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ON ONE GENERAL BOUNDARY VALUE PROBLEM WITH DISPLACEMENT FOR ONE SECOND ORDER HYPERBOLIC EQUATION

Abstract

In the present paper a general boundary value problem with shift for model hyperbolic type equation

$$U_{xx} - U_{yy} = 0 \tag{1}$$

is considered. The boundary value problem for the equation (1) is investigated in domain D*, when on* $\bar{J} = \{(x,y): y = 0, 0 \le x \le 1\}$ *the value of the function* $U(x,y)$ *is given, but in characteristics* AC *and* BC *- the condition that pointwisely connect the value* $U(x,y)$ *and its derivative in some direction.*

As a model of second order hyperbolic partial differential equation of two independent variables x, y consider the wave equation

$$U_{xx} - U_{yy} = 0. \tag{1}$$

Let D be a finite simply connected domain of plane of variables x, y bounded by characteristics $AC: x + y = 0$, $BC: x - y = 1$ of equation (1) and segment AB of the axis $y = 0; J \equiv AB$ is unit interval $0 < x < 1$. As regular solution of equation (1) in domain D we'll understand a function $U(x,y) \in C(\bar{D}) \cap C^2(D)$ satisfying equation (1) in D.

Consider the following boundary-value problem for equation (1) in domain D, when the value of function $U(x,y)$ is given on J, and on $AC \cup BC$ the condition pointwise connecting the values of $U(x,y)$ and its derivatives in some direction.

Problem C. Find a regular in domain D solution $U(x,y)$ of equation (1), satisfying the boundary conditions

$$U(x,0) = \tau(x),\ \forall x \in \bar{J}, \tag{2}$$

$$\alpha(x)U\left(\frac{x}{2}, -\frac{x}{2}\right) + \alpha_1(x)\frac{d}{dx}U\left(\frac{x}{2}, -\frac{x}{2}\right) + \beta(x)U\left(\frac{x+1}{2}, \frac{x-1}{2}\right) + \\ + \beta_1(x)\frac{d}{dx}U\left(\frac{x+1}{2}, \frac{x-1}{2}\right) = \delta(x),\ \forall x \in \bar{J}, \tag{3}$$

where $\tau(x)$, $\alpha(x), \beta(x), \alpha_1(x), \beta_1(x), \delta(x)$ are given sufficiently smooth functions.

Note that the considered problem is the generalization of two problems from paper [2] and analogous problem from [3]. We'll go over to investigation of problem C.

Regular in domain D general solution $U(x,y)$ of equation (1) satisfying boundary condition (2) can be represented in the following form

$$U(x,y) = \frac{1}{2}\left[\tau(x+y) + \tau(x-y)\right] + \frac{1}{2}\int_{x-y}^{x+y} \nu(t)dt, \tag{4}$$

where $\nu(x) \in C^1(J)$ is an arbitrary function.

Taking into account (4) in boundary condition (3) we'll obtain

$$[\alpha(x) + \beta(x)]\tau(x) + [\alpha_1(x) + \beta_1(x)]\tau'(x) - [\alpha_1(x) - \beta_1(x)]\nu(x) -$$

$$-\alpha(x)\int_0^x v(t)dt-\beta(x)\int_x^1 v(t)dt=2\delta(x)-\alpha(x)\tau(0)-\beta(x)\tau(1). \qquad (5)$$

Denoting by $v_1(x)=\int_0^x v(t)dt$ the primitive $v(x)$ we'll write equation (5) in the following form:

$$\begin{gathered}[\alpha_1(x)-\beta_1(x)]v_1'(x)+[\alpha(x)-\beta(x)]v_1(x)+\beta(x)v_1(1)= \\ =[\alpha(x)+\beta(x)]\tau(x)+[\alpha_1(x)+\beta_1(x)]\tau'(x)+\alpha(x)\tau(0)+\beta(x)\tau(1)-2\delta(x),\end{gathered} \qquad (6)$$

$$v_1(0)=0. \qquad (7)$$

By virtue of (4) solvability of the problem C is equivalent to solvability of Cauchy's problem (6), (7) for loaded differential equation (6) with respect to the unknown function $v_1(x)$. Substituting the value $v_1'(x)=v(x)$ obtained from (6), (7) in (4) we'll define solution of the problem C.

We'll investigate solvability of problem (6), (7) for different cases of representation of coefficients of equation (6).

The first case. Let conditions (a) be satisfied: $\alpha_1(x)\neq\beta_1(x)$, $\forall x\in\bar{J}, \alpha(x), \beta(x)$, $\delta(x), \tau(x)$ are given functions, where

$$\tau(x)\in C(\bar{J})\cap C^2(J),\ \alpha(x), \beta(x), \delta(x)\in C(\bar{J})\cap C^1(J),$$
$$\alpha_1(x), \beta_1(x)\in C^1(\bar{J}).$$

We'll rewrite equation (6) in the following form

$$[\alpha_1(x)-\beta_1(x)]v_1'(x)+[\alpha(x)-\beta(x)]v_1(x)+\beta(x)v_1(1)=f(x), \qquad (8)$$

where

$$f(x)=[\alpha(x)+\beta(x)]\tau(x)+[\alpha_1(x)+\beta_1(x)]\tau'(x)+\alpha(x)\tau(0)+\beta(x)\tau(1)-2\delta(x).$$

By virtue of conditions (a) equation (8) can be written in the following form:

$$v_1'(x)+p(x)v_1(x)=q(x), \qquad (9)$$

where

$$q(x)=\frac{\alpha(x)+\beta(x)}{\alpha_1(x)-\beta_1(x)}\tau(x)+\frac{\alpha_1(x)+\beta_1(x)}{\alpha_1(x)-\beta_1(x)}\tau'(x)+\frac{\alpha(x)\tau(0)}{\alpha_1(x)-\beta_1(x)}+$$
$$+\frac{\beta(x)\tau(1)}{\alpha_1(x)-\beta_1(x)}-\frac{2\delta(x)}{\alpha_1(x)-\beta_1(x)}-\frac{\beta(x)v_1(1)}{\alpha_1(x)-\beta_1(x)},\ p(x)=\frac{\alpha(x)-\beta(x)}{\alpha_1(x)-\beta_1(x)}.$$

Assuming that right-hand side $q(x)$ of equation (9) is known, taking into account initial condition (7), the general solution of equation (9) (see [4]) after some transformations is obtained in the following form

$$v_1(x)=\frac{\alpha_1(x)+\beta_1(x)}{\alpha_1(x)-\beta_1(x)}\tau'(x)-\frac{\alpha_1(0)+\beta_1(0)}{\alpha_1(0)-\beta_1(0)}\tau(0)\vartheta^{-1}(x)+$$
$$+2\vartheta^{-1}(x)\int_0^x\frac{\alpha_1(t)[\beta(t)-\beta_1'(t)]-\beta_1(t)[\alpha(t)-\alpha_1'(t)]}{[\alpha_1(t)-\beta_1(t)]^2}\tau(t)\vartheta(t)dt+$$
$$+\tau(0)\vartheta^{-1}(x)\int_0^x\frac{\alpha(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt+\tau(1)\vartheta^{-1}(x)\int_0^x\frac{\beta(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt-$$
$$+2\vartheta^{-1}(x)\int_0^x\frac{\delta(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt+v_1(1)\vartheta^{-1}(x)\int_0^x\frac{\beta(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt, \qquad (10)$$

where

$$\vartheta(x)=\exp\left[\int_0^x\frac{\alpha(t)-\beta(t)}{\alpha_1(t)-\beta_1(t)}dt\right],\ \vartheta^{-1}(x)=\exp\left[-\int_0^x\frac{\alpha(t)-\beta(t)}{\alpha_1(t)-\beta_1(t)}dt\right].$$

From formula (10) at $x=1$ we'll obtain

$$\nu_1(1)\cdot\left[1+\vartheta^{-1}(1)\int_0^1\frac{\beta(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt\right]=\frac{\alpha_1(1)+\beta_1(t)}{\alpha_1(t)-\beta_1(t)}\tau(1)-$$
$$-\frac{\alpha_1(0)+\beta_1(0)}{\alpha_1(0)-\beta_1(0)}\tau(0)\vartheta^{-1}(1)+\tau(0)\vartheta^{-1}(1)\int_0^1\frac{\alpha(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt+$$
$$+\tau(1)\vartheta^{-1}(1)\int_0^1\frac{\beta(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt-2\vartheta^{-1}(1)\int_0^1\frac{\delta(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt+\qquad(11)$$
$$+2\vartheta^{-1}(1)\int_0^1\frac{\alpha_1(t)[\beta(t)-\beta_1'(t)]-\beta_1(t)[\alpha(t)-\alpha_1'(t)]}{[\alpha_1(t)-\beta_1(t)]^2}\tau(t)\vartheta(t)dt.$$

The expression in square brackets in the left-hand side of relation (11) we'll denote by C_1 and right-hand side – by C_2. Then at $C_1\neq 0$ from (11) we'll uniquely define $\nu_1(1)=\frac{C_2}{C_1}$. If simultaneously $C_1=0$ and $C_2=0$ then relation (11) is satisfied for any finite $\nu_1(1)$. By fulfilling conditions $C_1=0, C_2\neq 0$, equation (11) isn't solvable. Consequently, in the first case solvability of the problem C depends on solvability of equation (11) with respect to $\nu_1(1)$. From (10) the unknown function $\nu(x)$ is defined in the form

$$\nu(x)=\frac{\alpha_1(x)+\beta_1(x)}{\alpha_1(x)-\beta_1(x)}\tau'(x)-2\frac{\alpha(x)\beta_1(x)-\beta(x)\alpha_1(x)}{[\alpha_1(x)-\beta_1(x)]^2}\tau(x)+$$
$$+\frac{\alpha(x)\tau(0)}{\alpha_1(x)-\beta_1(x)}+\frac{\beta(x)\tau(1)}{\alpha_1(x)-\beta_1(x)}-\frac{\beta(x)\nu_1(1)}{\alpha_1(x)-\beta_1(x)}-\frac{2\delta(x)}{\alpha_1(x)-\beta_1(x)}+$$
$$+\frac{\alpha_1(0)+\beta_1(0)}{\alpha_1(0)-\beta_1(0)}\tau(0)p(x)\vartheta^{-1}(x)-\tau(0)p(x)\vartheta^{-1}(x)\int_0^x\frac{\alpha(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt-$$
$$-\tau(1)p(x)\vartheta^{-1}(x)\int_0^x\frac{\beta(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt+\nu_1(1)p(x)\vartheta^{-1}(x)\int_0^x\frac{\beta(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt+$$
$$+2p(x)\vartheta^{-1}(x)\int_0^x\frac{\delta(t)\vartheta(t)}{\alpha_1(t)-\beta_1(t)}dt-$$
$$-2p(x)\vartheta^{-1}(x)\int_0^x\frac{\alpha_1(t)[\beta(t)-\beta'_1(t)]-\beta_1(t)[\alpha(t)-\alpha'_1(t)]}{[\alpha_1(t)-\beta_1(t)]^2}\tau(t)\vartheta(t)dt,\qquad(12)$$

where $\nu_1(1)$ is solution of equation (11), $p(x)=\frac{\alpha(x)-\beta(x)}{\alpha_1(x)-\beta_1(x)}$. Note that in particular if condition $\frac{\beta(t)}{\alpha_1(t)-\beta_1(t)}\geq 0$ is satisfied then $C_1>0$ and $\nu_1(1)$ is uniquely defined from (11).

So substituting the obtained value for $\nu(x)$ from (12) into formula (4) we'll find solution of the problem C. From the stated above follows

Theorem 1. *Let conditions (a) be satisfied. Then:*

1) *if* $C_1 \neq 0$ *then problem C is uniquely solvable and solution is represented by formula (4) where* $v(x)$ *is defined by formula (12), and* $v_1(1)=\dfrac{C_2}{C_1}$ *is solution of equation (11);*
2) *if simultaneously* $C_1=0, C_2=0$ *then problem C is solvable and solution is represented by formula (4), where* $v(x)$ *is defined by formula (12) and* $v_1(1)$ *is an arbitrary constant;*
3) *if conditions* $C_1=0, C_2\neq 0$ *are satisfied then problem C isn't solvable.*

Second case. Let conditions (b) be satisfied: $\alpha_1(x)=\beta_1(x), \alpha(x)\neq\beta(x), \forall x\in\bar{J}$, $\delta(x),\tau(x)$ are given functions where $\alpha(x),\beta(x),\alpha_1(x),\beta_1(x),\delta(x)\in C^1(\bar{J})\cap C^2(J)$ and $\tau(x)\in C^1(\bar{J})\cap C^3(J)$ at $\alpha_1(x)=\beta_1(x)\neq 0, \tau(x)\in C^1(\bar{J})\cap C^2(J)$, at $\alpha_1(x)=\beta_1(x)\equiv 0$.

In this case equation (6) takes the form

$$[\alpha(x)-\beta(x)]v_1(x)+\beta(x)v_1(1)=g(x), \tag{13}$$

where $g(x)=[\alpha(x)+\beta(x)]\tau(x)+2\alpha_1(x)\tau'(x)+\alpha(x)\tau(0)+\beta(x)\tau(1)-2\delta(x)$.

Taking into account condition (7), from (13) we'll obtain the following relation

$$\beta(0)v_1(1)=g(0). \tag{14}$$

It's obvious that at $\beta(0)\neq 0$ from (14) $v_1(1)=\dfrac{g(0)}{\beta(0)}$ is uniquely defined. Substituting this value in (13) we find

$$v_1(x)=[g(x)-\beta(x)v_1(1)][\alpha(x)-\beta(x)]^{-1} \tag{15}$$

and in turn $v(x)=v'_1(x)$. If conditions $\beta(0)=0, g(0)=0$ are satisfied simultaneously then relation (14) is valid for any finite $v_1(1)$. Fulfilling conditions $\beta(0)=0, g(0)\neq 0$ equation (14) isn't solvable.

Consequently, in the second case solvability of the problem C depends on solvability of equation (14) with respect to $v_1(1)$.

From (13) at $x=1$ we also obtain

$$\alpha(1)v_1(1)=g(1). \tag{16}$$

Relative to the equation (16) the following statements hold:

1) at $\alpha(1)\neq 0$ is uniquely define $v_1(1)=\dfrac{g(1)}{\alpha(1)}$;
2) at $\alpha(1)=0,\ g(1)=0$ equation (16) is solvable for any $v_1(1)$;
3) at $\alpha(1)=0,\ g(1)\neq 0$ equation (16) isn't solvable for any $v_1(1)$;

From above stated follows

Theorem 2. *Let conditions (b) be satisfied. Then:*

1) *if conditions* $\beta(0)\neq 0,\ \alpha(1)g(0)=\beta(0)g(1)$ *or* $\alpha(1)\neq 0,\ \alpha(1)g(0)=\beta(0)g(1)$ *are fulfilled, then the problem C is uniquely solvable and solution is represented by the formula (4), where* $v(x)=v'_1(x)$ *is defined by the formula (15), and* $v_1(1)$ *is a unique solution of one of the equations (14) or (16);*
2) *if conditions* $\alpha(1)=\beta(0)=g(0)=g(1)=0$ *are fulfilled simultaneously, then problem C is solvable and solution is represented by the formula (4), where* $v(x)=v'_1(x)$ *is defined from expression (15), and* $v_1(1)$ *is an arbitrary constant;*

3) *if conditions* $\beta(0)=0, g(0)\neq 0$ *or* $\alpha(1)=0, g(1)\neq 0$ *are satisfied then problem C isn't solvable.*

Third case. Let conditions (c) be satisfied: $\alpha_1(x)=\beta_1(x)\neq 0$, $\alpha(x)=\beta(x), \forall x\in \bar{J}$, $\delta(x), \tau(x)$ are given functions, where

$$\tau(x)\in C(\bar{J})\cap C^2(J),\ \alpha(x),\alpha_1(x),\delta(x)\in C(\bar{J})\cap C^1(J).$$

In this case equation (6) takes the form

$$\tau'(x)+\frac{\alpha(x)}{\alpha_1(x)}\tau(x)=\frac{\alpha(x)}{2\alpha_1(x)}\left[\nu_1(1)-\tau(0)-\tau(1)\right]+\frac{\delta(x)}{\alpha_1(x)}. \tag{17}$$

From (17) for $\tau(x)$ we'll obtain the following representation

$$\tau(x)=\tau(0)w^{-1}(x)+\frac{1}{2}\left[1-w^{-1}(x)\right]\left[c_1-\tau(0)-\tau(1)\right]+w^{-1}(x)\int_0^x\frac{\delta(t)}{\alpha_1(t)}w(t)dt, \tag{18}$$

where $c_1=const$, $w(x)=\exp\left[\int_0^x\frac{\alpha(t)}{\alpha_1(t)}dt\right]$, $w^{-1}(x)=\exp\left[-\int_0^x\frac{\alpha(t)}{\alpha_1(t)}dt\right]$.

The following theorem holds

Theorem 3. *Let conditions (c) be satisfied. Then if* $\tau(x)$ *represented in the form of (18) and equality* $c_1=\nu_1(1)$ *holds then problem C has infinitely many solutions, depending on arbitrary function* $\nu(x)\in C^1(J)$ *and solutions are represented by the formula (4).*

Fourth case. Let conditions (d) be satisfied:
$\alpha_1(x)=\beta_1(x)=0, \alpha(x)=\beta(x)\neq 0, \forall x\in\bar{J}, \delta(x), \tau(x)$ are given functions, where $\tau(x), \alpha(x), \delta(x)\in\ \in C(\bar{J})\cap C^2(J)$. Then equation (6) takes the form

$$\tau(x)=\frac{1}{2}\nu_1(1)-\frac{1}{2}\tau(0)-\frac{1}{2}\tau(1)+\frac{\delta(x)}{\alpha(x)}. \tag{19}$$

On the other hand, by force of condition (3) and properties of characteristic quadrangular we have

$$\tau(x)=\frac{\delta(x)}{\alpha(x)}+\tau(0)-\frac{\delta(0)}{\alpha(0)}. \tag{20}$$

From (19), (20) we obtain relation

$$\nu_1(1)=3\tau(0)+\tau(1)-\frac{2\delta(0)}{\alpha(0)}. \tag{21}$$

Consequently, the following theorem holds

Theorem 4. *Let the conditions (d) be satisfied. Then if the conditions (20), (21) are fulfilled then problem C has infinitely many solutions, depending on arbitrary function* $\nu(x)\in C^1(J)$ *and solutions are represented by the formula (4).*

If the trivial case holds

$$\alpha(x)=\beta(x)=\alpha_1(x)=\beta_1(x)=\delta(x)\equiv 0, \forall x\in\bar{J},$$

then it's obvious, that problem C has uncountable set of solutions depending on arbitrary function $\nu(x)\in C^1(J)$ and solutions are represented by the formula (4).

In conclusion we note that the considered problem C can be used at statement and solvability of different boundary-value problems for mixed equations (see for example [1], [2], [3]).

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THE APRIORI ESTIMATE OF HÖLDER'S NORM OF SOLUTIONS OF DEGENERATE ELLIPTICO-PARABOLIC EQUATIONS OF THE SECOND ORDER

Abstract

In the article a class of degenerate elliptico-parabolic equations of the second order in divergent form is considered. For weak solutions of these eqautions the apriori estimate of Hölder's norm is proved.

Introduction. Let R_n be an n - dimensional Euclidean space of the points $x=(x_1,...,x_n)$, $\Omega\subset R_n$ be a bounded n - dimensional domain with the boundary $\partial\Omega$, $B_R^{x^0}$ be an n - dimensional open ball of the radius $R<1$ with center at the point x^0 such that $\overline{B}_R^{x^0}\subset\Omega$, $B_R^0=B_R$, $Q_T=\{(x,t): x\in\Omega, 0<t<T<\infty\}$, $S_T=\{(x,t): x\in\partial\Omega, 0\le t\le T\}$, $\Gamma(Q_T)$ be a parabolic boundary of Q_T, i.e. $\Gamma(Q_T)=S_T\cup\{(x,t): x\in\Omega, t=0\}$, $Q_R^{x^0}=B_R^{x^0}\times(0,T)$, $Q_1^{x^0}=B_{R'}^{x^0}\times\left(\frac{7T}{8},T\right)$, $Q_2^{x^0}=B_{R'}^{x^0}\times\left(\frac{T}{4},\frac{7T}{24}\right)$, $Q_R^0=Q_R$, $Q_1^0=Q_1$, $Q_2^0=Q_2$, where $R'=\frac{R}{4}$.

Consider the following equation in Q_T

$$Lu=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)+\frac{\partial}{\partial t}\left(\varphi(T-t)\frac{\partial u}{\partial t}\right)-\frac{\partial u}{\partial t}=0\,, \qquad (1)$$

with supposition that $\|a_{ij}(x,t)\|$ is a real symmetrical matrix with measurable in Q_T elements, where for all $(x,t)\in Q_T$ and arbitrary n - dimensional vector ξ

$$\gamma|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi\le\gamma^{-1}|\xi|^2;\ \gamma\in(0,1]-const\,. \qquad (2)$$

Besides with respect to the function $\varphi(z)$ for $z>0$ the condition

$$\varphi(0)=0,\varphi(z)>0,\ \varphi'(z)>0,\ \varphi''(z)>0,\ \varphi'(z)\ge\varphi(z)\varphi''(z),\ \frac{\varphi'(z)}{\sqrt{\varphi(z)}}\ge\beta \qquad (3)$$

be satisfied, where β is a positive constant.

The aim of the present article is the proof of the interior apriori estimate of Hölder's norm for weak solutions of equations (1). Note that for the second order parabolic equations in divergent form the corresponding result is derived in [1-2]. We indicate monograph [3] in which the the Hölder continuity of solutions of the second order quasilinear parabolic equations was proved. As to the second order parabolic equations in divergent structure we mention in this connection papers [4-6]. Note that in proof of apriori estimate of Hölder's norm is the analogue of the classical Harnack's inequality for non-negative solutions of equations (1) established in [7] is a basis instrument.