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ON FRACTIONAL MAXIMAL FUNCTIONS AND FRACTIONAL INTEGRALS, GENERATED BY BESSEL DIFFERENTIAL OPERATORS

Abstract

In this work we consider the generalized Bessel shift operator, generated by $B=\frac{d^2}{dx^2}+\frac{\gamma}{x}\frac{d}{dx}$ *Bessel differential operators, by means of which fractional Bessel maximal functions (fractional* B *maximal functions) and Bessel fractional integrals (*B *fractional integrals) are defined and investigated. The boundedness of fractional* B *maximal functions and* B *fractional integrals from* $L_p^\gamma(R_+)$ *space to space* $L_p^\gamma(R_+)$ *is proved. We also prove weight inequalities for the* B *fractional maximal functions and* B *fractional integrals.*

Let $R_+=]0,\infty[$, $\gamma>0$, $E_+(x,r)=\{y\in R_+ : |x-y|<r\}$, $E_+(0,r)=(0,r)$. We will denote by $L_p^\gamma(R_+)$ the space of measurable functions $f(x)$, $x\in R_+$ with the finite norm

$$\|f\|_{L_p^\gamma(R_+)}=\left(\int_{R_+}|f(x)|^p x^\gamma dx\right)^{1/p},\quad 1\le p\le\infty .$$

We put $L_\infty^\gamma(R_+)=L_\infty(R_+)$, where $L_\infty(R_+)$ the class of all essential bounded functions f with the finite norm

$$\|f\|_{L_\infty^\gamma(R_+)}=\|f\|_{L_\infty(R_+)}=\operatorname*{esssup}_{x\in R_+}|f(x)| .$$

Denote the T^y the B-shift operator acting according to the law

$$T^y f(x)=C_\gamma\int_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos\alpha}\right)\sin^{\gamma-1}\alpha d\alpha ,$$

where $C_\gamma=\pi^{-\frac{1}{2}}\Gamma(\gamma+1/2)\Gamma^{-1}(\gamma)$.

We remark that T^y is closely connected with the $B=\frac{d^2}{dx^2}+\frac{\gamma}{x}\frac{d}{dx}$ (see [1] for details).

Let ω be a positive measurable function on R_+. Denote by $L_{p,\omega}^\gamma(R_+)$ the set of measurable functions $f(x)$, $x\in R_+$ with the finite norm

$$\|f\|_{L_{p,\omega}^\gamma(R_+)}=\left(\int_{R_+}|f(x)|^p\omega(x)x^\gamma dx\right)^{\frac{1}{p}}<\infty ,\ 1\le p\le\infty .$$

Let's determine B-fractional integrals, B-maximal functions and fractional B-maximal functions by the following way:

$$I_B^\alpha f(x)=\int_0^\infty T^y x^{\alpha-1-\gamma}f(y)y^\gamma dy ,\quad 0<\alpha<1+\gamma ,$$

$$M_B f(x)=\sup_{\varepsilon>0}|(0,\varepsilon)|_\gamma^{-1}\int_0^\varepsilon |T^y f(x)|y^\gamma dy,$$

$$M_B^\beta f(x)=\sup_{\varepsilon>0}|(0,\varepsilon)|_\gamma^{\frac{\beta}{1+\gamma}-1}\int_0^\varepsilon |T^y f(x)|y^\gamma dy\ ,\ \ 0\le\beta<1+\gamma .$$

Here $|E|_\gamma=\int_E x^\gamma dx$, $E\subset R_+$.

Note that, in $\beta=0$, $M_B^0 f(x)=M_B f(x)$.

The following theorem is valid

Theorem 1. *Let* $0\le\beta<1+\gamma$, $\frac{1}{p}-\frac{1}{q}=\frac{\beta}{1+\gamma}$, $1\le p\le q<\infty$.

1) If $p=1$, $f\in L_1^\gamma(R_+)$, *then for all* $\lambda>0$

$$\int_{\{x\in R_+:M_B^\beta f(x)>\lambda\}} x^\gamma dx\le\left(\frac{C}{\lambda}\int_{R_+}|f(x)|x^\gamma dx\right)^q,$$

where C *does not depend on* f.

2) If $1<p\le\frac{1+\gamma}{\beta}$, $f\in L_p^\gamma(R_+)$, *then* $M_B^\beta f\in L_q^\gamma(R_+)$ *and*

$$\left(\int_{R_+}\left(M_B^\beta f(x)\right)^q x^\gamma dx\right)^{1/q}\le C\left(\int_{R_+}|f(x)|^p x^\gamma dx\right)^{1/p},$$

where C *-depends only on* p,γ.

Proof of theorem 1. Denote $d\mu(x)=x^\gamma dx$. Then $\mu E_+(0,r)=|(0,r)|_\gamma=Cr^{1+\gamma}$,

$$\mu E_+(x,r)=\int_{\{y\in R_+:|x-y|<r\}}=\begin{cases}\int_0^{x+r}y^\gamma dy,\ r>x,\\ \int_{x-r}^{x+r}y^\gamma dy,\ r\le x\end{cases}=\begin{cases}Cr^{1+\gamma},\ r>x,\\ Crx^\gamma,\ r\le x.\end{cases}$$

We will show that

$$M_B^\beta f(x)\le CM^\beta f(x).$$

Here $M^\beta f(x)$ fractional maximal function in measure $d\mu(x)=x^\gamma dx$ and defined in the following

$$M^\beta f(x)=\sup_{\varepsilon>0}\mu(E_+(x,\varepsilon))^{\frac{\beta}{1+\gamma}-1}\int_{E_+(x,\varepsilon)}|f(y)|d\mu(x).$$

Denote by

$$M_{B,\varepsilon}^\beta f(x)=|(0,\varepsilon)|_\gamma^{\frac{\beta}{1+\gamma}-1}\int_0^\varepsilon |T^y f(x)|y^\gamma dy .$$

Consider

$$T^y\chi_{E_+(0,\varepsilon)}(x)=C_\gamma\int_0^\pi \chi_{E_+}(0,\varepsilon)\left(\sqrt{x^2-2xy\cos\alpha+y^2}\right)d\alpha .$$

Here χ_A -characteristic function the set A, $A\subset R_+$.

Further

$$\chi_{(0,\varepsilon)}\left(\sqrt{x^2-2xy\cos\alpha+y^2}\right)=\begin{cases}1, & x^2-2xy\cos\alpha+y^2\le\varepsilon^2,\\ 0, & x^2-2xy\cos\alpha+y^2>\varepsilon^2.\end{cases}$$

Note that

$$|x-y|>\varepsilon\Rightarrow x^2-2xy\cos\alpha+y^2>\varepsilon^2.$$

Then from inequality $|x-y|>\varepsilon$ it follows that $T^y\chi_{(0,\varepsilon)}(x)=0$. That means support function $T^y\chi_{(0,\varepsilon)}(x)$ belongs to intervals $E_+(x,\varepsilon)$.

Then taking into account property B-convolution and take into account support function $T^y\chi_{(0,\varepsilon)}(x)$ belong to intervals $E_+(x,\varepsilon)$ we have

$$M^{\beta}_{B,\varepsilon}f(x)\le Cr^{\beta-1-\gamma}\int_0^{\varepsilon}T^y|f(x)|y^\gamma dy=$$
$$=Cr^{\beta-1-\gamma}\int_{R_-}T^y|f(x)|\chi_{(0,\varepsilon)}(y)y^\gamma dy=$$
$$=Cr^{\beta-1-\gamma}\int_{R_+}|f(y)|T^y\chi_{(0,\varepsilon)}(x)y^\gamma dy=$$
$$=Cr^{\beta-1-\gamma}\int_{E_+(x,\varepsilon)}|f(y)|T^y\chi_{(0,\varepsilon)}(x)y^\gamma dy.$$

Generally speaking, for all $x,y\in R^n_{k,+}$

$$T^y\chi_{(0,\varepsilon)}(x)=C_\gamma\int_0^{\pi}\chi_{(0,\varepsilon)}\left(\sqrt{x^2-2xy\cos\alpha+y^2}\right)\sin^{\gamma-1}\alpha d\alpha\le C_\gamma\int_0^{\pi}\sin^{\gamma-1}\alpha d\alpha=1.$$

Note that, the number C_γ is often called the normalizing constant.

We estimate $T^y\chi_{(0,\varepsilon)}(x)$

$$T^y\chi_{(0,\varepsilon)}(x)=C_\gamma\int_{\{\alpha\in(0,\pi):x^2-2xy\cos\alpha+y^2<\varepsilon^2\}}\sin\gamma-1\ \alpha d\alpha=A(x,y,\varepsilon).$$

If the last integral replace $t=\cos\alpha$, then we get

$$A(x,y,\varepsilon)=C_\gamma\int_{\max\left\{-1,\frac{x^2+y^2-\varepsilon^2}{2xy}\right\}}^{1}\left(1-t^2\right)^{\frac{\gamma}{2}-1}dt.$$

Note that, for all $x,y\in R_+$, $\dfrac{x^2+y^2-\varepsilon^2}{2xy}\ge-1$ is equivalent to $x+y\ge\varepsilon$. Then, in the case $|x-y|<\varepsilon<\sqrt{x^2+y^2}$

$$A(x,y,\varepsilon)\le C(\gamma)\int_{\frac{x^2+y^2-\varepsilon^2}{2xy}}^{1}\left(1-t^2\right)^{\frac{\gamma}{2}-1}dt=C(\gamma)\int_0^{\frac{\varepsilon^2-|x-y|^2}{2xy}}t^{\frac{\gamma}{2}-1}dt\le$$

$$\le C\left(\frac{\varepsilon^2-|x-y|^2}{xy}\right)^{\gamma}\le C\frac{\varepsilon^{\gamma/2}\left(\varepsilon-|x-y|\right)^{\gamma/2}}{(xy)^{\gamma/2}}$$

and also

$$A(x,y,\varepsilon)\le C(\gamma)\int_0^{\frac{\varepsilon^2-|x-y|^2}{2xy}} t^{\frac{\gamma}{2}-1}dt\le C(\gamma)\int_0^1 t^{\frac{\gamma}{2}-1}dt\le C(\gamma).$$

Therefore in the case $|x-y|<\varepsilon<\sqrt{x^2+y^2}$

$$A(x,y,\varepsilon)\le C(\gamma)\min\left\{1,\frac{\varepsilon^{\gamma/2}(\varepsilon-|x-y|)^{\gamma/2}}{(xy)^{\gamma/2}}\right\}.$$

And in the case $\sqrt{x^2+y^2}<\varepsilon<x+y$,

$$A(x,y,\varepsilon)\le C(\gamma)\int_{-1}^1(1-t^2)^{\frac{\gamma}{2}-1}dt\le C(\gamma),$$

and also

$$A(x,y,\varepsilon)\le C(\gamma)\int_{\frac{x^2+y^2-\varepsilon^2}{2xy}}^1(1-t^2)^{\frac{\gamma}{2}-1}dt\le$$

$$\le C(\gamma)\int_{\frac{x^2+y^2-\varepsilon^2}{2xy}}^0(1+t)^{\frac{\gamma}{2}-1}dt+C(\gamma)\int_0^1(1-t)^{\frac{\gamma}{2}-1}dt=$$

$$=C(\gamma)\int_{\frac{(x+y)^2-\varepsilon^2}{2xy}}^1 t^{\frac{\gamma}{2}-1}dt+C(\gamma)\int_0^1 t^{\frac{\gamma}{2}-1}dt\le C(\gamma)\int_0^1 t^{\frac{\gamma}{2}-1}dt\le$$

$$\le C(\gamma)\int_0^{\frac{\varepsilon^2-|x-y|^2}{2xy}} t^{\frac{\gamma}{2}-1}dt=C\left(\frac{\varepsilon^2-|x-y|^2}{xy}\right)^{\gamma}\le C\frac{\varepsilon^{\gamma/2}(\varepsilon-|x-y|)^{\gamma/2}}{(xy)^{\gamma/2}}.$$

Thus

$$A(x,y,\varepsilon)\le C(\gamma)\min\left\{1,\frac{\varepsilon^{\gamma/2}(\varepsilon-|x-y|)^{\gamma/2}}{(xy)^{\gamma/2}}\right\}.$$

Note that, in the case $y\ge x$, $A(x,y,\varepsilon)\le C\min\left\{1,\frac{\varepsilon^\gamma}{x^\gamma}\right\}$, and in the case $0<y<x$, $\varepsilon<x$

$$\frac{\varepsilon-|x-y|}{y}<\frac{\varepsilon}{x}$$

and therefore in the case $A(x,y,\varepsilon)\le C\frac{\varepsilon^\gamma}{x^\gamma}$.

And in the case $0<y<x$, $\varepsilon\ge x$, $A(x,y,\varepsilon)\le C$.

It means

$$A(x,y,\varepsilon)\le C\min\left\{1,\frac{\varepsilon^\gamma}{x^\gamma}\right\}.$$

Therefore, for all $y\in E_+(x,\varepsilon)$

$$T^{\gamma}\chi_{(0,\varepsilon)}(x)\le C(\gamma)\min\{1,\varepsilon^{\gamma}x^{-\gamma}\}.$$

Further

$$M_B^{\beta}f(x)\le M_{1,B}^{\beta}f(x)+M_{2,B}^{\beta}f(x),$$

where

$$M_{1,B}^{\beta}f(x)=\sup_{\varepsilon>x}|(0,\varepsilon)|_{\gamma}^{\frac{\beta}{1+\gamma}-1}\int_{E_+(x,\varepsilon)}|f(y)|T^{y}\chi_{(0,\varepsilon)}(x)y^{\gamma}dy,$$

$$M_{2,B}^{\beta}f(x)=\sup_{\varepsilon\le x}|(0,\varepsilon)|_{\gamma}^{\frac{\beta}{1+\gamma}-1}\int_{E_+(x,\varepsilon)}|f(y)|T^{y}\chi_{(0,\varepsilon)}(x)y^{\gamma}dy.$$

In case $\varepsilon>x$ taking into account $\mu E_+(x,\varepsilon)=C\varepsilon^{1+\gamma}$, $|(0,\varepsilon)|_{\gamma}=\varepsilon^{1+\gamma}$ and $T^{y}\chi_{(0,\varepsilon)}\le 1$ we have

$$M_{1,B}^{\beta}f(x)=\sup_{\varepsilon>x}|(0,\varepsilon)|_{\gamma}^{\frac{\beta}{1+\gamma}-1}\int_{E_-(x,\varepsilon)}|f(y)|T^{y}\chi_{(0,\varepsilon)}(x)y^{\gamma}dy\le$$

$$\le C\sup_{\varepsilon>x}\mu E_+(x,\varepsilon)^{\frac{\beta}{1+\gamma}-1}\int_{E_+(x,\varepsilon)}|f(y)|d\mu(y)=CM^{\beta}f(x).$$

In case $\varepsilon\le x$ taking into account $\mu E_+(x,\varepsilon)=C\varepsilon x^{\gamma}$, $|(0,\varepsilon)|_{\gamma}=C\varepsilon^{1+\gamma}$, and $T^{y}\chi_{(0,\varepsilon)}\le C\frac{\varepsilon^{\gamma}}{x^{\gamma}}$ we have

$$M_{2,B}^{\beta}f(x)\le\sup_{\varepsilon\le x}|(0,\varepsilon)|_{\gamma}^{\frac{\beta}{1+\gamma}-1}\int_{E_+(x,\varepsilon)}|f(y)|T^{y}\chi_{(0,\varepsilon)}(x)y^{\gamma}dy\le$$

$$\le\sup_{\varepsilon>0}\left(\frac{\varepsilon}{x}\right)^{\beta\gamma}\mu B(x,\varepsilon)^{\frac{\beta}{1+\gamma}-1}\int_{E_+(x,\varepsilon)}|f(y)|d\mu(y)\le CM^{\beta}f(x).$$

Hence it follows that

$$M_B^{\beta}f(x)\le M_{1,B}^{\beta}f(x)+M_{2,B}^{\beta}f(x)\le CM^{\beta}f(x).$$

It is known, [2] that, the for measure μ, satisfying the doubling condition

$$\int_{B(x,2\varepsilon)}d\mu(y)\le C\int_{B(x,\varepsilon)}d\mu(y)$$

is valid

$$\left(\int_{R_+}|M^{\beta}f(x)|^{q}d\mu(x)\right)^{1/q}\le C_{p,\mu}\left(\int_{R_+}|f(x)|^{p}dx\right)^{1/p},$$

$$\mu\{x:M^{\beta}f(x)>\alpha\}\le\left(\frac{C}{\alpha}\int_{R_+}|f(x)|d\mu(x)\right)^{1/q}$$

then, taking into account, that the for measure $\mu(x)=x^{\gamma}dx$ correctly the doubling condition we have

$$\|M_B^{\beta}f\|_{L_q^{\gamma}(R_+)}\le\|M^{\beta}f\|_{L_q^{\gamma}(R_+)}\le C_p\|f\|_{L_p^{\gamma}(R_+)}$$

and

$$\left|\left\{x \in R_+ : M_B^\beta f(x) > \alpha\right\}\right|_\gamma \le$$

$$\le \mu\left\{x \in R_+ : M^\beta f(x) > \lambda\right\} \le \left(\frac{C}{\lambda}\int_{R_+}|f(x)|d\mu(x)\right)^q .$$

This completes the proof.

Corollary 1 [3]. *1) If* $f \in L_1^\gamma(R_+)$, *then for all* $\lambda > 0$

$$\int_{\{x\in R_+ : M_B f(x) > \lambda\}} x^\gamma dx \le \frac{C}{\lambda}\int_{R_+}|f(x)|x^\gamma dx,$$

where C *does not depend on* f.

2) If $1 < p \le \infty$, $f \in L_p^\gamma(R_+)$, *then* $M_B f \in L_p^\gamma(R_+)$ *and*

$$\int_{R_+}(M_B f(x))^p x^\gamma dx \le C\int_{R_+}|f(x)|^p x^\gamma dx,$$

where C *- depends only on* p *and* γ.

Corollary 2. *If* $f \in L_p^\gamma(R_+)$, $1 \le p \le \infty$, *then*

$$\lim_{\varepsilon\to 0}|(0,\varepsilon)|_\gamma^{-1}\int_0^\varepsilon T^y f(x) y^\gamma dy = f(x)$$

for a.e. $x \in R_+$.

Analogously to the well-known A_p classes of weight functions introduced by B.Muckechoupt we consider, for $1 \le p < \infty$, the following classes:

Definition 1. *The weight function* ω *belongs to the class* $A_p^\gamma(R_+)$ *for* $1 < p < \infty$ *if*

$$\sup_{x,r\in R_+}|E_+(x,r)|_\gamma^{-1}\int_{E_+(x,r)}\omega(y)y^\gamma dy\left(|E_+(x,r)|^{-1}\int_{E_+(x,r)}\omega^{-\frac{1}{p-1}}(y)y^\gamma dy\right)^{p-1} < \infty$$

and ω *belongs to* $A_1^\gamma(R_+)$ *if there exists a positive constant* C *such that for any* $x \in R_+$ *and* $r > 0$

$$|E_+(x,r)|^{-1}\int_{E_+(x,r)}\omega^{-\frac{1}{p-1}}(y)y^\gamma dy \le C\operatorname*{ess\,sup}_{y\in E_+(x,r)}\omega(y).$$

The properties of the class $A_p^\gamma(R_+)$ are analogous to those of the B.Muckenhoupt classes. In particular, if $\omega \in A_p^\gamma(R_+)$, then $\omega \in A_{p-\varepsilon}^\gamma(R_+)$ for a certain sufficiently small $\varepsilon > 0$ and $A_{p_1}^\gamma(R_+)$ for any $p_1 > p$.

Note that, $x^\alpha \in A_p^\gamma(R_+)$, $1 < p < \infty$ if and only if $-(1+\gamma) < \alpha < (1+\gamma)(p-1)$ and $x^\alpha \in A_1^\gamma(R_+)$, if and only if $-(1+\gamma) < \alpha \le 0$.

Theorem 2. *Let* $1 < p < \frac{1+\gamma}{\beta}$, $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{1+\gamma}$. *Then the following two condition are equivalent:*

(i) There is a constant $C > 0$ *such that for any* $f \in L_{p,\omega}^\gamma(R_+)$ *the inequality*

$$\left(\int_{R_+}\left(M_B^\beta\left(f\omega^\beta\right)(x)\right)^q \omega(x)x^\gamma dx\right)^{\frac{1}{q}} \le C\left(\int_{R_+}|f(x)|^p \omega(x)x^\gamma dx\right)^{\frac{1}{p}}$$

holds.

(ii) $\omega \in A^\gamma_{1+\frac{q}{p'}}(R_+),\ p' = \dfrac{p}{p-1}$.

Theorem 3. *Let* $q = \dfrac{1+\gamma}{1+\gamma-\beta}$. *Then the following two condition are equivalent:*

(i) $$\int_{\{x\in R_+ : M_B^\beta(f\omega^\beta)(x)>\lambda\}} \omega(x)x^\gamma dx \le C\lambda^{-q}\left(\int_{R_+}|f(x)|\omega(x)x^\gamma dx\right)^q$$

with a constant C *independent of* f *and* $\lambda > 0$.

(ii) $\omega \in A_1^\gamma(R_+)$.

We will prove a lemma stating a pointwise estimate for the function $I_B^\alpha f(x)$. Estimates of this type for Riesz potentials were obtained in [5], [6].

Lemma 1. *Let* $0 < \alpha < 1+\gamma$, $1 \le p < \dfrac{\lambda}{\alpha}$. *Then there exists a positive number* C *such that for every* $r > 0$ *and* $x \in R_+$

$$\left|I_B^\alpha f(x)\right| \le C\left(r^\alpha M_B f(x) + r^{\alpha-\frac{\lambda}{p}} M_B^{\frac{\lambda}{p}} f(x)\right). \tag{1}$$

Proof. Let $r > 0$ be arbitrary and write $I_B^\alpha f(x)$ in the from

$$I_B^\alpha f(x) = \int_0^\infty T^y x^{\alpha-1-\gamma} f(y) y^\gamma dy = \int_0^\infty y^{\alpha-1-\gamma} T^y f(x) y^\gamma dy =$$

$$= \int_0^r y^{\alpha-1} T^y f(x) dy + \int_r^\infty y^{\alpha-1} T^y f(x) dy = J_1 + J_2 .$$

Then for J_1 we have the estimate

$$|J_1| \le \int_0^r y^{\alpha-1} T^y |f(x)| dy = \sum_{k=0}^\infty \int_{2^{-k-1}r<y<2^{-k}r} y^{\alpha-1} T^y |f(x)| dy \le$$

$$\le \sum_{k=0}^\infty \left(2^{-k-1}r\right)^{\alpha-1-\gamma} \int_0^{2^{-k}r} T^y |f(x)| y^\gamma dy =$$

$$= Cr^\alpha \sum_{k=0}^\infty 2^{-k\alpha}\left(2^{-k}r\right)^{-1-\gamma} \int_0^{2^{-k}r} T^y |f(x)| y^\gamma dy \le Cr^\alpha M_B f(x).$$

Further

$$|J_2| \le \int_r^\infty y^{\alpha-1} T^y |f(x)| dy = \sum_{k=0}^\infty \int_{2^k r<y<2^{k+1}r} y^{\alpha-1} T^y |f(x)| dy \le$$

$$\le \sum_{k=0}^\infty \left(2^k r\right)^{\alpha-1-\gamma} \int_0^{2^{k+1}r} T^y |f(x)| y^\gamma dy =$$

$$=C\sum_{k=0}^{\infty}\left(2^k r\right)^{\alpha-\frac{\lambda}{p}}\left(2^{k+1}r\right)^{\frac{\lambda}{p}-1-\gamma}\int_0^{2^{k+1}r}T^y|f(x)|y^\gamma dy\le Cr^{\alpha-\frac{\lambda}{p}}M_B^{\frac{\lambda}{p}}f(x),$$

since by our assumption we have $\alpha-\dfrac{\lambda}{p}<0$.

Now lemma 1 follows immediately from these two estimates for J_1 and J_2.

Theorem 4. *Let* $0<\lambda\le 1,\ 1<p<\dfrac{\lambda}{\alpha},\ 1\le r\le\infty$ *and* $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{\alpha}{\lambda}+\dfrac{\alpha p}{\lambda r}$. *Then for every function* $f\in L_p^\gamma(R_+)$ *such that* $M_B^{\frac{\lambda}{p}}f\in L_r^\gamma(R_+)$ *the following estimate holds:*

$$\left\|I_B^\alpha f\right\|_{L_q^\gamma(R_+)}\le C\left\|M_B^{\frac{\lambda}{p}}f\right\|_{L_r^\gamma(R_+)}^{\frac{\alpha p}{\lambda}}\|f\|_{L_p^\gamma(R_+)}^{1-\frac{\alpha p}{\lambda}}. \tag{2}$$

Proof. Taking

$$r=r(x)=\left(\frac{M_B^{\frac{\lambda}{p}}f(x)}{M_Bf(x)}\right)^{\frac{p}{\lambda}}$$

in (1), we obtain that

$$\left|I_B^\alpha f(x)\right|\le C\left(M_B^{\frac{\lambda}{p}}f(x)\right)^{\frac{\alpha p}{\lambda}}\left(M_Bf(x)\right)^{1-\frac{\alpha p}{\lambda}} \tag{3}$$

for every $x\in R_+$.

Now inequality (2) follows if we take the q-th power in (3) and apply Hölder's inequality to the right-hand side of the inequality obtained:

$$\int_{R_+}\left|I_B^\alpha f(x)\right|^q x^\gamma dx\le C\int_{R_+}\left(M_B^{\frac{\lambda}{p}}f(x)\right)^{\frac{\alpha pq}{\lambda}}\left(M_Bf(x)\right)^{q-\frac{\alpha pq}{\lambda}}x^\gamma dx\le$$

$$\le C\left(\int_{R_+}\left(M_B^{\frac{\lambda}{p}}f(x)\right)^{\frac{\alpha pqs'}{\lambda}}x^\gamma dx\right)^{1/s'}\left(\int_{R_+}\left(M_Bf(x)\right)^{\left(q-\frac{\alpha pq}{\lambda}\right)s}x^\gamma dx\right)^{1/s},$$

where $\left(q-\dfrac{\alpha pq}{\lambda}\right)s=p,\ s'=\dfrac{s}{s-1}=\dfrac{\lambda r}{\alpha pq},\ \dfrac{1}{q}=\dfrac{1}{p}-\dfrac{\alpha}{\lambda}+\dfrac{\alpha p}{\lambda r}$.

Therefore

$$\left(\int_{R_+}\left|I_B^\alpha f(x)\right|^q x^\gamma dx\right)^{1/q}\le C\left(\int_{R_+}\left(M_Bf(x)\right)^p x^\gamma dx\right)^{1/s}\left(\int_{R_+}\left(M_B^{\frac{\lambda}{p}}f(x)\right)^r x^\gamma dx\right)^{\frac{\alpha pq}{\lambda r}}\le$$

$$\le C\left(\int_{R_+}|f(x)|^p x^\gamma dx\right)^{1/s}\left(\int_{R_+}\left(M_B^{\frac{\lambda}{p}} f(x)\right)^r x^\gamma dx\right)^{\frac{\alpha pq}{\lambda r}}$$

or

$$\left\|I_B^\alpha f\right\|_{L_q^\gamma(R_+)} \le C\|f\|_{L_p^\gamma(R_+)}^{1/sq}\left\|M_B^{\frac{\lambda}{p}} f\right\|_{L_r^\gamma(R_+)}^{\frac{\alpha p}{\lambda}} \le \left\|M_B^{\frac{\lambda}{p}} f\right\|_{L_r^\gamma(R_+)}^{\frac{\alpha p}{\lambda}}\|f\|_{L_p^\gamma(R_+)}^{1-\frac{\alpha p}{\lambda}}.$$

This completes the proof.

One of the fundamental results of this section and, in fact, of the paper is given by

Theorem 5. *Let* $1<p<\frac{1+\gamma}{\alpha}, \frac{1}{q}=\frac{1}{p}-\frac{\alpha}{1+\gamma}$ *and* $f\in L_p^\gamma(R_+)$. *Then* $I_B^\alpha f\in L_q^\gamma(R_+)$ *and the following inequality is valid*

$$\left\|I_B^\alpha f\right\|_{L_q^\gamma(R_+)} \le C\|f\|_{L_p^\gamma(R_+)},$$

where C *does not depend on* f.

Proof. Taking $r=\infty, \lambda=1+\gamma$, we obtain that

$$\left\|I_B^\alpha f\right\|_{L_q^\gamma(R_+)} \le C\left\|M_B^{\frac{1+\gamma}{p}} f\right\|_{L_\infty^\gamma(R_+)}^{\frac{\alpha p}{1+\gamma}}\|f\|_{L_p^\gamma(R_+)}^{1-\frac{\alpha p}{1+\gamma}} = C\|f\|_{L_p^\gamma(R_+)}^{1-\frac{\alpha p}{1+\gamma}} \operatorname*{ess\,sup}_{x\in R_+}\left(M_B^{\frac{1+\gamma}{p}} f(x)\right)^{\frac{\alpha p}{1+\gamma}}.$$

Indeed, from Hölder's inequality we have $M_B^{\frac{1+\gamma}{p}} f(x)\le C\|f\|_{L_p^\gamma(R_+)}$. Then

$$\left\|I_B^\alpha f\right\|_{L_q^\gamma(R_+)} \le C\|f\|_{L_p^\gamma(R_+)}^{1-\frac{\alpha p}{1+\gamma}}\|f\|_{L_\infty^\gamma(R_+)}^{\frac{\alpha p}{1+\gamma}} = C\|f\|_{L_p^\gamma(R_+)}.$$

This completes the proof.

Theorem 6. *Let* $0<\alpha<1+\gamma$, $q=\frac{1+\gamma}{1+\gamma-\alpha}$. *Then there is a constant* $C>0$ *such that for each* $f\in L_1^\gamma(R_+)$ *and* $t>0$

$$\int_{\{x\in R_+: I_B^\alpha f(x)>t\}} x^\gamma dx \le \left(\frac{C}{t}\int_{R_+}|f(x)|x^\gamma dx\right)^q,$$

where C *does not depend on* f.

Proof. Let us note that if (1) holds and if $p=1, \lambda=1+\gamma$ and $r=\infty$, then by the argument used in the proofs of lemma 1 and theorem 2 it is possible to obtain instead of (3) the estimate

$$\left|I_B^\alpha f(x)\right| \le C\left(M_B^{1+\gamma} f(x)\right)^{\frac{\alpha p}{1+\gamma}}\left(M_B f(x)\right)^{1-\frac{\alpha}{1+\gamma}} \le C\|f\|_{L_1^\gamma(R_+)}^{\frac{\alpha}{1+\gamma}}\left(M_B f(x)\right)^{1-\frac{\alpha}{1+\gamma}}. \qquad (4)$$

Further, from inequality (4) and from theorem 1 we derive that

$$\int\limits_{\left\{x\in R_+ : I_B^\alpha f(x)>t\right\}} x^\gamma dx \le \int\limits_{\left\{x\in R_+ : M_B f(x) > C^{-q} t^{-q} \|f\|_{L_1^\gamma(R_+)}^{\frac{\alpha q}{1+\gamma}}\right\}} x^\gamma dx \le \left(\frac{C}{t}\|f\|_{L_1^\gamma(R_+)}\right)^q .$$

In this we give a full description of measures for which weighted estimates for the fractional integral $I_B^\alpha f(x)$ hold, using the method of G.Welland [7].

We start with a lemma

Lemma 2. *For any* ε, $0<\varepsilon<\min(\alpha, 1+\gamma-\alpha)$, *there exists a constant* $c_\varepsilon>0$ *such that for any nonnegative function* $\phi : R_+ \to R$ *and for any point* $x \in R_+$ *the following inequality holds:*

$$I_B^\alpha \phi(x) \le c_\varepsilon \sqrt{M_B^{\alpha-\varepsilon}\phi(x) M_B^{\alpha+\varepsilon}\phi(x)} . \tag{5}$$

Proof. Let r be an arbitrary positive real number. Similarly as in the proof of lemma 1, we write the integral as the sum of two integrals:

$$I_B^\alpha \phi(x) = \int_0^r y^{\alpha-1} T^y \phi(x) dy + \int_r^\infty y^{\alpha-1} T^y \phi(x) dy .$$

For $0<\varepsilon<\alpha$ we have

$$\int_0^r y^{\alpha-1} T^y \phi(x) dy = \sum_{k=0}^\infty \int\limits_{2^{-k-1} r<y<2^{-k} r} y^{\alpha-1} T^y \phi(x) dy \le$$

$$\le \sum_{k=0}^\infty \left(2^{-k-1} r\right)^{\alpha-1-\gamma} \int_0^{2^{-k} r} T^y \phi(x) y^\gamma dy =$$

$$= C r^\varepsilon \sum_{k=0}^\infty 2^{-k\varepsilon} \left(2^{-k} r\right)^{\alpha-\varepsilon-1-\gamma} \int_0^{2^{-k} r} T^y \phi(x) y^\gamma dy \le C r^\varepsilon M_B^{\alpha-\varepsilon} \phi(x) .$$

On the other hand, for $0<\varepsilon<1+\gamma-\alpha$ we have

$$\int_0^r y^{\alpha-1} T^y \phi(x) dy = \sum_{k=0}^\infty \int\limits_{2^k r<y<2^{k-1} r} y^{\alpha-1} T^y \phi(x) dy \le$$

$$\le \sum_{k=0}^\infty \left(2^k r\right)^{\alpha-1-\gamma} \int_0^{2^{k+1} r} T^y \phi(x) y^\gamma dy =$$

$$= C r^{-\varepsilon} \sum_{k=0}^\infty \left(2^k r\right)^{\alpha+\varepsilon-1-\gamma} \int_0^{2^{k+1} r} T^y \phi(x) y^\gamma dy \le C r^{-\varepsilon} M_B^{\alpha+\varepsilon} \phi(x) .$$

Consequently, we obtained that for any ε, $0<\varepsilon<\min(\alpha, 1+\gamma-\alpha)$, there exists a constant $c_\varepsilon>0$ such that for every non-negative function ϕ and for any $x\in R_+$ and $r>0$ we have

$$I_B^\alpha \phi(x) \le c_\varepsilon \left(r^\varepsilon M_B^{\alpha-\varepsilon} \phi(x) + r^{-\varepsilon} M_B^{\alpha+\varepsilon} \phi(x)\right). \tag{6}$$

Taking

$$r^\varepsilon = \left(\frac{M_B^{\alpha+\varepsilon}\phi(x)}{M_B^{\alpha-\varepsilon}\phi(x)}\right)^{\frac{1}{2}}$$

in (6), we obtain (5).

Lemma 2 is proved.

Theorem 7. *Suppose that* $1 < p < \frac{1+\gamma}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{1+\gamma}$. *Then the inequality*

$$\left(\int_{R_+} \left| I_B^\alpha \left(f\omega^\alpha \right)(x) \right|^q \omega(x) x^\gamma dx \right)^{\frac{1}{q}} \le c \left(\int_{R_+} \left| f(x) \right|^p \omega(x) x^\gamma dx \right)^{\frac{1}{p}} \tag{7}$$

holds for any $L^\gamma_{p,\omega}(R_+)$ *with a constant* $c > 0$ *independent of* f *if and only if*

$$\omega \in A^\gamma_\beta(R_+), \ \beta = 1 + \frac{q}{p'}. \tag{8}$$

Proof. If $\omega \in A^\gamma_\beta(R_+)$, then $\omega \in A^\gamma_{\beta-\eta}(R_+)$ for sufficiently small positive η. Therefore it is possible to choose $\varepsilon, 0 < \varepsilon < \min(\alpha, 1+\gamma-\alpha)$, in such a way simultaneously $\omega \in A^\gamma_{\beta_1}(R_+)$ with $\beta_1 = 1 + \frac{p}{p'(1-p(\alpha+\varepsilon))}$ and $\omega \in A^\gamma_{\beta_2}(R_+)$ with $\beta_2 = 1 + \frac{p}{p'(1-p(\alpha+\varepsilon))}$. If we now take

$$\frac{1}{q_\varepsilon} = \frac{1}{p} - (\alpha+\varepsilon), \ \frac{1}{q_\varepsilon} = \frac{1}{p} - (\alpha-\varepsilon),$$

then we obtain that $\omega \in A_{1+\frac{q_\varepsilon}{p'}}$ and $\omega \in A_{1+\frac{\overline{q}_\varepsilon}{p'}}$.

Denoting

$$p_1 = \frac{2q_\varepsilon}{q} \text{ and } p_2 = \frac{2\overline{q}_\varepsilon}{q}$$

we have

$$\frac{1}{p_1} + \frac{1}{p_2} = 1.$$

Put

$$F_1(x) = \left(M_B^{\alpha+\varepsilon}\left(f\omega^\alpha \right)(x) \right)^{\frac{q}{2}} \omega(x)^{\frac{1}{p_1}}$$

and

$$F_2(x) = \left(M_B^{\alpha-\varepsilon}\left(f\omega^\alpha \right)(x) \right)^{\frac{q}{2}} \omega(x)^{\frac{1}{p_2}}, \quad (p_1 \to p_2, F_1 \to F_2).$$

Further, (5) together with Hölder's inequality implies the estimate

$$\int_{R_+} \left| I_B^\alpha \left(f\omega^\alpha \right)(x) \right|^q w(x) x^\gamma dx \le c_\varepsilon \int_{R_+} F_1(x) F_2(x) x^\gamma dx \le$$

$$\le c_\varepsilon \left(\int_{R_+} \left(M_B^{\alpha+\varepsilon}\left(f\omega^\alpha \right)(x) \right)^{\frac{qp_1}{2}} \omega(x) x^\gamma dx \right)^{\frac{1}{p_1}} \left(\int_{R_+} \left(M_B^{\alpha-\varepsilon}\left(f\omega^\alpha \right)(x) \right)^{\frac{qp_2}{2}} \omega(x) x^\gamma dx \right)^{\frac{1}{p_2}} =$$

$$= c_\varepsilon \left(\int_{R_+} \left(M_B^{\alpha+\varepsilon}\left(f\omega^\alpha \right)(x) \right)^{q_\varepsilon} \omega(x) x^\gamma \right)^{\frac{1}{p_1}} \left(\int_{R_+} \left(M_B^{\alpha-\varepsilon}\left(f\omega^\alpha \right)(x) \right)^{\overline{q}_\varepsilon} \omega(x) x^\gamma \right)^{\frac{1}{p_2}}.$$

Finally, using theorem 1 we conclude that

$$\left\| I_B^\alpha \left(f\omega^\alpha \right) \right\|_{L^\gamma_{q,\omega}(R_+)} \le c \left\| f \right\|_{L^\gamma_{p,\omega}(R_+)}.$$

The implication (7)⇒(8) follows from the pointwise inequality

$$M_B^\alpha\left(f\omega^\alpha\right)(x)\le c_1 I_B^\alpha\left(|f|\omega^\alpha\right)(x)$$

and theorem 1.

Theorem 7 is proved.

For Riesz potentials, Theorem 6 is due to B.Muckenhoupt and R.L.Wheeden [8]. It was proved by using above described method by G.Welland [7].

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