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ON THE BEHAVIOR OF SOLUTIONS OF NON-LINEAR PARABOLIC EQUATIONS IN NON-REGULAR DOMAINS

Abstract

The behavior of solutions of nonlinear equations in unbounded domains with nonsmooth boundary and in unbounded domain with non-compact boundary depending on geometry of domain.

In the work a priori estimations are obtained for generalized solutions of the mixed problems in the bounded and unbounded domains depending on geometry of domain being analogously to Saint-Venant's known principle in the theory of elasticity. On the basis of the obtained estimations the alternative theorems type of the Phragmen-Lindelof theorem on the behavior of solutions with unbounded integral of energy in unbounded domains, estimations near finite boundary points, theorems on the removable singularity of solutions on the boundary of boundary domain are established.

The classes of uniqueness in unbounded domains of generalized solutions of the mixed problems being analogous to classes of A.N.Tikhonov, S.Teklind for the heat equations are obtained.

For the linear equations, energetic estimations analogous to Saint-Venant principle are received by O.A.Oleynik and so on [3], for some classes of non-linear equations- by A.E.Shishkov, [4].

Let Ω be some boundary domain in R^3, $Q=\Omega\times(0,T)$, $T<\infty$, $\partial Q=\Gamma_0\cup\Gamma_T\cup\Gamma$, where $\Gamma_0(\Gamma_T)=\partial Q\cap\{(x,t):t=0(T)\}$. In Q we consider the next problem:

$$u_t+\Delta^2 u-\Delta f(u)=0\,, \tag{1}$$

$$u(x,0)=0\,, \tag{2}$$

$$u|_\Gamma=\left.\frac{\partial u}{\partial n}\right|_\Gamma=0\,. \tag{3}$$

The equations of kind (1) with corresponding selection of non-linearity describe various processes of hydrodynamics and in the theory of combustion. So, at $f(u)=-u+\alpha_1 u^2+\alpha_2 u^3$ (α_1,α_2 are constants, $\alpha_2>0$) the equation (1) is called the equation of Chan-Hillard's and describes the process of formation of a flame in phase transitions [5] and at $f(u)=\alpha u-\frac{1}{2}u^2$ $(\alpha>0)$ is called the equation of Curamoto-Sivashinsky's and described the process of cellular instability at hardening the diluted binary mixes [6].

First of all let's consider the equation of Chan-Hillard's type. Let $Q(r)=Q\cap\{B_0(r)\times(0,T)\}$, $S(r)=\partial Q\cap\{B_0(r)\times(0,T)\}$ where $B_0(r)=\{x:g(x)<r\}$, $g(x)$ is a parametric function with the next properties: $g(x)$ is non-negative in $\overline{\Omega}$, $g(x)\in C^m_{loc}(\overline{\Omega})$, such that $g(0)=0$ and for $x\in\overline{\Omega}$ holds the next estimation

$$|\nabla g(x)|\ge h_1>0,\ \left|\nabla^j g(x)\right|\le h_2(g(x))^{-j+1},\quad j=1,2,...,m,\qquad 0<h_2<\infty$$

Geometry ∂Q will be described by a non-linear basic frequency $\lambda_p(r,\tau)$ of sections $\sigma(r,\tau)=S(r)\cap Q(\tau)$

$$\lambda_p^p(r,\tau)=\inf\left(\int\limits_{\sigma(r,\tau)}|\nabla_s v|^p d\sigma\right)\left(\int\limits_{\sigma(r,\tau)}|v|^p d\sigma\right)^{-1},$$

where the lower bound in taken from all differentiable in some neighborhood of $\sigma(r,\tau)$ functions converging to 0 on ∂Q. $\nabla_s v(x)$ is the projection of vectors $\nabla v(x)$ on hyperplane tangent to $\sigma(r,\tau)$ at the point x.

The function $u(x,t)\in L_p\left(0,T;\overset{0}{W}{}^2_{2,loc}(Q(r))\right)\cap W_2^1(0,T;L_{2,loc}(Q(r)))$ is called a generalized solution of problem (1)-(3) if the integral identity

$$\int\limits_Q \frac{\partial u}{\partial t} v dxdt+\int\limits_Q \sum_{|\alpha|=2} D^\alpha u D^\alpha v dxdt-\int\limits_Q \sum_{|\alpha|=2} f(u) D^\alpha v dxdt=0 \tag{4}$$

is fulfilled for any function $v(x,t)\in L_2\left(0,T;\overset{0}{W}{}_2^2(Q'(r))\right)$, where $Q'(r)$ is any bounded sub-domain $Q(r)$.

Lemma 1. *Let* $J(t)$ *be a continuous non-decreasing function on* (t_0,∞) *satisfying the inequality*

$$J(t)\le\theta J(t\psi(t)),\quad 0<\theta<1,\ \psi(t)=1+\varphi(t),\quad \varphi(t)>0$$

and measurable function $\varphi(t)$ *is such that*

$$(\varphi(t))^{-1}\inf_{t<\tau<t\psi(t)}\varphi(t)\ge\nu>0. \tag{5}$$

Then for $J(t)$ *it holds the estimation*

$$J(t)\le\theta\exp\left(-\nu\ln\theta^{-1}\int\limits_{t_0}^{t}\frac{d\tau}{\tau\varphi(\tau)}\right)J(t_0).$$

Let's denote

$$\lambda^2_{\mu(s)}(r,\tau)\equiv\lambda_2^2(r,\tau)+\mu(s),\quad s,\tau>0.\ \ J_{\mu(s),2}(r,\tau)=\int\limits_{\Omega_\tau(r)}\left(\left|\nabla^2 u\right|^2+\mu^2(s)u^2\right)dx,$$

$$\Omega_\tau(r)=Q(r)\cap\{(x,t):t=\tau\},\quad M_\tau(s,s_2)=\Omega_\tau(s_1)\setminus\Omega_\tau(s_2).$$

Suppose that the condition is fulfilled on $\mu(\tau)$

$$0<h\le\mu(\tau\psi(\tau))(\mu(\tau))^{-1}\le H<\infty,\quad \forall\tau>\tau_0, \tag{6}$$

$$\theta\exp\left[\left(2\mu^2(\tau\psi(\tau))\right)-2\mu^2(\tau)\right]T\le\omega<1,\quad \forall\tau>\tau_0. \tag{7}$$

Lemma 2. *For the function* $u(x)\in\overset{\circ}{W}{}^m_p(\Omega)$ *the inequality is true at* $0<s_1<s_2<\infty,\ j\le m$

$$\int\limits_{M_\tau(s_1,s_2)}\left|\nabla_x^j u\right|^2\lambda_{\mu(s)}(g(x),\tau)dx\le\frac{h_2}{h_1}\int\limits_{M_\tau(s_1,s_2)}\left(\left|\nabla_x^{j-1}u\right|^2+\mu^2(s)\left|\nabla^j u\right|^2\right)dx. \tag{8}$$

This lemma is proved with the help of transition to integration on superficial levels of function $g(x)$.

Lemma 3. *For the function* $u(x)\in\overset{\circ}{W}{}^m_p(\Omega)$ *the inequality*

$$\mu(r)^{\frac{2(m-j)}{m}}\int\limits_{\Omega_\tau(r)}\left|\nabla_x^j u\right|^2 dx\le\int\limits_{\Omega_\tau(r)}\left|\nabla^m u\right|^2 dx+C_1\mu^2(r)\int\limits_{\Omega_\tau(r)}|u|^2 dx \tag{9}$$

is true.

The inequality (9) is obtained from Nirenberg-Galyardo interpolational inequality in the form

$$\int_{\Omega_\tau(r)} \left|\nabla^j u\right|^2 dx < \varepsilon \int_{\Omega_\tau(r)} \left|\nabla^m u\right|^2 dx + C\varepsilon^{-\frac{j}{m-j}} \int_{\Omega_\tau(r)} |u|^2 dx, \quad \varepsilon > 0 .$$

Let define the function $\psi(\tau)$ by the inequality

$$\inf_{\substack{\tau < g(x) < \tau\psi(\tau) \\ 0 < t < T}} \lambda_{\mu(\tau)}(g(x), t)\tau(\psi(\tau) - 1) \geq h_0 > 0, \quad \forall \tau > \tau_0, \tag{10}$$

here $h_0 > 0$ such that $\psi(\tau) - 1 > 0, \quad \forall \tau > \tau_0$.

Let put in integral identity (4) as test function

$$v(x,t) = u(x,t)\eta_\tau(x)\exp\left(-2\mu^2(\tau)t\right) \equiv u(x,t)\left[1 - \xi\left(\frac{\varphi(\tau) - g(x)\tau^{-1}}{\psi(\tau) - 1}\right)\right]\exp\left(-2\mu^2(\tau)t\right),$$

where $\mu(\tau)$ is above determined function, $\xi(s)$ is a special cut off function. After substitution, making corresponding estimations, using lemma 2,3 and also imbedding theorem $W_2^1(\Omega) \to L_p(\Omega)$, $p \leq \frac{2n}{n-2}$ when $n = 3$ we'll get the next estimation for $\forall \varepsilon > 0$, $\omega < 1$

$$J_{\mu(\tau),2}(\tau) \leq (\omega + \varepsilon)J_{\mu(\tau),2}(\tau\psi(\tau)), \quad \forall \tau > \tau_0(\varepsilon). \tag{11}$$

Using lemma 1 on the basis of (11) we'll get next theorem

Theorem 1. *Let suppose that $u(x,t)$ is a generalized solution of problems (1)-(3). $\mu(\tau)$, $\psi(\tau) > 1$ are measurable local bounded functions that satisfy the conditions (6), (7), (10) and $\varphi(\tau) \equiv \psi(\tau) - 1$, the condition (5) of lemma 1 with some $\nu > 0$. Then for the integral of energy $J_{\mu(\tau),2}$ it holds the next estimation $\forall \varepsilon > 0$, $\forall \tau > \tau_0(\varepsilon)$*

$$J_{\mu(\tau),2}(\tau) \leq \Phi_\varepsilon(\tau, \tau_0)J_{\mu(\tau),2}(\tau_0) \equiv (\omega + \varepsilon)\exp\left(-\nu\ln(\omega + \varepsilon)^{-1}\int_{\tau_0}^{\tau} \frac{d\tau}{(\tau\psi(\tau) - 1)}\right)J_{\mu(\tau),2}(\tau_0). \tag{12}$$

Corollary 1. *The generalized solution $u(x,t)$ of problems (1)-(3) will be unique.*

This fact directly follows from theorem 1. The estimation (12) gives a class of uniqueness of generalized solutions. For the classical solutions it is Tikhonov's class. From this estimation by the known scheme [3] proved the existence of a generalized solution in the same class of increasing functions in which the uniqueness is fixed.

For the arbitrary $S \in \partial\Omega$ let's denote by $\overset{\circ}{W}{}_p^m(\Omega, S)$ the closure in the norm $W_p^m(\Omega)$ the set of functions from $C^\infty(\Omega)$ converging to zero at the neighborhood of $\partial\Omega \setminus S$.

Theorem 2. *Let $u(x,t)$ be a generalized solution of problems (1)-(3) from*
$L_p\left(0, T; \overset{\circ}{W}{}_2^2(\Omega_t, \Gamma_0)\right) \cap W_2^1\left(0, T; L_{2,loc}(\Omega_t)\right)$.

Then special set Γ_0 of the solution $u(x,t)$ is removed, i.e.

$$u(x,t) \in L_p\left(0, T; \overset{\circ}{W}{}_{2,loc}^2(\Omega_t)\right) \cap W_2^1\left(0, T; L_{2,loc}(\Omega_t)\right).$$

Remark 1. The theorem 1 is true for generalized solutions of problems (1)-(3) in case when equation (1) is non-divergent

$$\frac{\partial u}{\partial t}+\sum_{|\alpha|\le 4}a_\alpha(x)D^\alpha u(x,t)=\Delta f(u).$$

Here with respect $a_\alpha(x)$ the fulfillment of conditions of uniform ellipticity and $a_\alpha(x)\in C^{|\alpha|-2}(\overline{\Omega})$ at $|\alpha|\ge 2$ is supposed. $a_\alpha(x)$ are measurable and bounded at $|\alpha|<2$.

Let's consider the problem (1)-(3) in a unbounded domain Q of Euclidean space $R^{n+1}_{x,t}$ having a non-compact boundary.

Lemma 4. *Let $J(t)$ be a continuous non-decreasing on (t_0,∞) function satisfying the inequality*

$$J(t)\le \theta J(t\psi(t)),\quad 0<\theta<1,\ \psi(t)=1+\varphi(t),\quad \varphi(t)>0$$

and a measurable function $\varphi(t)$ is such that

$$(\varphi(t))^{-1}\inf_{t<\tau<t\psi(t)}\varphi(\tau)\ge \nu>0. \tag{13}$$

Then for $J(t)$ holds the estimation $J(t)\ge\theta\exp\left(\nu\ln\theta^{-1}\int_{t_0}^{t}\frac{d\tau}{\tau\varphi(\tau)}\right)J(t_0)$.

In this case the next theorem is true.

Theorem 3. *Let $u(x,t)$ be a generalized solution of problems (1)-(3) in an unbounded domain Q such that $u(x,t)\in L_{6,loc}(Q)$. $\mu(\tau),\psi(\tau)>1$ are measurable local bounded functions and satisfy the conditions (6), (7), (10) and $\varphi(\tau)=\psi(\tau)-1$, condition (13) of lemma 4 with some $\nu>0$. Then for integral of energy $J_{\mu,2}(\tau)$ is true: either $J_{\mu,2}(\tau)<c<\infty$ or $J_{\mu,2}(\tau)$ is increasing so quickly that for $\forall\varepsilon>0$, $\forall\tau>\tau_0(\varepsilon)$ holds the estimation at $\omega<1$*

$$J_{\mu,2}(\tau)\ge\Phi_\varepsilon(\tau,\tau_0)J_{\mu,2}(\tau_0)\equiv(\omega+\varepsilon)\exp\left(\nu\ln(\omega+\varepsilon)^{-1}\int_{\tau_0}^{\tau}\frac{d\tau}{\tau\psi(\tau)-1}\right)J_{\mu,2}(\tau_0).$$

Remark 2. Above considered results are true for the Cyramoto-Sivashinsky equation.

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