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ON SOME PROPERTIES OF BOUNDARY VALUE PROBLEMS FOR OPERATOR-DIFFERENTIAL EQUATIONS ON SEMI-AXIS

Abstract

In the paper the theorems on interior compactness of spaces of solutions of homogeneous equation of a class of the operator-differential equations of elliptic type and on minimally of elementary solutions are obtained.

On a separable Hilbert space H consider a polynomial operator bundle of order $n=2k$

$$P(\lambda)=(-1)^k \lambda^n E+\lambda^{n-1} A_1+\ldots+\lambda A_n+A^n, \tag{1}$$

where

1) A is a normal operator with complete inverse A^{-1}, i.e. $A^{-1}\in\sigma_\infty(H)$, whose spectrum is contained in the angle sector

$$S_\varepsilon=\left\{\lambda \middle| \ |\arg\lambda|\le\varepsilon,\ 0\le\varepsilon<\frac{\pi}{n}\right\};$$

2) the operators $B_j=A_j\cdot A^{-j}$ $(j=1,\ldots,n)$ are bounded in H, i.e. $B_j\in L(H)$;

3) the operator $E+B_n$ is inverteble in the space H.

By fulfilling the condition 1)-3) the operator bundle $P(\lambda)$ has a discrete specter with a unique limit point in infinity (see, foe example [1]).

Let $\{e_n\}$ be an orthonormal system of eigen-vectors of the operator A, i.e. $Ae_n=\lambda e_n$. Then by determination, for all real γ we have:

$$A^\gamma=\sum\lambda_n^\gamma(\cdot,e_n)e_n,$$

where $\lambda_n^\gamma=|\lambda_n|^\gamma e^{i\gamma\arg\lambda_n}$, when $-\varepsilon\le\arg\lambda_n\le\varepsilon$. For the operator A^γ we suppose

$$D(A^\gamma)=\left\{\varphi\in H:\sum_{n=1}^{\infty}|\lambda_n|^{2\gamma}|(\varphi,e_n)|^2<\infty\right\}$$

and

$$H_\gamma=D(A^\gamma),\ (x,y)_{H_\gamma}=(A^\gamma x,A^\gamma y).$$

Let's denote by $L_2(R_+,H)$ a Hilbert space of vector-functions $f(t)$ with the values from H, measurable α and for which

$$\|f\|^2_{L_2(R_+;H)}=\int_0^\infty\|f(t)\|_H^2\,dt<\infty$$

is finite and let's define the spaces

$$W_2^n(R_+;H)=\left\{u(t):u^{(n)}(t)\in L_2(R_+;H),\ A^n u\in L_2(R_+;H)\right\}$$

and

$$\overset{\circ}{W}{}_2^n(R_+;H)=\left\{u:u\in W_2^n(R_+,H),\ u^{(i)}(0)=0,\ i=1,\ldots,n-1\right\}$$

with the norm

$$\|u\|^2_{W_2^n(R_+;H)}=\left\|u^{(n)}\right\|^2_{L_2(R_+;H)}+\left\|A^n u\right\|^2_{L_2(R_+;H)}.$$

The space $W_2^n(a;b;H)$ when $0<a<b<\infty$ is defined similarly.

Now and further the derivatives are understood in the sense of distribution theory (see [2]).

Let's connect the bundle (1) with a boundary-value problem

$$P(d/dt)u(t)=0\,, \tag{2}$$

$$u^{(\nu)}=\varphi_\nu\ (\nu=0,1,\dots,k-1),\ \left(k=\frac{n}{2}\right). \tag{3}$$

From the results of paper [3] yields

Theorem 1. *Let conditions 1) and 2) be fulfilled and it holds the inequality*

$$a(\varepsilon)=\sum_{j=1}^{n}c_{n-j}(\varepsilon)\|B_j\|<1\,,$$

where

$$C_0(\varepsilon)=\begin{cases}1, & \text{for } 0\le\varepsilon<\dfrac{\pi}{2n}\\ \left(\sqrt{2}\cos k\varepsilon\right)^{-1}, & \text{for } \dfrac{\pi}{2n}\le\varepsilon<\dfrac{\pi}{n},\end{cases}$$

for $j=1,2,\dots,k$

$$c_j(\varepsilon)=\begin{cases}(2\cos k\varepsilon)^{-\frac{j}{k}}, & \text{for } 0\le\varepsilon<\dfrac{\pi}{2n}\\ \left(2^{\frac{j+k}{n}}\cos k\varepsilon\right)^{-1}, & \text{for } \dfrac{\pi}{2n}\le\varepsilon<\dfrac{\pi}{n},\end{cases}$$

and for $j=k+1,\dots,n-1$

$$c_j(\varepsilon)=\begin{cases}2^{\frac{(n-j)(k(j-k)-2)}{n}}(\cos k\varepsilon)^{-\frac{j-n}{k}}, & \text{for } 0\le\varepsilon<\dfrac{\pi}{2n},\\ 2^{\frac{(n-j)(k(j-k)-1)}{n}}(\cos k\varepsilon)^{-1}, & \text{for } \dfrac{\pi}{2n}\le\varepsilon<\dfrac{\pi}{n}.\end{cases}$$

Then the problem (2)-(3) has a unique regular solution $u(t)\in W_2^n(R_+;H)$ at any set of vector $\varphi_n\in H_{n-\nu-\frac{1}{2}}$ $(\nu=0,1,\dots,k-1)$, moreover the inequality

$$\|u\|_{W_2^n(R_+;H)}\le const\sum_{\nu=0}^{k-1}\|\varphi_\nu\|_{H_{n-\nu-\frac{1}{2}}} \tag{4}$$

holds.

Under regular solution of the problem (1)-(2) we understand a function from the space $W_2^n(R_+;H)$, which satisfies the equation (2) almost everywhere and boundary conditions is understood in the sense of convergence of the norm $H_{n-\nu-\frac{1}{2}}$:

$$\lim_{t\to0}\left\|u^{(\nu)}(t)-\varphi_\nu\right\|_{H_{n-\nu-\frac{1}{2}}}=0\,.$$

In paper [4] P.D. Lax gives the definition of interior compactness for a space of vector-functions defined on some infinite interval, moreover this theory is constructed in connection with Phragmen-Lindelöf principle for the solutions of elliptic equations.

Let $S \subset L(R_+;H)$ be some space of vector-functions invariant with respect to shift, i.e. $u(t) \in S$, then for any $\eta > 0$ vector-function $u(t+\eta)$ also belongs to the space S.

Further, if the vector-function $u(t) \in S$, then for any $(a,b) \subseteq R_+$ we introduce the denotation

$$\|u\|^2_{(a;b)} = \int_a^b \|u(t)\|^2 dt .$$

Definition 1. *The space S is called the interior compact if the set $Q_M = \{u : u \in S, \|u\|_{(a,b)} \le M\}$ is precompact in the norm $\|u\|_{(a',b')}$ for any $M > 0$ and (a',b') such that $a < a' < b' < b$.*

Let's consider a space of vector-function $W_2^{n-1}(a,b;H)$ in which the norm is represented in the following form

$$\|u\|_{W_2^{n-1}(a,b;H)} = \left(\int_a^b \left(\left\|u^{(n-1)}\right\|_H^2 + \left\|A^{n-1}u\right\|^2 \right) dt \right)^{\frac{1}{2}} .$$

It is clear that for any $0 \le a < b \le \infty$, $W_2^n(a,b;H) \subset W_2^{n-1}(a,b;H)$ as a compact set.

Let $\ker(P,R_+)$ be a set of solutions of the equations (1) from the space $W_2^n(R_+;H)$. It is obvious that $\ker(P,R_+)$ is a linear close subspace $W_2^n(R_+;H)$ and $\ker(P,R_+) \subset W_2^{n-1}(R_+;H)$. Let's denote by $L(P,R_+)$ the closure $\ker(P,R_+)$ by norm of the space $W_2^{n-1}(R_+;H)$.

In this paper the properties of interior compactness of the space $\ker(P,R_+)$constructed by the norm $\|u\|_{W_2^{n-1}(R_+;H)}$ which is weaker than the norm of the space $W_2^n(R_+;H)$ is established. Further using some results of paper [4] we get the theorem on minimality of elementary decreasing solutions and miltiminimality of root vectors responding to the eigen-values from the left half-plane.

Let's note, that similar questions are investigated, for example, in paper [5-10] and others.

The following theorem is proved.

Theorem 2. *Let the conditions 1)-3) be fulfilled and for all the vector-fucntions $u(t) \in \overset{\circ}{W}{}_2^n(R_+;H)$ the inequality*

$$\left\| P\left(\frac{d}{dt}\right)u(t) \right\|_{L_2(R_+;H)} \ge const \|u(t)\|_{W_2^n(R_+;H)} \quad (5)$$

holds.

Then the space $L(P,R_+)$ is interior compact. There exists the number $\mu_0 > 0$, that for any $u(t) \in L(P,R_+)$ the estimations

$$\int_0^\infty e^{2\mu_0 t} \left(\left\|u^{(n-1)}\right\|_H^2 + \left\|A^{n-1}u\right\|^2 \right) dt < \infty \quad (6)$$

are fulfilled.

Proof. Let $u(t)\in \ker(P,R_+)$. Then it is evident that vector-function $v(t)= =\varphi(t)u(t)\mathring{W}_2^n(R_+;H)$, where $\varphi(t)$ is a scalar function from space $C_\infty(a,b)$ $(0\le a<b<\infty)$, such that $\varphi(t)=1$, when $t\in(a',b')$ $(a<a'<b'<b)$. (Here a, a', b' and b are any numbers satisfying the indicated conditions). Then for vector-functions $v(t)$ the condition (5) is fulfilled, i.e.

$$\left\|P\left(\frac{d}{dt}\right)v(t)\right\|_{L_2(R_+;H)}\ge const\|v(t)\|_{W_2^n(R_+;H)}.$$

Hence we have

$$\left\|P\left(\frac{d}{dt}\right)v(t)\right\|_{L_2(R_+;H)}=\left\|P\left(\frac{d}{dt}\right)\varphi u\right\|_{L_2(R_+;H)}\ge const\|\varphi u\|_{W_2^n(R_+;H)}=$$

$$=const\left(\int_a^b\left(\left\|(\varphi u)^{(n)}\right\|_H^2+\left\|\left(A^n\varphi u\right)\right\|_H^2\right)dt\right)^{\frac{1}{2}}\ge const\left(\int_{a'}^{b'}\left(\left\|(\varphi u)^{(n)}\right\|_H^2+\left\|\left(A^n\varphi u\right)\right\|_H^2\right)dt\right)^{\frac{1}{2}}=$$

$$const\left(\int_{a'}^{b'}\left(\left\|u^{(n)}\right\|_H^2+\left\|\left(A^n u\right)\right\|_H^2\right)dt\right)^{\frac{1}{2}}=const\|u\|_{W_2^n(a';b';H)}. \qquad (7)$$

On the other hand we have:

$$P\left(\frac{d}{dt}\right)v(t)=(-1)^k\frac{d^n(\varphi u)}{dt^n}+A^n(\varphi u)+\sum_{j=1}^n A_j\frac{d^{n-j}(\varphi u)}{dt^{n-j}}=$$

$$=\left[(-1)^k\frac{d^n(u)}{dt^n}+A^n(u)+\sum_{j=1}^n A_j u^{(n-j)}\right]\varphi(t)+\sum_{q=1}^n(-1)^k\binom{q}{n}\varphi^{(q)}u^{(n-q)}+$$

$$+\sum_{j=1}^{n-1}A_j\sum_{p=1}^{n-j}\binom{p}{n-j}\varphi^{(p)}u^{(n-j-p)}=\varphi(t)P\left(\frac{d}{dt}\right)u(t)+\sum_{q=0}^{n-2}(-1)^k\binom{n}{q+1}\varphi^{(q+1)}u^{(n-1-q)}+$$

$$+\sum_{j=1}^{n-1}A_j\sum_{p=1}^{n-j-1}\binom{p+1}{n-j}\varphi^{(p+1)}u^{((n-1)-(j+p))}. \qquad (8)$$

As $u(t)\in\ker(P,R_+)$ then $\varphi(t)P\left(\frac{d}{dt}\right)u(t)=0$.

Let's estimate other two summands from above

$$\left\|\sum_{q=0}^{n-1}(-1)^k\binom{q+1}{n}\varphi^{(q+1)}u^{(n-1-q)}\right\|_{L_2(R_+;H)}=\left\|\sum_{q=0}^{n-1}(-1)^k\binom{q+1}{n}\varphi^{(q+1)}u^{(n-1-q)}\right\|_{L_2(a;b;H)}\le$$

$$\le\sum_{q=0}^{n-1}\binom{q+1}{n}\max_{t\in(a;b)}\left|\varphi^{(q+1)}(t)\right|\cdot\left\|A^{-q}\left(A^q u^{(n-1)-q}\right)\right\|_{L_2(a;b;H)}\le$$

$$\le const\sum_{q=0}^{n-1}\binom{q+1}{n}\cdot\left\|A^{-q}\right\|\left\|A^q u^{((n-1)-q)}\right\|_{L_2(a;b;H)}.$$

In the last equality using the theorem on intermediate derivatives we have:

$$\left\|\sum_{q=0}^{n-1}(-1)^k\binom{q+1}{n}\varphi^{(q+1)}u^{(n-1-q)}\right\|\le const\cdot\|u\|_{W_2^{n-1}(a;b;H)}. \qquad (9)$$

Similarly we have:

$$\left\|\sum_{j=1}^{n-1} A_j\left(\sum_{p=0}^{n-j-1}\binom{p+1}{n-j}\right)\varphi^{(p+1)}u^{((n-1)-(p+j))}\right\|_{L_2(R_+;H)} \le$$

$$\le\left\|\sum_{j=1}^{n-1} A_j A^{-j}\left(\sum_{p=0}^{n-j-1}\binom{p+1}{n-j}\right)\varphi^{(p+1)}A^j u^{((n-1)-(p+j))}\right\|_{L_2(R_+;H)} \le$$

$$\le\sum_{j=1}^{n-1}\left\|A_j A^{-j}\right\|\left(\sum_{p=0}^{n-j-1}\binom{p+1}{n-j}\right)\cdot\max_{t\in(a,b)}\left|\varphi^{(p+1)}\right|\cdot\left\|A^j u^{((n-1)-(p+j))}\right\|_{L_2(a;b;H)} \le$$

$$\le\sum_{j=1}^{n-1}\left\|A_j A^{-j}\right\|\sum_{p=0}^{n-j-1}\binom{p+1}{n-j}\cdot c_{p+1}\cdot\left\|A^{-p}A^{j+p}u^{((n-1)-(p+j))}\right\|_{L_2(a;b;H)}.$$

Since by theorem on intermediate derivatives [2, p.29]

$$\left\|A^{-p}A^{j+p}u^{(n-1)-(p+j)}\right\|_{L_2(a;b;H)} \le\left\|A^{-p}\right\|\cdot\left\|A^{j+p}u^{((n-1)-(p+j))}\right\|_{L_2(a;b;H)} \le const\|u\|_{W_2^{n-1}(a;b;H)},$$

$$(p=0,1,\dots,n-j-1,\ j=0,1,\dots,n-2),$$

then we get, that

$$\left\|\sum_{j=0}^{n-2} A_j\left(\sum_{p=0}^{n-j-1}\binom{n-j}{p+1}\right)\varphi^{(p-1)}u^{((n-1)-(p+j))}\right\|_{L_2(R_+;H)} \le const\|u\|_{W_2^{n-1}(a;b;H)}. \qquad (10)$$

Taking into account the equality (8) and inequalities (9) and (10) in the inequality (7) we have:

$$\|u\|_{W_2^n(a';b';H)} \le const\|u\|_{W_2^{n-1}(a;b;H)}. \qquad (11)$$

Now we'll prove the interior compactness of the space $L(P,R_+)$.

For this assume, that M is any number more than zero and denote it by

$$Q_M=\left\{u: u\in L(P,R_+),\ \|u\|_{W_2^{n-1}(a;b;H)}\le M\right\}.$$

We must prove, that Q_M is a compact set in the norm of the space $W_2^{n-1}(a';b',H)$, where $0\le a<a'<b'<b<\infty$ are any numbers. Denote by $Q'_M=Q_M\cap\ker(P,R_+)$. Let's show that Q'_M is compact with respect to the norm $\|u\|_{W_2^{n-1}(a';b')}$. As for $u\in\ker(P,R_+)$ the inequality (11) holds, then the set Q'_M is bounded by the norm $\|u\|_{W_2^{n-1}(a';b',H)}$, i.e. it is a bounded set in the space $W_2^n(a';b',H)$. From the complete continuity of the operator A^{-1} it follows, that the space $W_2^n(a';b',H)$ is embedded to the space $W_2^{n-1}(a';b',H)$ compactly [11, p.79]. Consequently, Q'_M is a compact set by the norm $\|u\|_{W_2^{n-1}(a';b',H)}$. Thus we have proved that the set $L(P,R_+)$ is an interior compact set. Then using the fact, that the space $L(P,R_+)$ is invariant with respect to shift, applying theorem 1.1 from paper [4] we receive that the interior compactness implies realizability of a Phragmen-Lindelöf principle, i.e. there exists the number $\mu_0>0$ such that for any $u\in L(P,R_+)$ the estimation (6) is fulfilled.

The theorem is proved.

2^0. Let $\{x_{m,j,0},x_{m,j,1},\dots,x_{m,j,q_j}\}$, $m=1,2,\dots$, $j=1,2$ be a chain of eigen and joined vectors from a canonical system ([12]) of root vectors of the bundle $P(\lambda)$ responding the eigen value λ_m from the left half-plane. Then vector-functions $u_{m,j,s}(t)=e^{\lambda_m t}\sum_{h=0}^{s}\frac{t^{s-h}}{(s-h)!}x_{m,j,h}(t)$ satisfy the equation (2) and called elementary solutions.

From theorem 2 it follows

Theorem 3. *By fulfilling the conditions of theorem 2 a system of elementary solutions of the equation (2) is minimal in the space* $W_2^n(R_+;H)$.

Proof. We fix $\eta_0>0$ and in the space $L(P,R_+)$ consider a shift operator $Tu(t)=u(t+\eta_0)$. It is clear that for $u(t)\in W_2^n(R_+;H)$, then this operator coincides with the shift operator T_1 in the space $\ker(P,R_+)$, i.e. T_1 is a contraction of the operator T on $\ker(P,R_+)$. Since all elementary solutions $u_{m,j,s}(t)$ are root vectors of the operator T_1, consequently of the operator T, too. As by theorem 2 the space $L(P,R_+)$ is interior compact, then according to theorem 1,2 from [4] the shift operator T is a compact operator in the space $L(P,R_+)$. Then (for example [12]) the root vectors of the operator T are minimal in the space $L(P,R_+)$, consequently the root vectors of the operator T_1 are also minimal in $L(P,R_+)$. But the norm in the space $L(P,R_+)$ is weaker than the norms in the space $\ker(P,R_+)$, then the root vectors of the operator T_1 are minimal in the space $\ker(P,R_+)$. As the elementary solutions are root vectors of the operator T_1, then they are minimal in the space $W_2^n(R_+;H)$.

The theorem is proved.

Theorem 4. *It the conditions of theorem 1 be fulfilled. Then the system* $x_{m,j,s}$ $\left(m=\overline{1,\infty},\ j=\overline{1,j_p},\ s=\overline{0,m_{qp}}\right)$ *of root vectors responding the eigen-values from the left half-plane* k *is multiminimal.*

Proof. It is easy to see, that traces of elementary solutions in zero produce derivatives Keldysh of chains [12] from the canonical system $x_{m,j,s}$ $\left(m=\overline{1,\infty}\right)$, i.e.

$$x_{m,j,s}^{(\nu)}=\frac{d^{\nu-1}}{dt^{\nu-1}}u_{m,j,s}(t)\Big|_{t=0},\quad \nu=0,1,\ldots,k-1.$$

From the theorem on traces ([2], p.32) and from the inequality (4) it follows, that the operator $\Gamma:u(t)\to\left(u^{(\nu)}(0)\right)_{\nu=0}^{k-1}$ is bounded inverse from the $\ker(P,R_+)$ on the space $\widetilde{H}=\bigoplus_{\nu=0}^{k-1}H_{n-\nu-\frac{1}{2}}$, i.e.

$$c_2\|u\|_{W_2^n(R_+;H)}\le\|\Gamma u\|_{\widetilde{H}}\le c_1\|u\|_{W_2^n(R_+;H)}.$$

By theorem 3, the system $\{u_{m,j,s}(t)\}$ is minimal in the space $\ker(P,R_+)$, and operator Γ transforms it to the system $x_{m,j,s}^{\nu}$ $(\nu=0,1,\ldots,k-1)$.

Consequently, the system $\{x_{m,j,s}^{\nu}\}$ is also minimal in the space $\widetilde{H}=\bigoplus_{\nu=0}^{k-1}H_{n-\nu-\frac{1}{2}}$.

The theorem is proved.

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