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THE APRIORI ESTIMATE OF HÖLDER'S NORM OF SOLUTIONS OF DEGENERATE ELLIPTICO-PARABOLIC EQUATIONS OF THE SECOND ORDER

Abstract

In the article a class of degenerate elliptico-parabolic equations of the second order in divergent form is considered. For weak solutions of these eqautions the apriori estimate of Hölder's norm is proved.

Introduction. Let R_n be an n - dimensional Euclidean space of the points $x=(x_1,...,x_n)$, $\Omega\subset R_n$ be a bounded n - dimensional domain with the boundary $\partial\Omega$, $B_R^{x^0}$ be an n- dimensional open ball of the radius $R<1$ with center at the point x^0 such that $\overline{B}_R^{x^0}\subset\Omega$, $B_R^0=B_R$, $Q_T=\{(x,t): x\in\Omega, 0<t<T<\infty\}$, $S_T=\{(x,t): x\in\partial\Omega, 0\le t\le T\}$, $\Gamma(Q_T)$ be a parabolic boundary of Q_T, i.e. $\Gamma(Q_T)=S_T\cup\{(x,t): x\in\Omega, t=0\}$, $Q_R^{x^0}=B_R^{x^0}\times(0,T)$, $Q_1^{x^0}=B_{R'}^{x^0}\times\left(\frac{7T}{8},T\right)$, $Q_2^{x^0}=B_{R'}^{x^0}\times\left(\frac{T}{4},\frac{7T}{24}\right)$, $Q_R^0=Q_R$, $Q_1^0=Q_1$, $Q_2^0=Q_2$, where $R'=\frac{R}{4}$.

Consider the following equation in Q_T

$$Lu=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)+\frac{\partial}{\partial t}\left(\varphi(T-t)\frac{\partial u}{\partial t}\right)-\frac{\partial u}{\partial t}=0\,, \qquad (1)$$

with supposition that $\|a_{ij}(x,t)\|$ is a real symmetrical matrix with measurable in Q_T elements, where for all $(x,t)\in Q_T$ and arbitrary n- dimensional vector ξ

$$\gamma|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi\le\gamma^{-1}|\xi|^2;\ \gamma\in(0,1]-const\,. \qquad (2)$$

Besides with respect to the function $\varphi(z)$ for $z>0$ the condition

$$\varphi(0)=0,\varphi(z)>0,\ \varphi'(z)>0,\ \varphi''(z)>0,\ \varphi'(z)\ge\varphi(z)\varphi''(z),\ \frac{\varphi'(z)}{\sqrt{\varphi(z)}}\ge\beta \qquad (3)$$

be satisfied, where β is a positive constant.

The aim of the present article is the proof of the interior apriori estimate of Hölder's norm for weak solutions of equations (1). Note that for the second order parabolic equations in divergent form the corresponding result is derived in [1-2]. We indicate monograph [3] in which the the Hölder continuity of solutions of the second order quasilinear parabolic equations was proved. As to the second order parabolic equations in divergent structure we mention in this connection papers [4-6]. Note that in proof of apriori estimate of Hölder's norm is the analogue of the classical Harnack's inequality for non-negative solutions of equations (1) established in [7] is a basis instrument.

1. Some notations and definitions.

Let $W_2^{1,1}$ be a Banach space of functions $u(x,t)$ given on Q_T for which

$$\|u\|_{W_2^{1,1}(Q_T)} = \left[\int_{Q_T} \left(u^2 + \sum_{i=1}^{n} u_i^2 + u_t^2 \right) dxdt \right]^{\frac{1}{2}}.$$

is finite.

The function $u(x,t) \in W_2^{1,1}(Q_T)$ is called a weak solution of equation (1) if $\int_Q \left(\sum_{i,j=1}^{n} a_{ij} u_i w_j + \varphi(T-t) u_t + u_t w \right) dxdt = 0$ for every infinite differentiable function $\omega(x,t)$ vanishing on $\Gamma(Q_T)$.

Here for $i = 1,...,n$ $u_i = \dfrac{\partial u}{\partial x_i}$.

Denote by $C_0(Q_T)$ a space of continuous in Q_T bounded function with norm

$$\|u\|_{C_0(Q_T)} = \sup_{Q_T} |u(x,t)|.$$

Denote by $C_\alpha(Q_T)$ a Banach space of functions $u(x,t)$ satisfying in Q_T the Hölder's parabolic condition with exponent α, $0 < \alpha < 1$. The norm in this space is introduced by the following way

$$\|u\|_{C_\alpha(Q_T)} = \|u\|_{C_0(Q_T)} + \sup_{\substack{(x^1,t^1) \in Q_T \\ (x^2,t^2) \in Q_T \\ (x^1,t^1) \neq (x^2,t^2)}} \frac{\left| u(x^1,t^1) - u(x^2,t^2) \right|}{\left[\rho((x^1,t^1),(x^2,t^2)) \right]^\alpha},$$

where $\rho((x^1,t^1),(x^2,t^2)) = \sqrt{|t^1 - t^2| + |x^1 - x^2|^2}$.

2. Lemma on oscillation.

For brevity we'll write sup and inf instead of *ess* sup and *ess* inf respectively.

We introduce the following notations:

$$M = \sup_{Q_R} u, \quad m = \inf_{Q_R} u, \quad M_1 = \sup_{Q_1} u, \quad m_1 = \inf_{Q_1} u,$$

$$\operatorname*{osc}_{Q_R} = M - m, \quad \operatorname*{osc}_{Q_1} u = M_1 - m_1.$$

Lemma. *Let u be a week solution of equation (1) in Q_R and relative to the coefficients of the operator L the conditions (2)-(3) be satisfied. Then the following inequality*

$$\operatorname*{osc}_{Q_1} u \le \theta \operatorname*{osc}_{Q_R} u \tag{4}$$

is valid, where constant $\theta \in (0,1)$ depends only on T, γ, n and the function φ.

Proof. If $u(x,t)$ is a non-negative weak solution of equation (1), then

$$\mu = \frac{1}{mes Q_2} \int_{Q_2} u(x,t) dxdt \le \sup_{Q_2} u.$$

On the other hand, according proved in [7] Harnack's inequality for these solutions

$$\sup_{Q_2} u \le C_1 \inf_{Q_1} u\,, \tag{5}$$

where constant $C_1 > 1$ depends only on T,γ,n and the function φ.

From inequality (5) it follows that

$$\inf_{Q_1} u \ge C_1^{-1}\mu\,. \tag{6}$$

Let now $u(x,t)$ be an arbitrary (not necessarily non-negative) weak solution of equation (1) in Q_T.

Allowing that $M-u$ and $u-m$ are non-negative weak solutions of equation (1) in Q_2, then from (6) we conclude

$$M - M_1 \ge C_1^{-1}(M-\mu), \quad m_1 - m \ge C_1^{-1}(\mu - m).$$

Adding these last relations we have the required estimate (4). The lemma is proved.

Remark 1. The statement of this lemma is correct for cylinders $Q_1^{x^0}$ and $Q_R^{x^0}$. It follows from that inequality (5) remains valid for cylinders $Q_1^{x^0}$ and $Q_2^{x^0}$ respectively.

Remark 2. Let $s \in \left[\frac{15}{16},1\right], Q_1^{x^0}(s) = B_{R/4} \times \left[\left(s-\frac{1}{16}\right)T, sT\right]$. Then $Q_1^{x^0} \supset Q_1^{x^0}(s)$ and therefore according to (5)

$$\sup_{Q_2^{x^0}} u \le C_1 \inf_{Q_1^{x^0}} u \le C_1 \inf_{Q_1^{x^0}(s)} u$$

provided u is non-negative weak solutions of equations (1).

Thus lemma on oscillation holds for cylinders $Q_1^{x^0}(s)$ and $Q_R^{x^0}(s) = B_R^{x^0} \times (0,sT)$ respectively with constant θ which is independent of s.

3. The apriori estimate of Hölder's norm.

We denote by Ω_ρ the set $\{x : x \in \Omega, dist(x,\partial\Omega) > \rho\}$ for $\rho > 0$ and let

$$Q^\rho = \Omega_\rho \times \left(\rho^2, T\right].$$

Theorem. Let u be a weak solution of equation (1) in Q_T and conditions (2)-(3) hold with respect to coefficients of the operator L. Then for arbitrary $\rho > 0$ the following inequality

$$\|u\|_{C_\alpha(Q^\rho)} \le C_2 \|u\|_{C_0(Q_T)} \tag{7}$$

is valid, where α depends only on $\gamma, n, T, diam\Omega$ and the function φ, but C_2 in addition and on ρ.

Proof. Let $\left(x^1,t^1\right)$ and $\left(x^2,t^2\right)$ be two points belonging to Q^ρ. Without losing generality we can assume that $\rho \le \min\left\{1, \frac{T}{32}\right\}$.

We consider separately two cases.

1. At least one of above mentioned points belongs to $\Omega_\rho \times \left[\frac{15}{16}T, T\right]$.
2. Both points belong to $\Omega_\rho \times \left(\rho^2, \frac{15}{16}T\right)$.

Case 1. Let for definiteness $t^2 \le t^1$ and $(x^1,t^1) \in \Omega_\rho \times \left[\frac{15}{16}T, T\right]$. Assume at first, that $(x^2,t^2) \in B_\rho^{x^1} \times (t^1-\rho^2, t^1)$. Denote by m_0 a natural number, such that

$$\begin{aligned}(x^2,t^2) &\in B^{x^1}_{\rho/4^{m_0}} \times \left(t^1 - \frac{\rho^2}{16^{m_0}}, t^1\right),\\ (x^2,t^2) &\in B^{x^1}_{\rho/4^{(m_0+1)}} \times \left(t^1 - \frac{\rho^2}{16^{(m_0+1)}}, t^1\right).\end{aligned} \tag{8}$$

Hence it follows that

$$\rho\left[(x^1,t^1),(x^2,t^2)\right] \ge \frac{\rho^2}{16^{(m_0+1)}}. \tag{9}$$

Applying the lemma (allowing for remarks to it) to cylinders $B^{x^1}_{\rho/4^m} \times \left(t^1 - \frac{\rho^2}{16^m}, t^1\right)$ and $B^{x^1}_{\rho/4^{(m+1)}} \times \left(t^1 - \frac{\rho^2}{16^{(m+1)}}, t^1\right)$, $m = 0,1,\dots,m_0-1$ we have

$$\underset{B^{x^1}_{\rho/4^{m_0}} \times \left(t^1-\frac{\rho^2}{16^{m_0}},t^1\right)}{osc} u \le \theta^{m_0} \underset{B^{x^1}_\rho \times (t^1-\rho^2,t^1)}{osc} u. \tag{10}$$

Allowing that

$$\underset{B^{x^1}_\rho \times (t^1-\rho^2,t^1)}{osc} u \le 2\|u\|_{C_0(Q_T)}$$

from (8),(9) and (10) we conclude

$$\begin{aligned}\left|u(x^1,t^1)-u(x^2,t^2)\right| &\le \underset{B^{x^1}_{\rho/4^{m_0}} \times \left(t^1-\frac{\rho^2}{16^{m_0}},t^1\right)}{osc} u \le 2\theta^{m_0}\|u\|_{C_0(Q_T)} = \frac{2}{\theta}\theta^{m_0+1}\|u\|_{C_0(Q_T)} = \\ &= \frac{2}{\theta}\left(\frac{1}{16^{(m_0+1)}}\right)^{\frac{\ln\theta^{-1}}{\ln 16}}\|u\|_{C_0(Q_T)} = \frac{2}{\theta}\cdot\frac{1}{\rho^{2\ln\theta^{-1}/\ln 16}}\left[\frac{\rho^2 1}{16^{(m_0+1)}}\right]^{\frac{\ln\theta^{-1}}{\ln 16}} \times \\ &\times \|u\|_{C_0(Q_T)} \le \frac{2}{\theta\,\rho^{2\ln\theta^{-1}/\ln 16}}\left[\rho\left((x^1,t^1),(x^2,t^2)\right)\right]^{\frac{\ln\theta^{-1}}{\ln 16}} \times \\ &\times \|u\|_{C_0(Q_T)} = C_3\left[\rho\left((x^1,t^1),(x^2,t^2)\right)\right]^{\alpha_1}\|u\|_{C_0(Q_T)},\end{aligned} \tag{11}$$

where constant $\alpha_1 = \frac{\ln\theta^{-1}}{\ln 16}$ depends on θ, but C_3 - in addition and on ρ.

If $(x^2,t^2) \overline{\in} B^{x^1}_\rho \times (t^1-\rho^2,t^1)$, then

$$\rho\left((x^1,t^1),(x^2,t^2)\right) \ge \rho^2,$$

and therefore

$$\left|u\left(x^1,t^1\right)-u\left(x^2,t^2\right)\right|\le\frac{2}{\rho^{2\alpha_1}}\left[\rho\left(\left(x^1,t^1\right),\left(x^2,t^2\right)\right)\right]^{\alpha_1}\|u\|_{C_0(Q_T)}. \qquad (12)$$

Case 2. Notice that for $(x,t)\in\Omega_\rho\times\left(0,\frac{31}{32}T\right)$ equation (1) is elliptic, which the constant of ellipticity depends only on γ,T and the function φ. Therefore according to apriori estimate of Hölder's norm for elliptic equation of the second order in divergent structure [8] we conclude

$$\left|u\left(x^1,t^1\right)-u\left(x^2,t^2\right)\right|\le C_4\left[\rho\left(\left(x^1,t^1\right),\left(x^2,t^2\right)\right)\right]^{\alpha_2}\|u\|_{C_0(Q_T)}, \qquad (13)$$

where the constant $\alpha_2\in(0,1)$ depends only on γ,n,T and the function φ, but C_4 - in addition and on ρ.

Let now $\alpha_3=\min\{\alpha_1,\alpha_2\}$, $C_5=\max\left\{C_3,\frac{2}{\rho^{2\alpha_1}},C_4\right\}$. Then from (11)-(13) the required estimate (7) with $\alpha=\alpha_3$ and $C_2=C_5+1$ follows. The theorem is proved.

The author thanks her supervisor prof. I.T. Mamedov.

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Received April 9, 2001; Revised August 28, 2001.
Translated by author.

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ON THE COMPLEXITY CHARACTERISTICS OF SOME PROBLEMS

Abstract

The complexity characteristics of optimal algorithms theory for the problems being solved approximately are considered. The connections of the complexity category with other complexity characteristics, namely with entropy on assumption of the existence of algorithm of approximate solution of given precision are investigated. With application of the obtained result the estimations of entropy for some concrete tasks are obtained.

Investigation of the complexity issue of processes is of great interest in various fields of science, technology and economics. The study difficulties of many problems arising as example in projecting of modern technology, in planning on the different levels in controlling of the systems, in scientific basing of the evolutionary processes prediction and others are mainly of the same nature. Namely in all these processes it's required to establish the principle solvability of arising problems, and for algorithmically solvable problems it should be found out their physical realizability. In other words, it's necessary to estimate the minimal volumes of those and other resources required for realization of a solution. The absence of precise definition of the "complexity" notion didn't usually impede its recognition and investigation on empirical level. Starting with 50-s of the XX-century with increase of production scale and control levels, the necessity of investigation of appearing mathematical models arised. And consequently the mathematical models of applied problems appeared for statements and solutions of which the heuristic representations of complexity weren't sufficient. Hence, the necessity of introducing of the precise definition of "complexity" arised. A.N. Kolmogorov was the first who suggested the way to forming of complexity category, to investigation of its properties and perspectives of its applications [1]. Later on the other directions of investigations related to computational complexity, information complexity, energetic complexity, complexity of schemes and others appeared.

One of the basic applications of the "complexity" category is the theory of optimal algorithms for problems that are solved approximately. Here, by complexity of a problem is meant the true intrinsic difficulty of obtaining its solution not depending on a method of determination of solution. The key issue in such problems is the choice of the best algorithm for the solution of a problem. The choice of the best algorithm is a multicriterial optimization problem among which we can note: simplicity of program realization, time, the volume of occupied memory, stability and so on. The investigation that is necessary for characterization and construction of an optimal (in some sense) algorithm for the given concrete problem is a difficult mathematical task and requires the significant effort. Only in very rare cases we can find the exact value of mathematical complexity of the problem, more often one is able to estimate it. And besides it's natural and is dictated with the notion itself.

Now we introduce the necessary definitions and notions.

Let Y and Z be linear metric spaces with the metrics ρ_Y, ρ_Z respectively. By an information operator on Y we understand any (maybe nonlinear) operator $N : Y \to Z$, with

$$N(y) = (N_1(y), \dots, N_n(y)),\ y \in Y$$

The least non-negative number n, $0 \le n \le +\infty$ for which it will be found the linear subspace Γ from Z of dimension n containing $N(Y)$ (in particular it can be