z)=(-1)^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}(z) \qquad (1.22)$$

([12], p.218) from (1.15), (1.18), (1.21) and (1.22) for Green's function of the problem (1.1)-(1.2) for odd n we obtain

$$G(k,x,y)=-\frac{i}{2}(2\pi)^{-\frac{n}{2}}|x'-y'|^{1-\frac{n}{2}}\Bigg\{\frac{p(k)e^{-(x_{n+1}+y_{n+1})p(k)}}{2sh2hp(k)}\left(\sqrt{p^2(k)+k^2}\right)^{\frac{n}{2}-1}\times$$

$$\times H^{(1)}_{\frac{n}{2}}\left(r\sqrt{p^2(k)+k^2}\right)+\sum_{\nu=1}^{\infty}\frac{1}{2h}\left[\frac{\pi\nu}{2h}\cos(h-y_{n+1})\frac{\pi\nu}{2h}+p(k)\sin(h-y_{n+1})\frac{\pi\nu}{2h}\right]\times$$

$$\times\left[\frac{\frac{\pi\nu}{2h}\cos(h-x_{n+1})\frac{\pi\nu}{2h}+p(k)\sin(h-x_{n+1})\frac{\pi\nu}{2h}}{p^2(k)+\frac{\pi^2\nu^2}{4h^2}}\right]\times \qquad (1.23)$$

$$\times\left(\sqrt{k^2-\frac{\pi^2\nu^2}{4h^2}}\right)^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}\left(r\sqrt{k^2-\frac{\pi^2\nu^2}{4h^2}}\right)\Bigg\}.$$

Now let n be an odd number. Using the formula (1.14) from (1.13) we obtain

$$G(k,x,y)=(2\pi)^{\left(\frac{n}{2}+1\right)}|x'-y'|^{1-\frac{n}{2}}\left[\frac{1}{2}\int_0^{\infty}\hat{G}(k,\rho,x_{n+1},y_{n+1})\rho^{\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(\rho r)d\rho+\right.$$
$$\left.+\frac{1}{2}\int_0^{\infty}\hat{G}(k,\rho,x_{n+1},y_{n+1})\rho^{\frac{n}{2}}H^{(2)}_{\frac{n}{2}-1}(\rho r)d\rho\right]. \qquad (1.24)$$

In this case the Hankel function $H^{(1,2)}_{\frac{n}{2}-1}(z)$ at the point $z=0$ has a branching point. Therefore making the cross-cut $(-\infty,0)$ and using the formula (1.22) in the form of

$$z^{\frac{n}{2}-1}H^{(2)}_{\frac{n}{2}-1}(z)=(-z)^{\frac{n}{2}-1}H^{(1)}_{\frac{n}{2}-1}(-z)$$

from (1.24) we obtain

$$G(k,x,y)=\frac{(2\pi)^{-\left(\frac{n}{2}+1\right)}}{2}|x'-y'|^{1-\frac{n}{2}}\int_{-\infty}^{\infty}\hat{G}(k,\rho,x_{n+1},y_{n+1})\rho^{\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(\rho r)d\rho .$$

Thus we obtain the formula of the form of (1.15). Now if we'll have in exactly the same way as in calculation of the integral $I^{(1)}(k,x,y)$ we obtain in this case the formula (1.23).

Consequently, for any natural n the expansion of the Green function $G(k,x,y)$ of the problem (1.1)-(1.2) in the series (1.5) is obtained.

The series in (1.5) uniformly converges to k in every compact. Really, for large ν and $|k|\le A$ where A is some constant, the following inequality takes place.

$$0<\arg\sqrt{1-\left(\frac{2hk}{\pi\nu}\right)^2}<\frac{\pi}{4}. \qquad (1.25)$$

For $r>0$ from (1.25) and from the asymptotic estimation (1.17) it follows that

$$\left|H^{(1,2)}_{\frac{n}{2}-1}\left(r\sqrt{k^2-\frac{\pi^2\nu^2}{4h^2}}\right)\right|\le Ce^{-\frac{\sqrt{2}}{2}r\nu}.$$

Consequently, for $p(k)\ne i\frac{\pi\nu}{2h}$ the series in (1.5) uniformly converges to k. Theorem 1 is proved.

§ 2. Limiting absorption principle.

Consider the boundary value problem (1.1)-(1.2) with a real parameter k in Π. In this case the solution of the problem isn't unique. By limiting absorption principle for selection unique solution of the problem (1.1)-(1.2) the parameter k in the problem (1.1)-(1.2) is replaced by $k + i\varepsilon = k_\varepsilon$, then select unique bounded solution of the problem (1.1)-(1.2) we pass the limit when $\varepsilon \to 0$. The obtained function is a solution of the problem (1.1)-(1.2). The solution of the problem (1.1)-(1.2) with a complex parameter is given by the formula

$$u(k_\varepsilon, x) = \int_{\Pi}\!\cdots\!\int G(k_\varepsilon, x, y) f(y) dy, \tag{2.1}$$

where the convolution (2.1) is accomplished on the layer Π and $G(k_\varepsilon, x, y)$ is determined by the formula (1.5) in which we must substitute k by k_ε. From theorem 1 it follows that the series in (1.5) with the parameter k_ε uniformly converges in ε. Therefore we can pass to the limit when $\varepsilon \to 0$ in (2.1). The obtained limiting function satisfies the problem (1.1)-(1.2) with the real parameter k. Thus the following theorem takes place.

Theorem 2. *For the solution of the problem (1.1)-(1.2) for $n \geq 3$ for any real k and for $n = 1,2$ $k \neq \frac{\pi \nu}{2h}$, the limiting absorption principle holds.*

Author thanks to prof. B.A. Iskenderov for the statement of the problem and helpful comments.

References

[1]. Brekhovskikh L.M. *Waves in laminar mediums.* Moscow, "Nauka", 1973, 341p. (in Russian)
[2]. Sveshnikov A.G. *Radiation principle.* DAN SSSR, 1950, v.73, №5, p.917-920. (in Russian)
[3]. Iskenderov B.A., Akimov A.B. *Limiting absorption and limiting amplitude principles and partial conditions of radiation for a bounded value problem in n dimensional layer for Helmholtz equation.* Differential equations, 1977, v.13, №8, p.1503-1505. (in Russian)
[4]. Sveshnikov A.G. *Limiting absorption principle for wave guide.* DAN SSSR, 1951, v.80, №3, p. 345-347. (in Russian)
[5]. Iskenderov B.A., Abbasov Z.G., Eyvazov E.Kh. *Radiation principles for the Helmholtz equation in a cylindrical domain.*, DAN Azerb. SSR, 1980, v.36, №4, p.8-11. (in Russian)
[6]. Iskenderov B.A. Principles of radiation for elliptic equation in the cylindrical domain. Colloquia Math. Soc. Janos Bolyai. Szeged, Hungary, 1988, p.249-261.
[7]. Iskenderov B.A., Eyvazov E.Kh., Efendiyeva A.N. *Radiation principle for the higher order elliptic equations in a cylindrical domain.* Differential equations., 1987, v.23, №10, p.1804-1807. (in Russian)
[8]. Iskenderov B.A. . *Radiation principles for the higher order elliptic equations in a cylindrical domain.* Journal of computational mathematics and mathematical physics, 1996,v.36, №1, p. 73-91. (in Russian)
[9]. Iskenderov B.A., Mekhtiyeva A.I. *Radiation principle for the Helmholtz equation in a many-dimensional layer with impedance boundary conditions.* Differential equations, 1993, v.29, №8, p.1462-1464. (in Russian)
[10]. Gelphand I.M., Shilov G.E. *Generalized functions.* Issue 3, Some questions of the theory of differential equations, Moscow, Fizmatgiz., 1958, 274p. (in Russian)
[11]. Shilov G.E. *Mathematical analysis.* (Special course), Moscow, Fizmatgiz., 1965, 327p. (in Russian)
[12]. Nikiforov A,F., Uvarov V.B. *Basis of theory of special functions.* Moscow., Fizmatgiz., 1974, 303p. (in Russian)

Seymur E. Suleymanov
Institute of Mathematics & Mechanics of NAS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

Shahrizad D. Mamedova
Azerbaijan State Oil Academy.
20, Azadlyg av., 370601, Baku, Azerbsijan.

Received January 5, 2001; Revised June 29, 2001.
Translated by Mitzoyeva K.S.

VAHABOV N.G.

LOCALIZATION OF SPECTRUM AND ITS APPLICATIONS, III NUMERICAL RANGE AND SPECTRUM OF OPERATOR-FUNCTIONS

Abstract

In the paper the classes of operator-functions are selected in a Banach space for which the analogs of theorems on behavior of numerical ranges different geometrical operations are proved. These geometrical properties of numerical ranges of operator-functions are applied to obtain the localization relations for spectrum of operator-functions by its numerical ranges.

Introduction. One of the problems of spectral theory is obtaining for operator functions analogies of classical theorems on localization of spectrum of operator by its numerical ranges. As was noted by Hadeler K. [4] it is impossible for arbitrary operator functions and he proved the analogy of Wintner-Stone's theorem on localization of spectrum of linear multiparametric operator in a real Hilbert space. The sufficient condition of localizability of spectrum of o.f. given by Hadeler suggests the selection of one-(multi) parameter operators in Banach space on which a series of facts of the classical theory of numerical ranges can be carried over. On the other hand technique of work A.Brown's and R.Duglas' [5] allows to adapt and use the scheme of Hadeler's proof for holomorphic o.f. in a complex Banach space.

The basic aim of the present paper is to obtain for o.f. analogies of the theorems on behavior of numerical ranges at different geometric operators and apply them to the questions on localization of spectrum of operator functions. The notice of these results is given in [14b].

Attraction of geometric properties of numerical ranges of o.f. and Banach spaces brings to light the main role of Teoplitz's theorem on localization of point spectrum and Wintner-Lumer's theorem on localization of approximative points spectrum in the questions on localizability of spectrum (and its parts) by numerical ranges. It turned out that all the localization theorems can be derived from these two theorems and if we use the adaptation of Berberian's construction for the spaces with semiinner product-just from Teoplitz's theorem.

We'll describe contents of the paper which is a continuation of the first two parts [14c]. In §5 the three types of domains of regularity of o.f. in Banach space are introduced and influence of geometric properties of the space and numerical ranges on its hierarchy is considered. The three natural classes of holomorphic o.f. in Banach space for which the analogs of geometric and spectral properties of numerical ranges of operators are selected.

In §6 the one-parameter analogs of geometric properties of numerical ranges of operators: G. Lumer's (K.Mc. Gregor) theorem on closed convex hull of Lumerian (Bauerian) numerical range; B. Bollobas's theorem on behavior of Bauerian numerical range relative to conjugation and S.Berberian's and G.Orland's theorem on extension (by Berberian) of Hausdorff numerical range are proved. Then using these geometric properties of numerical congruences and also the localization correlation for the compression spectrum [14c, §1, proposition 1] the one-parameter analog of theorems: Winter-Lumer's theorem on localization of approximative point spectrum by Lumerian numerical range; Lumer's theorem on localization of spectrum by algebraic numerical range; William's theorem on localization of spectrum by Bauerian *numerical range and*