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ON THE COMPLEXITY CHARACTERISTICS OF SOME PROBLEMS

Abstract

The complexity characteristics of optimal algorithms theory for the problems being solved approximately are considered. The connections of the complexity category with other complexity characteristics, namely with entropy on assumption of the existence of algorithm of approximate solution of given precision are investigated. With application of the obtained result the estimations of entropy for some concrete tasks are obtained.

Investigation of the complexity issue of processes is of great interest in various fields of science, technology and economics. The study difficulties of many problems arising as example in projecting of modern technology, in planning on the different levels in controlling of the systems, in scientific basing of the evolutionary processes prediction and others are mainly of the same nature. Namely in all these processes it's required to establish the principle solvability of arising problems, and for algorithmically solvable problems it should be found out their physical realizability. In other words, it's necessary to estimate the minimal volumes of those and other resources required for realization of a solution. The absence of precise definition of the "complexity" notion didn't usually impede its recognition and investigation on empirical level. Starting with 50-s of the XX-century with increase of production scale and control levels, the necessity of investigation of appearing mathematical models arised. And consequently the mathematical models of applied problems appeared for statements and solutions of which the heuristic representations of complexity weren't sufficient. Hence, the necessity of introducing of the precise definition of "complexity" arised. A.N. Kolmogorov was the first who suggested the way to forming of complexity category, to investigation of its properties and perspectives of its applications [1]. Later on the other directions of investigations related to computational complexity, information complexity, energetic complexity, complexity of schemes and others appeared.

One of the basic applications of the "complexity" category is the theory of optimal algorithms for problems that are solved approximately. Here, by complexity of a problem is meant the true intrinsic difficulty of obtaining its solution not depending on a method of determination of solution. The key issue in such problems is the choice of the best algorithm for the solution of a problem. The choice of the best algorithm is a multicriterial optimization problem among which we can note: simplicity of program realization, time, the volume of occupied memory, stability and so on. The investigation that is necessary for characterization and construction of an optimal (in some sense) algorithm for the given concrete problem is a difficult mathematical task and requires the significant effort. Only in very rare cases we can find the exact value of mathematical complexity of the problem, more often one is able to estimate it. And besides it's natural and is dictated with the notion itself.

Now we introduce the necessary definitions and notions.

Let Y and Z be linear metric spaces with the metrics ρ_Y, ρ_Z respectively. By an information operator on Y we understand any (maybe nonlinear) operator $N : Y \to Z$, with

$$N(y) = (N_1(y),\dots,N_n(y)),\ y \in Y$$

The least non-negative number n, $0 \le n \le +\infty$ for which it will be found the linear subspace Γ from Z of dimension n containing $N(Y)$ (in particular it can be infinite dimensional) we call the cardinality of an information operator. By an algorithm φ of ε-approximation of elements of the arbitrary compact subspace $K \subset Y$ we mean any (maybe nonlinear) operator $\varphi : \Gamma \subset Z \to K$ assigning within the finite number of steps to information vector $N(y) \in \Gamma,\ y \in K$ with finite cardinality n an element $\varphi(N(y)) \in K$ as an ε-approximate solution

$$\rho_Y(y, \varphi(N(y))) \le \varepsilon .$$

Definition 1. [2] *An ε - entropy of the compact set K we call the logarithm (on the base of 2) of minimal number of balls with radius ε covering this set.*

The measure of the ball of the radius r in the space R^n is determined by the formula [3]

$$\mu(O_r) = \begin{cases} \dfrac{\pi^m}{m!} r^{2m}, & if\ \ n = 2m \\ \dfrac{2(2\pi)^m}{(2m+1)!!} r^{2m+1}, & if\ \ n = 2m+1. \end{cases}$$

The optimality criteria of an approximate solution for a large class of problems is the least quantity of calculations for determination of their solutions, i.e. construction of an optimal algorithm using information of minimal length. In the given case the complexity of the problem is accepted as the cardinality of optimal information.

The goal of paper is to obtain the estimations for complexity of the above mentioned problems. Under assumption of existence of algorithm of approximate solution of the given precision the connections of the complexity category of the given algorithm with other complexity characteristics and namely with entropy of given problem are investigated.

Assume that for a fixed non-negative number $\varepsilon \ge 0$ there exists the algorithm φ of ε-approximation of elements of the fixed compact set K from the space Y determined on the image of the information operator N with the finite cardinality $n(\varepsilon)$. Therefore we can take the space Γ with finite dimensional $n(\varepsilon)$. Denote by $e_1,\dots,e_{n(\varepsilon)}$ a basis in the space Γ. Then the arbitrary element $N(y)$ from $N(Y)$ we can represent in the form of

$$N(y) = \sum_{i=1}^{n(\varepsilon)} L_i(y) e_i ,$$

where L_i in general are nonlinear functionals.

Let $\psi : \Gamma \to R^{n(\varepsilon)}$ be a natural homeomorphism between the spaces of the same finite dimension establishing the correspondence $N(y) \leftrightarrow (L_1(y),\dots,L_{n(\varepsilon)}(y))$.

Assume that the information operator N is continuous, then the set $\psi(N(K))$ is also compact. Hence, there exists the finite number

$$M(\varepsilon) = \sup_{y \in K} \{|L_i(y)| : i = 1,\dots,n(\varepsilon)\} \tag{1}$$

such that we have

$$\psi(N(K)) \subset [-M(\varepsilon), M(\varepsilon)]^{n(\varepsilon)} .$$

Then by virtue of compactness for an arbitrary number $\delta > 0$ the sets $\psi(N(K))$ and $N(K)$ may be covered by a finite number of balls of the radius δ.

It's obvious that this number doesn't exceed the quantity

$$\frac{\mu\left([-M(\varepsilon),M(\varepsilon)]^{n(\varepsilon)}\right)}{\mu\left(O_{\delta/2}\right)},$$

and consequently doesn't exceed the number

$$\left(\frac{c\cdot M(\varepsilon)}{\delta}\right)^{n(\varepsilon)},$$

where $c=const>4$.

Really, for $n=2m$ we have

$$\frac{\mu\left([-M(\varepsilon),M(\varepsilon)]^{2m(\varepsilon)}\right)}{\mu\left(O_{\delta/2}\right)}=\frac{(2M)^{2m}\cdot m!\cdot 2^{2m}}{\pi^{m}\cdot\delta^{2m}}=\left(\frac{16}{\pi}\right)^{m}\cdot m!\left(\frac{M}{\delta}\right)^{2m}.$$

And for $n=2m+1$ we have

$$\frac{\mu\left([-M,M]^{2m+1}\right)}{\mu\left(O_{\delta/2}\right)}=\frac{(2M)^{2m+1}\cdot(2m+1)!!\cdot 2^{2m+1}}{2\cdot(2\pi)^{m}\cdot\delta^{2m+1}}=2\cdot(2m+1)!!\left(\frac{8}{\pi}\right)^{m}\cdot\left(\frac{M}{\delta}\right)^{2m+1}$$

We denote the centers of these balls by $N(y_i)$, $i=1,\ldots,\left(\frac{c\cdot M(\varepsilon)}{\delta}\right)^{n(\varepsilon)}$. Then for arbitrary $y\in K$ it's found $y_i\in K$ such that

$$\rho_{\Gamma}(N(y),N(y_i))\le\delta.$$

Assume that the algorithm φ is uniformly continuous. Then for arbitrary $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that for any y, y_i satisfying $\rho_{\Gamma}(N(y),N(y_i))\le\delta(\varepsilon)$

$$\rho_Y(\varphi(N(y)),\varphi(N(y_i)))\le\varepsilon$$

is fulfilled.

We take as ε previously fixed number ε for which there exists the algorithm φ of ε-approximation, and as $\delta(\varepsilon)$ we take previous radius δ. Then by virtue of the triangle inequality we have

$$\rho_Y(y,\varphi(N(y_i)))\le\rho_Y(y,\varphi(N(y)))+\rho_Y(\varphi(N(y)),\varphi(N(y_i))).$$

The first addend doesn't exceed ε by virtue of existence of the algorithm φ of ε-approximation, and the second addend by virtue of uniformly continuity of this algorithm. Thus

$$\rho_Y(y,\varphi(N(y_i)))\le 2\varepsilon.$$

In other words the set K is covered at most $\left(\frac{c\cdot M(\varepsilon)}{\delta(\varepsilon)}\right)^{n(\varepsilon)}$ balls of the radius 2ε with the centers $\varphi(N(y_i))$, $i=1,\ldots,\left(\frac{c\cdot M(\varepsilon)}{\delta(\varepsilon)}\right)^{n(\varepsilon)}$.

Thus the following theorem is proved.

Theorem 1. *Let ε be a non-negative number for which there exists the uniform continuous algorithm φ of ε-approximation of elements of the compact set K in the linear metric space Y, $n(\varepsilon)$ is finite cardinality of nonlinear continuous informational operator N, $M(\varepsilon)$ be a positive number determined in (1). Then there exists a positive number $\delta(\varepsilon)$ such that for 2ε - entropy the set K, the following inequality is valid.*

$$H_k(2\varepsilon) \le n(\varepsilon) \cdot \log \frac{cM(\varepsilon)}{\delta(\varepsilon)} \tag{2}$$

i.e. 2ε *-entropy of* K *is lower estimation of problem of* ε *-approximation of elements of this set.*

Applications of main result.

As it's known, many non-linear problems under mathematical modelling generate non-linear problems about the determination possibility of analytical solution of which no use to talk. Therefore the necessity to prove the existence of solution and to find the way for obtaining the approximate solution arises. This in its turn reduces to approximation and numerical approach and consequently to using of PC that requires the optimal algorithm. The proved theorem gives the possibility for the problems of above mentioned type to prove that the optimal algorithm can't give the necessary information the length of which doesn't satisfy the inequality (2).

We cite a few simple and characteristics examples that explain the importance of this theorem.

Corollary 1. *Let* A *be a closed one-to mapping operating from the linear metric space* (X, ρ_x) *to the linear metric space* (Y, ρ_Y)*, and* K *is a compact subset of* Y *for which the inequality (2) is valid. Then for* 2ε *-entropy of each pre-image (in the case of multivaluability of inverse mapping) of the set* K *the following inequality is valid,*

$$H_{A^{-1}K}(2\varepsilon) \le n(\varepsilon) \cdot \log \frac{c \cdot M(\varepsilon)}{\delta(\varepsilon)},$$

where under $A^{-1}K$ *we understand one of the pre-images of* K.

Really, from compactness of the set K and the lower semi-continuity of mapping A we can represent a pre-image of K as union (maybe infinite number) of the compact sets

$$A^{-1}K = \bigcup_j K_j, \text{ where } \quad A(K_i) = K \quad \forall i .$$

Then the sets K_j are covered by same quantity of balls as the set K of the radius 2ε with centers $A^{-1}(\varphi(N(y_i))), i = 1, \ldots, \left(\frac{c \cdot M(\varepsilon)}{\delta(\varepsilon)}\right)^{n(\varepsilon)}$.

Corollary 2. *Let* A *be a non-linear operator operating from the linear metric space* (X, ρ_x) *to the linear metric sequentially-dense space* (Y, ρ_Y) *such that the pre-image of arbitrary compact* $K \subset Y$ *is bounded and the constant* $\tilde{c} \ge 1$ *exists with the condition*

$$A^{-1}\left(K \cap Y_{n(\varepsilon)}\right) \subset X_{\tilde{c} \cdot n(\varepsilon)},$$

where X_i, Y_j *are finite-dimensional spaces correspondingly in* X *and* Y *with dimensions* i *and* j*. Then for* 2ε *- entropy of the set* $A^{-1}\left(K \cap Y_{n(\varepsilon)}\right)$ *the following inequality is valid*

$$H_{A^{-1}\left(K \cap Y_{n(\varepsilon)}\right)}(2\varepsilon) \le \tilde{c} \cdot n(\varepsilon) \cdot \log \frac{c \cdot P(\varepsilon)}{\delta(\varepsilon)} \tag{3}$$

where $P(\varepsilon)$ *is determined from the conditions imposed on the operator* A^{-1}.

Really, from definition of sequentially-dense of Y it can be represented as union $Y=\bigcup_n Y_n$ of finite-dimensional subspaces. In other words for arbitrary $y\in Y$ there exists the sequence $y_{n(\varepsilon,y)}\in Y_{n(\varepsilon,y)}$ such that

$$\rho_Y\left(y,y_{n(\varepsilon,y)}\right)\le\varepsilon .$$

Taking into account of compactness of K we can assume that

$$\sup_{y\in K} n(\varepsilon,y)=n(\varepsilon)$$

and for arbitrary $y\in K$

$$\rho_Y\left(y,y_{n(\varepsilon,y)}\right)\le\varepsilon .$$

Consequently, for the set K analogously to the proof of the theorem we can obtain

$$K\cap Y_{n(\varepsilon)}\subset\psi^{-1}\left([-M(\varepsilon),M(\varepsilon)]^{n(\varepsilon)}\right).$$

Taking notice of conditions imposed on the operator A^{-1} we can say that there exists a positive number $P(\varepsilon)$ such that

$$A^{-1}\left(K\cap Y_{n(\varepsilon)}\right)\subset[-P(\varepsilon),P(\varepsilon)]^{\tilde{c}\cdot n(\varepsilon)}.$$

Hence we obtain the truth of the inequality (3).

Corollary 3. *Let A be a non-linear operator operating from the linear metric space (X,ρ_x) to the linear metric sequentially-dense space (Y,ρ_Y) such that*

$$A^{-1}(A(X)\cap Y_n)\subset X_{m(n)}, \tag{4}$$

where for arbitrary n $m(n)$ is finite and for any $y_1,y_2\in A(x)$ and $x_1\in A^{-1}(y_1)$, $x_2\in A^{-1}(y_2)$

$$\rho_Y(y_1,y_2)\ge f(\rho_x(x_1,x_2)), \tag{5}$$

where $f:R_+\to R_+$ is a continuous function satisfying the conditions $f(0)=0$; $f(\tau)>0,\tau>0; f(\tau)\xrightarrow[\tau\to 0]{}0$.

Then for arbitrary compact $K\subset Y$ the inequality

$$H_{A^{-1}(K\cap A(X)\cap Y_{n(\varepsilon,K)})}\left(\tilde{\delta}(\varepsilon)\right)\le m(n(\varepsilon,K))\cdot\log\frac{c\cdot Q(\varepsilon)}{\delta(\varepsilon)}$$

is valid, where $Q(\varepsilon)$ is determined analogously to the quantity $P(\varepsilon)$ from corollary 2 and $\tilde{\delta}(\varepsilon)$ from continuity of the function f .

Analogously to the proof of corollary 2 from the consequentially-density of the space Y for the arbitrary compact $K\subset Y$ there is found a finite number $n(\varepsilon,K)$ such that for any $y\in K$ exists $y_{n(\varepsilon,K)}\in Y_{n(\varepsilon,K)}$ with

$$\rho_y\left(y,y_{n(\varepsilon,K)}\right)\le\varepsilon$$

and taking notice of the theorem's proof we have that the set $K\cap A(X)\cap Y_{n(\varepsilon,K)}$ is covered by $\left(\dfrac{c\cdot M(\varepsilon)}{\delta(\varepsilon)}\right)^{n(\varepsilon,K)}$ balls of the radius 2ε. Then taking into account the continuity of the function f, the condition (4) and the inequality (5) there are found the numbers $\tilde{\delta}(\varepsilon)$ and $Q(\varepsilon)$ such that the set $A^{-1}(K \cap A(x) \cap Y_{n(\varepsilon,K)})$ is covered by $\left(\frac{c \cdot Q(\varepsilon)}{\delta(\varepsilon)}\right)^{m(n(\varepsilon,K))}$ balls of the radius $\tilde{\delta}(\varepsilon)$ that affirms the truth of corollary 3.

Finally we introduce the application of theorem 1 for estimations of ε - entropy of one class of linear operator equations.

Let $d \in N$, $G = [0,2\pi]^d$ and $L_2(G)$ be the space of square-summable functions on G with the usual norm and the usual scalar product. For $n \in N$ and $s \in [0,2\pi]$ we set

$$e_0(s) = \frac{1}{\sqrt{2\pi}} \quad e_n(s) = \frac{1}{\sqrt{\pi}}\cos(ns) \quad e_{-n}(s) = \frac{1}{\sqrt{\pi}}\sin(ns).$$

Then for the multiindex $i = (i_1,\dots,i_d) \in Z^d$ the basis function $e_i \in L_2(G)$ is defined by

$$e_i(t) = e_{i_1}(t_1)\cdot \ldots \cdot e_{id}(t_d),$$

where $t = (t_1,\dots,t_d) \in G$.

The Fourier coefficients of the function $g \in L_2(G)$ are given by $\hat{g}(i) = (g, e_i)$, $i \in Z^d$. Moreover, for $i \in Z^d$ assume $|i| = \sqrt{i_1^2 + \ldots + i_d^2}$.

Let $r \in R_+$. As it's known we can determine the Sobolev space of the periodic functions on G having the square-summable partial derivatives to the order r as the following form

$$H^r(G) = \left\{ g : g \in L_2(G), \|g\|_r := \left(\sum_{i \in Z^d} \left(1 + |i|^2\right)^r \hat{g}(i)^2\right)^{1/r} < \infty \right\}.$$

For the first time in papers [5,4] in the case $\alpha = 1$ the class of the operator equations

$$u - T_k u = g \;\equiv\; u(t) - \int_G K(t,s)u(s)ds = g(t)$$

was considered, where $K(t,s) \in \mathcal{H}^r \equiv \left\{ K \in H^r(G^2) : \|K\|_r \le 1, \left\|(I - T_k)^{-1}\right\|_{L_2 \to L_2} \le 1 \right\}$ and $g \in B_{H^r}$ is a unique ball in the space H^r. Here the lower and upper estimations of minimal error attained by the constructed algorithm are found. The Fourier *coefficients of* kernel and a free term are taken as information operator of those algorithm.

And in paper [6] the following exact relation was established in the case of $d \ge 1$ between the minimal error and minimal information

$$n(\varepsilon) \approx \left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)^{r/d}.$$

Then using the obtained in this paper exact relation and theorem 1 we can find the following upper estimation for ε-entropy of the compact set $\mathcal{H}^r : B_{H_r}$:

$$H(\varepsilon) \le \left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)^{r/d} \cdot \log\frac{c \cdot M(\varepsilon)}{\delta(\varepsilon)}.$$

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LIMITING ABSORPTION PRINCIPLE FOR THE HELMHOLTZ EQUATION IN A MANY-DIMENSIONAL LAYER WITH A GENERAL BOUNDARY CONDITION

Abstract

In the paper the Green function of the boundary value problem is constructed and for this problem the limiting absorption principle is proved.

Introduction. Wave propagation in a homogeneous layer bounded from two sides by plane-parallel boundaries leads to different boundary value problems in a layer for the Helmholtz equation. The limiting absorption principle for the Helmholtz equation in a two-dimensional layer with the Dirichlet or Neumann boundary conditions is considered in L.M. Brekhovskikh's book [1], in a three-dimensional layer the limiting amplitude and partial conditions of radiation for this problem are considered in A.G.Sveshnikov's article [2]. In this paper A.G. Sveshnikov introduced also new conditions which ensure the uniqueness of solutions of a boundary value problem for the Helmholtz equation in a three-dimensional layer. Now these conditions are called A.G. Sveshnikov's partial conditions. The radiation principle in a many dimensional layer for the Helmholtz equation with Dirichlet and Neumann boundary conditions were studied in [3]. The radiation principles in a three-dimensional cylindrical domain are studied in [4], and in a many-dimensional cylindrical domain - in [5,6]. The radiation principles for the higher order elliptic equations with constant coefficients in a many-dimensional cylinder are studied in [7,8]. In [3,5-8] for the first time the resonance phenomenon was studied and the rate of increase of solutions of non-stationary problem is mentioned when $t \to \infty$. In [9] the radiation principles for the Holmholtz equation are studied in a many-dimensional layer with impedance boundary conditions.

In the present paper the Green function for the Helmholtz equation is constructed in a many-dimensional layer with a general boundary condition and the limiting absorption principle is studied. The results by limiting amplitude principle for this problem will be published later.

§ 1. Construction of the Green function.

Let

$$\Pi = \{x : (x', x_{n+1}),\ x' = (x_1, x_2, \ldots, x_n),\ -\infty < x_j < +\infty,\ j = 1,2,\ldots,n; -h < x_{n+1} < +h\}$$

be a layer in the $n+1$ dimensional Euclidean space R_{n+1}. Consider the following boundary value problem in Π

$$\left(\Delta + k^2\right)u(k,x) = f(x), \tag{1.1}$$

$$\left(\frac{\partial}{\partial x_{n+1}} + p(k)\right)u(k,x)\Big|_{x_{n+1}=\pm h} = 0, \tag{1.2}$$

where Δ is a Laplacian operator, $f(x)$ is a finite infinitely differentiable function with support in Π, k is a complex parameter with $\operatorname{Im} k > 0$, $p(k) = ak + b$, a and b are real numbers.

Definition 1. *Under the solution of the problem (1.1)-(1.2) we'll understand the decreasing on infinity function $u(k,x)$ satisfying the equation (1.1) and the boundary conditions (1.2) in sense of generalized functions ([10], p.40-187).*