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THE BOUNDEDNESS OF $B_{k,n}$ MAXIMAL FUNCTIONS IN SPACES $L_p^{\gamma_{k,n}}\left(R_{k,+}^n\right)$

Abstract

In this work we consider the generalized Bessel-Fourier shift operator, by means of which defined and investigated Hardy-Littlewood-Bessel-Fourier maximal functions ($B_{k,n}$-maximal functions). The bounded ness of $B_{k,n}$-maximal functions in $L_p^{\gamma_{k,n}}\left(R_{k,+}^n\right)\equiv L_p\left(R_{k,+}^n, x_{k,n}^{\gamma_{k,n}}dx\right)$, $0\le k\le n-1$, $1\le p\le\infty$ spaces have been proved.

The shift operator $f(x)\to f(x+y)$ and the techniques of Fourier analysis which is related with it playing a great role in the theory of functions.

The intrinsic generalization of the shift operators are Delsart-Levitan generalized shift operator (GSO) in particular Bessel GSO, which may be constructed by Sturm-Lioville arbitrary differential operator on R. The generalized shift operators forme the one parametric family, but nevertheless many problems of harmonic analysis maybe generalized using the generalized shifts instead of ordinary one.

In this work we consider the generalized Bessel-Fourier shift operator, by means of which Hardy-Littlewood-Bessel-Fourier maximal functions ($B_{k,n}$-maximal functions) are defined and investigated. The bounded ness of $B_{k,n}$- maximal functions in $L_p^{\gamma_{k,n}}\left(R_{k,+}^n\right)=L_p\left(R_{k,+}^n, x_{k,n}^{\gamma_{k,n}}dx\right)$, $0\le k\le n-1$ spaces have been proved. In case $k=0$ maximal functions $B_{0,n}\equiv B$ have been introduced and investigated by Guliev V.S. [2], but in case $k=n-1$ they have been considered in [3].

Let R^n be the n measure Euclidean space of points $x=(x_1,...,x_n)$, $|x|=\left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$,

$1\le k\le n-1$, $x'=x_{1,k}=(x_1,...,x_k)\in R^k$, $x''=x_{k,n}=(x_{k+1},...,x_n)\in R^{n-k}$, $x=(x',x'')=(x_{1,k},x_{k,n})\in R^n$, $R_{k,+}^n=\{x=(x_{1,k},x_{k,n})\in R^n;\ x_{k+1}>0,...,x_n>0\}$, $B(x,r)=\{y\in R_{k,+}^n; |x-y|<r\}$, $B_{k,+}(x,r)=$ $=\{y-x\in R_{k,+}^n; |x-y|<r\}$, $\gamma_{k,n}=(\gamma_{k+1},...,\gamma_n)$, $\gamma_{k+1}>0,...,\gamma_n>0$, $x_{k,n}^{\gamma_{k,n}}=x_{k+1}^{\gamma_{k+1}}\cdot\ldots\cdot x_n^{\gamma_n}$, $|\gamma_{k,n}|=\gamma_{k+1}+...+\gamma_n$.

In case $k=0$ $x=x''=x_{0,n}\in R^n$, $R_+^n\equiv R_{0,+}^n=\{\,x\in R^n,\ x_1>0\ ,\ ...\ ,\ x_n>0\,\}$, $\gamma=\gamma_{0,n}=(\gamma_1,...,\gamma_n)$.

By $L_p^{\gamma_{k,n}}=L_p^{\gamma_{k,n}}\left(R_{k,+}^n\right)$ we'll denote the spaces of measurable functions $f(x)$, $x\in R_{k,+}^n$ with finite norm

$$\|f\|_{L_p^{\gamma_{k,n}}\left(R_{k,+}^n\right)}=\|f\|_{p,\gamma_{k,n}}=\left(\int_{R_{k,+}^n}|f(x)|^p x_{k,n}^{\gamma_{k,n}}dx\right)^{\frac{1}{p}},\qquad 1\le p<\infty .$$

Assume $L_\infty^{\gamma_{k,n}}\left(R_{k,+}^n\right)=L_\infty\left(R_{k,+}^n\right)$, where $L_\infty\left(R_+^n\right)$ is the class of all essentially bounded functions f with the norm:

$$\|f\|_{L_\infty^{\gamma_{k,n}}(R_{k,+}^n)} = \|f\|_{L_\infty(R_{k,+}^n)} = \operatorname{ess}\sup_{x\in R_{k,+}^n}|f(x)|.$$

The operator of generalized shift or Bessel shift (GSO or shift $B_{k,n}$ in short) is determined by the following way (see [4], [5], [6]):

$$T^y f(x) = \frac{\prod_{i=k+1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{n-k}\prod_{i=k+1}^{n}\Gamma\left(\frac{\gamma_i}{2}\right)}\int_0^\pi\cdots\int_0^\pi f\Big(x_1-y_1,\dots,x_k-y_k,$$

$$\sqrt{x_{k+1}^2-2x_{k+1}y_{k+1}\cos\alpha_{k+1}+y_{k+1}^2},\dots,\sqrt{x_n^2-2x_ny_n\cos\alpha_n+y_n^2}\Big)\prod_{i=k+1}^{n}\sin^{\gamma_i-1}\alpha_i\,d\alpha_{k+1}\dots d\alpha_n.$$

By B_j we will denote Bessel singular differential operator $B_j = \frac{\partial^2}{\partial x_j^2}+\frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}$, $(\gamma_j>0,\ j=k+1,\dots,n)$, $B_{k,n}=(B_{k+1},\dots,B_n)$, but by $\Delta_{B_{k,n}}$ - the operator of Laplace-Bessel type, which is determined by the following way:

$$\Delta_{B_{k,n}} = \sum_{i=1}^{k}\frac{\partial^2}{\partial x_i^2}+\sum_{i=k+1}^{n}B_i\,.$$

Let's determine maximal function $B_{k,n}$ by the following way

$$M_{B_{k,n}}f(x) = \sup_{\varepsilon>0}\left|B_{k,+}(0,\varepsilon)\right|_{\gamma_{k,n}}^{-1}\int_{B_{k,+}(0,\varepsilon)}T^y|f(x)|\,y_{k,n}^{\gamma_{k,n}}dy\,.$$

Here $B_{k,+}(0,\varepsilon)=\{y\in R_{k,+}^n;|y|<\varepsilon\}$, $|E|_{\gamma_{k,n}}=\int_E x_{k,n}^{\gamma_{k,n}}dx$, $E\subset R_{k,+}^n$.

It is true the theorem

Theorem 0.1.

1) If $f\in L_1^{\gamma_{k,n}}(R_{k,+}^n)$, then for any $\alpha>0$

$$\left|\left\{x\in R_{k,+}^n: M_{B_{k,n}}f(x)>\alpha\right\}\right|_{\gamma_{k,n}} \le \frac{C}{\alpha}\int_{R_{k,+}^n}|f(x)|\,x_{k,n}^{\gamma_{k,n}}dx\qquad,$$

where C does not depend on f.

2) If $f\in L_p^{\gamma_{k,n}}(R_{k,+}^n)$, $1<p\le\infty$, then $M_{B_{k,n}}f(x)\in L_p^{\gamma_{k,n}}(R_{k,+}^n)$ and

$$\left\|M_{B_{k,n}}f\right\|_{p,\gamma_{k,n}} \le C_{p,\gamma_{k,n}}\|f\|_{p,\gamma_{k,n}},$$

where $C_{p,\gamma_{k,n}}$ depends only on p, $\gamma_{k,n}$ and dimension n.

Proof of theorem 1. Let's denote $S_{k,+}=\{x\in R_{k,+}^n;|x|=1\}$, $x_i=t\xi_i$, $\xi_i\in S_{k,+}$, $i=1,\dots,n$

$$\left|B_{k,+}(0,\varepsilon)\right|_{\gamma_{k,n}} = \int_{B_{k,+}(0,\varepsilon)}x_{k,n}^{\gamma_{k,n}}dx = \int_{S_{k,+}}\int_0^\varepsilon t^{n-1+|\gamma_{k,n}|}\,\xi_{k,n}^{\gamma_{k,n}}dt\,d\xi = C\varepsilon^{n+|\gamma_{k,n}|}.$$

Let's show that $\qquad M_{B_{k,n}}f(x)\le C\,M_\mu f(x).$

Here $M_\mu f(x)$ is the maximal function in measure and is determined by the following way:

$$M_\mu f(x) = \sup_{\varepsilon>0} \mu(B(x,\varepsilon))^{-1} \int_{B(x,\varepsilon)} |f(y)| d\mu(x),$$

where $\mu(B(x,\varepsilon)) = \int_{B(x,\varepsilon)} x_{k,n}^{\gamma_{k,n}} dx = |B(x,\varepsilon)|_{\gamma_{k,n}}$, $\quad d\mu(x) = x_{k,n}^{\gamma_{k,n}} dx$.

Let's denote

$$M_{B_{k,n},\varepsilon} f(x) = |B_{k,+}(0,\varepsilon)|_{\gamma_{k,n}}^{-1} \int_{B_{k,+}(0,\varepsilon)} T^y |f(x)| y_{k,n}^{\gamma_{k,n}} dy.$$

Let's consider

$$T^y \chi_{B_{k,+}(0,\varepsilon)}(x) = C_{\gamma_{k,n}} \int_0^\pi \cdots \int_0^\pi \chi_{B_{k,+}(0,\varepsilon)}\Big(x'-y', \sqrt{x_{k+1}^2 - 2x_{k+1}y_{k+1}\cos\alpha_{k+1} + y_{k+1}^2}, \ldots$$

$$\ldots, \sqrt{x_n^2 - 2x_n y_n \cos\alpha_n + y_n^2}\Big) \cdot \prod_{i=k+1}^n \sin^{\gamma_i - 1}\alpha_i \, d\alpha_{k+1} \ldots d\alpha_n .$$

Here χ_A is the characteristics function of set A, $A \subset R_{k,+}^n$,

$$C_{\gamma_{k,n}} = \frac{\prod_{i=k+1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{n-k} \prod_{i=k+1}^n \Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Further

$$\chi_{B_{k,+}(0,\varepsilon)}\Big(x'-y', \sqrt{x_{k+1}^2 - 2x_{k+1}y_{k+1}\cos\alpha_{k+1} + y_{k+1}^2}, \ldots, \sqrt{x_n^2 - 2x_n y_n \cos\alpha_n + y_n^2}\Big) =$$

$$= \begin{cases} 1, |x'-y'|^2 + \sum_{i=k+1}^n \left(x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2\right) \le \varepsilon^2, \\ 0, |x'-y'|^2 + \sum_{i=k+1}^n \left(x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2\right) > \varepsilon^2. \end{cases}$$

Let's note that

$$|x-y| > \varepsilon \Leftrightarrow |x'-y'|^2 + |x''-y''| > \varepsilon^2 \Rightarrow |x'-y'|^2 + \sum_{i=k+1}^n \left(x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2\right) > \varepsilon^2 .$$

Then from inequality $|x-y| > \varepsilon$ it follows that $T^y \chi_{B_{k,+}(0,\varepsilon)}(x) = 0$. And this means that the support of function $T^y \chi_{B_{k,+}(0,\varepsilon)}(x)$ belongs to ball $B(x,\varepsilon)$.

Then granting the property of convolution $B_{k,n}$ and taking into account the fact that the support of function $T^y \chi_{B_{k,+}(0,\varepsilon)}(x)$ belongs to ball $B(x,\varepsilon)$ we obtain:

$$M_{B_{k,n},\varepsilon} f(x) = |B_{k,+}(0,\varepsilon)|_{\gamma_{k,n}}^{-1} \int_{B_{k,+}(0,\varepsilon)} T^y |f(x)| y_{k,n}^{\gamma_{k,n}} dy =$$

$$= |B_{k,+}(0,\varepsilon)|_{\gamma_{k,n}}^{-1} \int_{R_{k,+}^n} T^y |f(x)| \chi_{B_{k,+}(0,\varepsilon)}(y) y_{k,n}^{\gamma_{k,n}} dy =$$

$$= |B_{k,+}(0,\varepsilon)|_{\gamma_{k,n}}^{-1} \int_{R_{k,+}^n} |f(y)| T^y \chi_{B_{k,+}(0,\varepsilon)}(x) y_{k,n}^{\gamma_{k,n}} dy =$$

$$= |B_{k,+}(0,\varepsilon)|_{\gamma_{k,n}}^{-1} \int_{B(x,\varepsilon)} |f(y)| T^y \chi_{B_{k,+}(0,\varepsilon)}(x) y_{k,n}^{\gamma_{k,n}} dy .$$

Generally speaking, for any $x, y \in R^n_{k,+}$

$$T^y \chi_{B_{k,+}(0,\varepsilon)}(x) = C_{\gamma_{k,n}} \int_0^\pi \cdots \int_0^\pi \chi_{B_{k,+}(0,\varepsilon)}\Big(x' - y', \sqrt{x_{k+1}^2 - 2x_{k+1}y_{k+1}\cos\alpha_{k+1} + y_{k+1}^2}, \dots$$

$$\dots, \sqrt{x_n^2 - 2x_n y_n \cos\alpha_n + y_n^2}\Big) \cdot \prod_{i=k+1}^{n} \sin^{\gamma_i - 1}\alpha_i d\alpha_{k+1}\dots d\alpha_n \le$$

$$\le C_{\gamma_{k,n}} \int_0^\pi \cdots \int_0^\pi \prod_{i=k+1}^{n} \sin^{\gamma_i - 1}\alpha_i d\alpha_{k+1}\dots d\alpha_n = 1 .$$

Let's note that the number

$$C_{\gamma_{k,n}} = \prod_{i=k+1}^{n} C_{0,\gamma_i} = \frac{\prod_{i=k+1}^{n} \Gamma\left(\frac{\gamma_i + 1}{2}\right)}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{n-k} \prod_{i=k+1}^{n} \Gamma\left(\frac{\gamma_i}{2}\right)} = \prod_{i=k+1}^{n} \frac{\Gamma\left(\frac{\gamma_i + 1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\gamma_i}{2}\right)}$$

is often called a norming constant.

Let's estimate $T^y \chi_{B_{k,+}(0,\varepsilon)}(x)$

$$T^y \chi_{B_{k,-}(0,\varepsilon)}(x) = C_{\gamma_{k,n}} \int_{\left\{(\alpha_{k+1},\dots,\alpha_n)\in(0,\pi)^{n-k} : |x'-y'|^2 + \sum_{i=k-1}^{n}\left(x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2\right) < \varepsilon^2\right\}} \int \sin^{\gamma_i - 1}\alpha_i d\alpha_{k+1}\dots d\alpha_n$$

$$\le C_{\gamma_{k,n}} \prod_{i=k+1}^{n} \int_{\left\{\alpha_i\in(0,\pi) : x_i^2 - 2x_i y_i\cos\alpha_i + y_i^2 \le \varepsilon^2\right\}} \sin^{\gamma_i - 1}\alpha_i d\alpha_i \le$$

$$\le \prod_{i=k+1}^{n} C_{0,\gamma_i} \int_{\left\{\alpha_i \in (0,\pi) : \frac{x_i^2 + y_i^2 - \varepsilon^2}{2x_i y_i} \le \cos\alpha_i\right\}} \sin^{\gamma_i - 2}\alpha_i d\cos\alpha_i = \prod_{i=k+1}^{n} A_i(x_i, y_i, \varepsilon) .$$

If in the i-the integral to substitute $t_i = \cos\alpha_i$ then we get

$$A_i(x_i, y_i, \varepsilon) = C_{0,\gamma_i} \int_{\max\left\{-1, \frac{x_i^2 + y_i^2 - \varepsilon^2}{2x_i y_i}\right\}}^{1} \left(1 - t_i^2\right)^{\frac{\gamma_i}{2} - 1} dt_i .$$

Let's note that for any $x, y \in R^n_{k,+}$ $\dfrac{x_i^2 + y_i^2 - \varepsilon^2}{2x_i y_i} \ge -1$ is $x_i + y_i \ge \varepsilon$, $i = k+1, \dots, n$.

Then in case $|x_i - y_i| < \varepsilon < \sqrt{x_i^2 + y_i^2}$, $i = k+1, \dots, n$

$$A_i(x_i, y_i, \varepsilon) \le C(\gamma_i) \int_{\frac{x_i^2 + y_i^2 - \varepsilon_2}{2x_i y_i}}^{1} \left(1 - t_i\right)^{\frac{\gamma_i}{2} - 1} dt_i = C(\gamma_i) \int_0^{\frac{\varepsilon^2 - |x_i - y_i|^2}{2x_i y_i}} t^{\frac{\gamma_i}{2} - 1} dt \le$$

$$\le C\left(\frac{\varepsilon^2 - |x_i - y_i|^2}{x_i y_i}\right)^{\gamma_i} \le C \frac{\varepsilon^{\gamma_i/2}\left(\varepsilon - |x_i - y_i|\right)^{\gamma_i/2}}{x_i y_i^{\gamma_i/2}} .$$

and also

$$A_i(x_i,y_i,\varepsilon)\le C(\gamma_i)\int_0^{\frac{\varepsilon^2-|x_i-y_i|^2}{2x_iy_i}} t^{\frac{\gamma_i}{2}-1}dt\le C(\gamma_i)\int_0^1 t^{\frac{\gamma_i}{2}-1}dt\le C(\gamma_i).$$

Consequently in case $|x_i-y_i|<\varepsilon<\sqrt{x_i^2+y_i^2}$, $i=k+1,...,n$

$$A_i(x_i,y_i,\varepsilon)\le C(\gamma_i)\min\left\{1,\frac{\varepsilon^{\gamma_i/2}(\varepsilon-|x_i-y_i|)^{\gamma_i/2}}{x_iy_i^{\gamma_i/2}}\right\},\quad i=k+1,...,n$$

and in case $\sqrt{x_i^2+y_i^2}<\varepsilon<x_i+y_i$, $i=k+1,...,n$

$$A_i(x_i,y_i,\varepsilon)\le C(\gamma_i)\int_{-1}^1 (1-t_i^2)^{\frac{\gamma_i}{2}-1}dt_i\le C(\gamma_i)$$

and also

$$A_i(x_i,y_i,\varepsilon)=C_{0,\gamma_i}\int_{\frac{x_i^2+y_i^2-\varepsilon^2}{2x_iy_i}}^1 (1-t_i^2)^{\frac{\gamma_i}{2}-1}dt_i\le C(\gamma_i)\int_{\frac{x_i^2+y_i^2-\varepsilon^2}{2x_iy_i}}^0 (1+t_i)^{\frac{\gamma_i}{2}-1}dt_i+C(\gamma_i)\int_0^1 (1-t_i)^{\frac{\gamma_i}{2}-1}dt_i=$$

$$=C(\gamma_i)\int_{\frac{(x_i+y_i)^2-\varepsilon^2}{2x_iy_i}}^1 t^{\frac{\gamma_i}{2}-1}dt+C(\gamma_i)\int_0^1 t^{\frac{\gamma_i}{2}-1}dt\le C(\gamma_i)\int_0^1 t^{\frac{\gamma_i}{2}-1}dt\le C(\gamma_i)\int_0^{\frac{\varepsilon^2-|x_i-y_i|^2}{2x_iy_i}} t^{\frac{\gamma_i}{2}-1}dt=$$

$$=C\left(\frac{\varepsilon^2-|x_i-y_i|^2}{x_iy_i}\right)^{\gamma_i}\le C\frac{\varepsilon^{\gamma_i/2}(\varepsilon-|x_i-y_i|)^{\gamma_i/2}}{x_iy_i^{\gamma_i/2}}.$$

Thus

$$A_i(x_i,y_i,\varepsilon)\le C(\gamma_i)\min\left\{1,\frac{\varepsilon^{\gamma_i/2}(\varepsilon-|x_i-y_i|)^{\gamma_i/2}}{x_iy_i^{\gamma_i/2}}\right\},\ i=k+1,...,n.$$

Let's note that, in case $y_i\ge x_i$, $A_i(x_i,y_i,\varepsilon)\le C\min\left\{1,\frac{\varepsilon^{\gamma_i}}{x_i^{\gamma_i}}\right\}$, in case $0<y_i<x_i$, $\varepsilon<x_i$

$$\frac{\varepsilon-|x_i-y_i|}{y_i}<\frac{\varepsilon}{x_i}$$

and therefore in this case $A_i(x_i,y_i,\varepsilon)\le C\frac{\varepsilon^{\gamma_i}}{x_i^{\gamma_i}}$.

And in case $0<y_i<x_i$, $\varepsilon\ge x_i$ $\quad A_i(x_i,y_i,\varepsilon)\le C$.

So $\qquad A_i(x_i,y_i,\varepsilon)\le C\min\left\{1,\frac{\varepsilon^{\gamma_i}}{x_i^{\gamma_i}}\right\}$.

Consequently, for any $y\in B(x,\varepsilon)$

$$T^y\chi_{B_{k,+}(0,\varepsilon)}(x)\le\prod_{i=k+1}^n C(\gamma_i)\min\{1,\varepsilon^{\gamma_i}x_i^{-\gamma_i}\}$$

And also

$$\mu B(x,\varepsilon)=\int\limits_{\{y\in R^n_{k,+}:|x-y|<\varepsilon\}} y^{\gamma_{k,n}}_{k,n}dy=\int\limits_{\{y\in R^n_{k,+}:|y'|^2+|x''-y''|^2<\varepsilon^2\}} y^{\gamma_{k,n}}_{k,n}dy\le$$

$$\le\prod_{i=1}^{k}\int\limits_{0<y_i<\varepsilon}dy_i\prod_{j=k+1}^{n}\int\limits_{y_j>0:|x_j-y_j|<\varepsilon}y_j^{\gamma_j}dy_j\le C\varepsilon^k\prod_{j=k+1}^{n}\begin{cases}\int\limits_{x_j-\varepsilon}^{x_j+\varepsilon}y_j^{\gamma_j}dy_j\,,\ \varepsilon<x_j\\ \int\limits_{0}^{x_j+\varepsilon}y_j^{\gamma_j}dy_j\,,\ \varepsilon\ge x_j\end{cases}\le$$

$$\le C\varepsilon^k\prod_{j=k+1}^{n}\begin{cases}\varepsilon x_j^{\gamma_j},\varepsilon<x_j\\ \varepsilon^{1+\gamma_j},\varepsilon\ge x_j\end{cases}=C\varepsilon^{n+\gamma_{k,n}}\prod_{j=k+1}^{n}\begin{cases}(x_j/\varepsilon)^{\gamma_j},\varepsilon<x_j\\ 1,\ \ \varepsilon\ge x_j\end{cases}.$$

Further

$$M_{B_{k,n}}f(x)\le\sum_{p=k}^{n}M_{p,B_{k,n}}f(x)=$$

$$=\sum_{p=k}^{n}\sup_{\substack{\varepsilon>x_{i_j},\,j=\overline{k+1,p}\\ \varepsilon\le x_{i_j},\,j=\overline{p+1,n}\\ i_j\in\{k+1,\dots,n\}}}\left|B_{k,+}(0,\varepsilon)\right|^{-1}_{\gamma_{k,n}}\int\limits_{B(x,\varepsilon)}|f(y)|T^y\chi_{B_{k,+}(0,\varepsilon)}(x)y^{\gamma_{k,n}}_{k,n}dy.$$

Not loosing generality we'll account that $i_j\equiv j,\ j=k+1,\dots,n$. Then

$$M_{p,B_{k,n}}f(x)=\sup_{\substack{\varepsilon>x_j,\,j=\overline{k+1,p}\\ \varepsilon\le x_j,\,j=\overline{p+1,n}\\ j\in\{k+1,\dots,n\}}}\left|B_{k,+}(0,\varepsilon)\right|^{-1}_{\gamma_{k,n}}\int\limits_{B(x,\varepsilon)}|f(y)|T^y\chi_{B_{k,+}(0,\varepsilon)}(x)y^{\gamma_{k,n}}_{k,n}dy.$$

In case $p=k$ granting that $\mu B(x,\varepsilon)\le C\varepsilon^{n+|\gamma_{k,n}|}$, $\left|B_{k,+}(0,\varepsilon)\right|_{\gamma_{k,n}}=\varepsilon^{n+|\gamma_{k,n}|}$ and $T^y\chi_{B_{k,+}(0,\varepsilon)}\le 1$ we'll get that

$$M_{k,B_{k,n}}f(x)=\sup_{\varepsilon>x_j,\,j=\overline{k+1,n}}\left|B_{k,+}(0,\varepsilon)\right|^{-1}_{\gamma_{k,n}}\int\limits_{B(x,\varepsilon)}|f(y)|T^y\chi_{B_{k,+}(0,\varepsilon)}(x)y^{\gamma_{k,n}}_{k,n}dy\le$$

$$\le C\sup_{\varepsilon>0}\frac{1}{\mu B(x,\varepsilon)}\int\limits_{B(x,\varepsilon)}|f(y)|d\mu(y)=CM_\mu f(x).$$

In case $p=n$ granting that $\mu B(x,\varepsilon)\le C\varepsilon^{n+|\gamma_{k,n}|}\dfrac{x^{\gamma_{k,n}}_{k,n}}{\varepsilon^{|\gamma_{k,n}|}}$, $\left|B_{k,+}(0,\varepsilon)\right|_{\gamma_{k,n}}=\varepsilon^{n+|\gamma_{k,n}|}$, and $T^y\chi_{B_{k,+}(0,\varepsilon)}\le C\dfrac{\varepsilon^{\gamma_{k,n}}}{x^{\gamma_{k,n}}_{k,n}}$ we'll get that

$$M_{n,B_{k,n}}f(x)\le\sup_{\varepsilon\le x_j,\,j=\overline{k+1,n}}\left|B_{k,+}(0,\varepsilon)\right|^{-1}_{\gamma_{k,n}}\int\limits_{B(x,\varepsilon)}|f(y)|T^y\chi_{B_{k,+}(0,\varepsilon)}(x)y^{\gamma_{k,n}}_{k,n}dy\le$$

$$\le\sup_{\varepsilon>0}\frac{1}{\mu B(x,\varepsilon)}\int\limits_{B(x,\varepsilon)}|f(y)|d\mu(y)=CM_\mu f(x).$$

In case $k<p<n$ granting that

$$\mu B(x,\varepsilon)\le C\varepsilon^{n+|\gamma_{k,n}|}\prod_{j=k+1}^{n}\begin{cases}(x_j/\varepsilon)^{\gamma_j},\varepsilon<x_j\\ 1,\ \ \varepsilon\ge x_j\end{cases}=C\varepsilon^{n+|\gamma_{k,n}|}\prod_{j=k+1}^{p}(x_j/\varepsilon)^{\gamma_j}=C\varepsilon^{n+|\gamma_{k,n}|}\frac{x^{\gamma_{k,p}}_{k,p}}{\varepsilon^{|\gamma_{k,p}|}},$$

$$T^{y}\chi_{B_{k,+}(0,\varepsilon)} \le C(\gamma)\prod_{i=k+1}^{n}\min\left\{1,\frac{\varepsilon^{\gamma_i}}{x_i^{\gamma_i}}\right\} = C(\gamma)\prod_{i=k+1}^{p}\frac{\varepsilon^{\gamma_i}}{x_i^{\gamma_i}} = C(\gamma)\frac{\varepsilon^{|\gamma_{k,p}|}}{x_{k,p}^{\gamma_{k,p}}}$$

and $\left|B_{k,+}(0,\varepsilon)\right|_{\gamma_{k,n}} = \varepsilon^{n+|\gamma_{k,n}|}$ we'll get that

$$M_{p,B_{k,n}}f(x) \le \sup_{\varepsilon\le x_j,\, j=\overline{k+1,n}} \left|B_{k,+}(0,\varepsilon)\right|_{\gamma_{k,n}}^{-1} \int_{B(x,\varepsilon)} |f(y)| T^{y}\chi_{B_{k,+}(0,\varepsilon)}(x) y_{k,n}^{\gamma_{k,n}} dy \le$$

$$\le \sup_{\varepsilon>0}\frac{1}{\mu B(x,\varepsilon)}\int_{B(x,\varepsilon)} |f(y)| d\mu(y) = CM_{\mu}f(x).$$

Hence follows

$$M_{B_{k,n}}f(x) \le \sum_{p=k}^{n} M_{p,B_{k,n}}f(x) \le CM_{\mu}f(x).$$

It is known (for example, see [7]) that for measures μ satisfying the doubling condition

$$\int_{B(x,2\varepsilon)} d\mu(y) \le C \int_{B(x,\varepsilon)} d\mu(y),$$

it is true

$$\int_{R^n} |M_{\mu}f(x)|^{p} d\mu(x) \le C_{p,\mu}\int_{R^n}|f(x)|^{p}dx \quad \text{and} \quad \mu\{x : M_{\mu}f(x) > \alpha\} \le \frac{C}{\alpha}\int_{R^n}|f(x)|d\mu(x)$$

then granting that for measures $d\mu(x) = x_{k,n}^{\gamma_{k,n}}dx$ it is correct the doubling condition we get

$$\left\|M_{B_{k,n}}f\right\|_{p,\gamma_{k,n}} \le \left\|M_{\mu}f\right\|_{p,\gamma_{k,n}} \le C_p\|f\|_{p,\gamma_{k,n}}$$

and

$$\left|\left\{x\in R_{k,+}^{n} : M_{B_{k,n}}f(x) > \alpha\right\}\right|_{\gamma_{k,n}} \le \mu\left\{x\in R_{k,+}^{n} : M_{\mu}f(x) > \alpha\right\} \le \frac{C}{\alpha}\int_{R_{k,+}^{n}}|f(x)|d\mu(x).$$

Theorem has been proved.

Corollary 0.1. *If* $f\in L_p^{\gamma_{k,n}}\left(R_{k,+}^{n}\right)$, *where* $1\le p\le\infty$, $0\le k\le n-1$ *then*

$$\lim_{\varepsilon\to 0}\left|B_{k,+}(0,\varepsilon)\right|_{\gamma_{k,n}}^{-1}\int_{B_{k,+}(0,\varepsilon)} T^{y}f(x) y_{k,n}^{\gamma_{k,n}}dy = f(x)$$

for almost all $x\in R_{k,+}^{n}$.

The proof of the corollary. By virtue of the locality of the problem, one can account that $f\in L_1^{\gamma_{k,n}}\left(R_{k,+}^{n}\right)$. In general case one can multiply f by the characteristic function of ball $B(0,r)$ and obtain required convergence almost everywhere interior to this ball and then tending r to infinity to get on all space $R_{k,+}^{n}$.

Assume for any $r\ge 0$ and for any $x\in R_{k,+}^{n}$

$$f_r(x) = \left|B_{k,+}(0,r)\right|_{\gamma_{k,n}}^{-1}\int_{B_{k,+}(0,r)} T^{y}f(x) y_{k,n}^{\gamma_{k,n}}dy.$$

Let $r_0>0$, $B = B_{k,+}(0,r_0)$. According to Minskovsky generalized inequality and to the continuity property with respect to the generalized shift (of shift $B_{k,n}$)

$$\|f_r - f\|_{L_1^{\gamma_{k,n}}(B_{k,+}(0,r_0))} = \left\| |B_{k,+}(0,r_0)|_{\gamma_{k,n}}^{-1} \int_{B_{k,+}(0,r_0)} \left(T^y f(\cdot) - f(\cdot)\right) y_{k,n}^{\gamma_{k,n}} dy \right\|_{L_1^{\gamma_{k,n}}(B)} \le$$

$$\le |B_{k,+}(0,r_0)|_{\gamma_{k,n}}^{-1} \int_{B_{k,+}(0,r_0)} \|T^y f(\cdot) - f(\cdot)\|_{L_1^{\gamma_{k,n}}(B)} y_{k,n}^{\gamma_{k,n}} dy \le \sup_{|y|\le r_0} \|T^y f(\cdot) - f(\cdot)\|_{L_1^{\gamma_{k,n}}(B)} \to 0$$

for $r_0 \to +0$. It means that there exists such sequence r_k, that $r_k \to +0, (k \to \infty)$ and

$$\lim_{k\to\infty} f_{r_k}(x) = f(x)$$

almost everywhere in $R_{k,+}^n$.

Now, let's prove that $\lim\limits_{r\to+0} f_r(x)$exists almost everywhere. For this purpose for any $x \in R_{k,+}^n$ let's denote

$$\Omega_f(x) = \left| \overline{\lim_{r\to+0}} f_r(x) - \underline{\lim_{r\to+0}} f_r(x) \right|$$

($\Omega_f(x)$ is the oscillation of family $\{f_r\}$ at the point x for $r \to +0$).

If g is a continuous function with compact support on $R_{k,+}^n$, then g_r is convergent to g uniformly, and consequently, Ω_g is identically equal to zero in this case.

Further, if $g \in L_1^{\gamma_{k,n}}(B)$, then according to the statement of theorem 1

$$\left| \left\{ x \in R_{k,+}^n : M_{B_{k,n}} g(x) \ge \varepsilon \right\} \right|_{\gamma_{k,n}} \le \frac{C}{\varepsilon} \| g \|_{1,\gamma_{k,n}},\ g \in L_1^{\gamma_{k,n}}(R_{k,+}^n).$$

On the other hand it is obvious that $\Omega g(x) \le 2M_{B_{k,n}} g(x)$. Thus

$$\left| \left\{ x \in R_{k,+}^n : \Omega g(x) \le \varepsilon \right\} \right|_{\gamma_{k,n}} \le \frac{2C}{\varepsilon} \| g \|_{1,\gamma_{k,n}},\ g \in L_1^{\gamma_{k,n}}(R_{k,+}^n) \qquad (1)$$

It is known that the set of all continuous functions with compact support in $R_{k,+}^n$ is dense in $L_p^{\gamma_{k,n}}(R_{k,+}^n)$. Therefore for any number $\varepsilon > 0$ there exist a continuous function with compact support in $R_{k,+}^n$, such that

$$\|f - f_\varepsilon\|_{L_p^{\gamma_{k,n}}(R_{k,+}^n)} < \varepsilon .$$

Denote $g_r := f - f_r$. Then $g_r \in L_p^{\gamma_{k,n}}(R_{k,+}^n)$ and $\|g_r\|_{L_p^{\gamma_{k,n}}(R_{k,+}^n)} < \varepsilon$.

Thus, if $f \in L_p^{\gamma_{k,n}}(R_{k,+}^n)$, then for any number $\varepsilon > 0$ there exists a continuous function f_ε with the compact support and function $g_\varepsilon \in L_p^{\gamma_{k,n}}(R_{k,+}^n)$ with condition $\|g_\varepsilon\|_{L_p^{\gamma_{k,n}}(R_{k,+}^n)} < \varepsilon$, such, that $f = f_\varepsilon + g_\varepsilon$.

Any function $f \in L_1^{\gamma_{k,n}}(R_{k,+}^n)$ can be written in form $f = h + g$, where h is continuous and has compact support on $R_{k,+}^n$, and norm of function g in $L_1^{\gamma_{k,n}}(R_{k,+}^n)$ can be chosen according our judgment. But $\Omega f \le \Omega h + \Omega g$ and $\Omega h \equiv 0$, however h is continuous. Therefore from (1) it follows, that

$$\left|\left\{x \in R_{k,+}^n : \Omega g(x) > \varepsilon\right\}\right|_{\gamma_{k,n}} \leq \frac{C}{\varepsilon}\| g \|_{L_1^{\gamma_{k,n}}\left(R_{k,+}^n\right)}.$$

As norm g in $L_1^{\gamma_{k,n}}\left(R_{k,+}^n\right)$ one can choose arbitrary small, then we get $\Omega f = 0$ almost everywhere on $R_{k,+}^n$, hence follows that $\lim\limits_{r \to 0} f_r(x)$ exists almost everywhere on $R_{k,+}^n$.

The corollary has been proved.

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