GADJIEVA E.A., GASANOVA T.KH.

APPROXIMATION BY TWO DIMENSIONAL BERNSTEIN-CHLODOWSKY POLYNOMIALS IN TRIANGLE WITH MOBILE BOUNDARY

Abstract

The authors present a construction of Bernstein-Chlodowsky Polynomials of function of two variable, corresponding to the triangular domain with mobile boundary and give the theorems on weighted approximation of continuous functions by these polynomials.

Key words. Bernstein-Chlodowsky Poynomials, linear Positive Operators, Korovkins theorem, weighted space, triangle with mobile boundary.

Bernstein-Chlodowsky polynomials were introduced by Chlodowsky in 1932 as a generalization of classical Bernstein polynomials to the interval $[0,b_n]$, where b_n tends to infinity with the n. Some basic results concerning these polynomials may be found in the books [1],[2]. A new results on approximation and on the order of approximation of functions by Bernstein-Chlodowsky polynomials and its generalizations were obtained in the papers [3],[4],[5].

This paper is devoted to the problem of weighted approximation of continuous and unbounded functions of two variables by two-dimensional Bernstein-Chlodowsky polynomials in triangle with mobile boundary. Note that two dimensional Bernstein polynomials, corresponding to the triangle domain (naturally, with a fixed boundary) were studied in [6].

We will use the following notations. Let $R_+^2=\{(x,y):x\ge 0,y\ge 0\}$ and for any $a>0$ $\Delta_a=\{(x,y)\in R_+^2 : x+y\le a\}$ is a triangle with the fixed boundary $x+y=a$. For $b_n\to\infty$ as $n\to\infty$, we will considered also a triangle Δ_{b_n} with the mobile boundary $x+y=b_n$, which tend to infinity as $n\to\infty$.

Let

$$\rho(x,y)=1+x^2+y^2$$

and denote by $C_\rho(R_+^2)$ the space of all functions f, which are continuous in R_+^2 and satisfies the inequality

$$|f(x,y)|\le M_f\rho(x,y), \tag{1}$$

where M_f is constant depending on function f only.

Obviously, $C_\rho(R_+^2)$is a linear normed space with norm

$$\|f(x,y)\|_\rho=\sup_{(x,y)\in R_+^2}\frac{|f(x,y)|}{1+x^2+y^2}. \tag{2}$$

Denote also by $C_\rho^0(R_+^2)$ a subspace of functions $f\in C_\rho(R_+^2)$, for which

$$\lim_{x+y\to\infty}\frac{f(x,y)}{1+x^2+y^2}=0. \tag{3}$$

Let $\{b_n\}$ is an increasing sequence of positive numbers such that

$$\lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{4}$$

Let $(x,y) \in \Delta_{b_n}$. Using the equality

$$1 = \sum_{k=0}^{n} C_n^k \left(1 - \frac{x+y}{b_n}\right)^{n-k} \sum_{j=0}^{k} C_k^j \left(\frac{x}{b_n}\right)^j \left(\frac{y}{b_n}\right)^{k-j} \tag{5}$$

consider the following Bernstein-Chlodowsky polynomials of functions of two variables, corresponding to the triangle Δ_{b_n}

$$B_n(f;x,y) = B_n(f(t,\tau);x,y) =$$
$$= \sum_{k=0}^{n} C_n^k \left(1 - \frac{x+y}{b_n}\right)^{n-k} \sum_{j=0}^{k} C_k^j \left(\frac{x}{b_n}\right)^j \left(\frac{y}{b_n}\right)^{k-j} f\left(\frac{j}{n} b_n, \frac{k-j}{n} b_n\right), \tag{6}$$

where $x + y \le b_n$ and $f \in C_\rho(R_+^2)$.

Lemma 1. *a) The polynomials (6) has the following properties*

$$B_n(1;x,y) = 1, \tag{7}$$
$$B_n(t;x,y) = x, \tag{8}$$
$$B_n(\tau;x,y) = y, \tag{9}$$
$$B_n(t^2;x,y) = x^2 + \frac{x(b_n - x)}{n}, \tag{10}$$
$$B_n(\tau^2;x,y) = y^2 + \frac{y(b_n - y)}{n}. \tag{11}$$

b) (6)is a sequence of linear positive operators, acting from $C(R_+^2)$ to $C(R_+^2)$.

Proof. (7) follows from (5), (8)-(11) may be proved by direct calculations. To prove part b) note that if $f \in C(R_+^2)$ then by (2)

$$\left| f\left(\frac{j}{n} b_n, \frac{k-j}{n} b_n\right)\right| \le \|f\|_\rho \cdot \left(1 + \frac{j^2}{n^2} b_n^2 + \frac{(k-j)^2}{n^2} b_n^2\right)$$

and therefore

$$|B_n(f;x,y)| \le \|f\|_\rho \left(B_n(1;x,y) + B_n(t^2;x,y) + B_n(\tau^2;x,y)\right)$$

Now, by (7),(10),(11)

$$|B_n(f;x,y)| \le \|f\|_\rho \left(1 + x^2 + \frac{x(b_n - x)}{n} + y^2 + \frac{y(b_n - y)}{n}\right) \le$$
$$\le M\|f\|_\rho \left(1 + x^2 + y^2\right),$$

since $\frac{b_n}{n}$ is bounded by (4). The positively of linear operator (6) is obvious because $x + y \le b_n$.

Lemma 2. *Let $f \in C(R_+^2)$. Then for any fixed $a > 0$, a polynomials $B_n(f;x,y)$ uniformly convergence to $f(x,y)$ in the triangle Δ_a, that is*

$$\lim_{n\to\infty} \max_{(x,y)\in\Delta_a} |B_n(f;x,y) - f(x,y)| = 0.$$

Proof. (7),(8) and (9) gives

$$\max_{(x,y)\in\Delta_a}\left|B_n(1;x,y)-1\right|=0,$$
$$\max_{(x,y)\in\Delta_a}\left|B_n(t;x,y)-x\right|=0,$$
$$\max_{(x,y)\in\Delta_a}\left|B_n(\tau;x,y)-y\right|=0.$$

From (10) and (11) we obtain

$$\max_{(x,y)\in\Delta_a}\left|B_n\left(t^2+\tau^2;x,y\right)-\left(x^2+y^2\right)\right|=\max_{(x,y)\in\Delta_a}\left|\frac{x(b_n-x)+y(b_n-y)}{n}\right|\le 2a\frac{b_n}{n}$$

and consequently

$$\lim_{n\to\infty}\max_{(x,y)\in\Delta_a}\left|B_n\left(t^2+\tau^2;x,y\right)-\left(x^2+y^2\right)\right|=0.$$

Therefore, as $n\to\infty$, uniformly in the triangle Δ_a

$$B_n(1;x,y)\to 1, B_n(t;x,y)\to x, B_n(\tau;x,y)\to y$$
$$B_n\left(t^2+\tau^2;x,y\right)\to x^2+y^2.$$

Hence satisfies all conditions of two dimensional Korovkin's type theorem (see [8]) and application of this theorem gives desired result.

Lemma is proved.

Using these propositions we can prove the weighted approximation theorems by polynomials (6) in triangle Δ_{b_n} with the mobile boundary.

Theorem 1. *Let $f\in C\left(R_+^2\right)$. Then for any $\alpha>0$*

$$\lim_{n\to\infty}\sup_{(x,y)\in\Delta_{b_n}}\frac{\left|B_n(f;x,y)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\alpha}}=0,$$

where b_n is a sequence, satisfying (4).

Proof. Given $\varepsilon>0$, these exists a positive number a such that

$$\left(1+x^2+y^2\right)^{-\alpha}<\varepsilon \quad \text{if} \quad x+y>a. \tag{12}$$

Since for a large n a triangle Δ_{b_n} consist Δ_a, we can write

$$\sup_{(x,y)\in\Delta_{b_n}}\frac{\left|B_n(f;x,y)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\alpha}}\le\sup_{(x,y)\in\Delta_a}\frac{\left|B_n(f;x,y)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\alpha}}+$$
$$+\sup_{(x,y)\in\Delta_{b_n}\setminus\Delta_a}\frac{\left|B_n(f;x,y)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\alpha}}=I_n'+I_n''.$$

Using part b) of lemma 1 and (12) we obtain

$$I_n''\le\left(1+x^2+y^2\right)^{-\alpha}\left(\frac{\left|B_n(f;x,y)\right|}{1+x^2+y^2}+\frac{\left|f(x,y)\right|}{1+x^2+y^2}\right)<C\varepsilon.$$

Moreover by lemma 2 $\lim_{n\to\infty}I_n'=0$ and the proof is completed.

Note. In one-dimensional case a Korovkins type theorems in the space $C_\rho(R)$ were proven in [7], where shown that in general the convergence in norm of the space $C_\rho(R)$ not holds. By this reason we can not take in the theorem 1 $\alpha=0$. But, as in [7], we can show that the convergence in the norm with $\alpha=0$ take place for any function, belonging to the subspace $C_\rho^0\left(R_-^2\right)$.

Theorem 2. *Let $f \in C_\rho^0(R_+^2)$. Then*

$$\lim_{n\to\infty} \sup_{(x,y)\in\Delta_{b_n}} \frac{|B_n(f;x,y)-f(x,y)|}{1+x^2+y^2}=0 .$$

Proof. By the definition of $C_\rho^0(R_+^2)$ we have

$$\lim_{x+y\to\infty}\frac{f(x,y)}{1+x^2+y^2}=0, \quad \lim_{n\to\infty}\frac{f\left(\frac{j}{n}b_n,\frac{k-j}{n}b_n\right)}{1+\frac{j^2}{n^2}b_n^2+\frac{(k-j)^2}{n^2}b_n^2}=0$$

and therefore for $\varepsilon>0$ we can found a number a such that

$$|f(x,y)|<\varepsilon(1+x^2+y^2) \text{ if } x+y>a \tag{13}$$

and a number n_0 such that

$$\left|f\left(\frac{j}{n}b_n,\frac{k-j}{n}b_n\right)\right|<\varepsilon\left(1+\frac{j^2}{n^2}b_n^2+\frac{(k-j)^2}{n^2}b_n^2\right) \quad \text{if} \quad n>n_0 . \tag{14}$$

Therefore

$$\sup_{(x,y)\in\Delta_{b_n}}\frac{|B_n(f;x,y)-f(x,y)|}{1+x^2+y^2}\le \sup_{(x,y)\in\Delta_a}\frac{|B_n(f;x,y)-f(x,y)|}{1+x^2+y^2}+$$

$$+\sup_{(x,y)\in\Delta_{b_n}\setminus\Delta_a}\frac{|B_n(f;x,y)-f(x,y)|}{1+x^2+y^2}=i_n'+i_n''$$

and by lemma 2 it sufficient to show that $i_n''\to 0$ as $n\to\infty$.

By (13) and (14)

$$i_n''\le\varepsilon+\sup_{(x,y)\in\Delta_{b_n}\setminus\Delta_a}\frac{|B_n(f;x,y)|}{1+x^2+y^2}\le$$

$$\le\varepsilon+\varepsilon\sup_{(x,y)\in\Delta_{b_n}\setminus\Delta_a}\frac{B_n(1;x,y)+B_n(t^2;x,y)+B_n(\tau^2;x,y)}{1+x^2+y^2}=$$

$$=\varepsilon\left(1+\sup_{(x,y)\in\Delta_{b_n}\setminus\Delta_a}\frac{1+x^2+\frac{x(b_n-x)}{n}+y^2+\frac{y(b_n-y)}{n}}{1+x^2+y^2}\right)\le C\varepsilon ,$$

where C independent on n.

Hence for $n>n_0$

$$i_n''<C\varepsilon ,$$

which gives a proof.

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Gadjieva E.A.
Azerbaijan Technical University.
25, H.Javid av., 370073, Baku, Azerbaijan.

Gasanova T.Kh.
Institute of Mathematics and Mechanics of AS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-91-92.

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