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IMBEDDING THEOREMS FOR ANISOTROPIC BANACH-VALUED SOBOLEV SPACES

Abstract

In this paper imbedding theorems on the anisotropic Banach-valued Sobolev function spaces $W_{\mathbf{p}}^{l}(G;B)$ are proved.

Imbedding theorems by means of integral representations of functions by their derivatives for the first time was established by S.L. Sobolev (see for instance [1]). Then integral representations method has been developed in papers by V.P. Il'in, and in particular was transferred to the case of representations by difference [2]. Integral representations on a flexible horn [3] were constructed by O.V. Besov. In the paper, by using integral representations on a flexible horn, imbedding theorems for anisotropic banach-valued Sobolev spaces $W_{\mathbf{p}}^{l}(G;B)$ were obtained.

Let N be a set of natural numbers, $N_0^n = N^n \bigcup \{0\}$, $n \in N$, $R_0^n \equiv R^n \setminus \{0\}$ e^i basis vector of a standard basis in R^n, $x = (x_1,...,x_n) = \sum_1^n x_i e_i$, $y \in R^n$, $(x,y) = \sum_1^n x_i \cdot y_i$, $|x| = (x,x)^{1/2}$.

By $\alpha = (\alpha_1,...,\alpha_n)$, $k = (k_1,...,k_n)$ we denote a multiindex with integer non-negative components $|\alpha| = \sum_1^n \alpha_i$, $(\alpha,k) = \sum_1^n \alpha_i \cdot k_i$.

We say that a Banach space B is ζ-convex (or is convex by Burkholder [8]), if there exists a symmetric function $\zeta(a,b)$ on $B \times B$ convex on each of variables and satisfying the conditions:

$$\zeta(0,0) > 0, \ \zeta(a,b) \le \|a+b\|_B \ \text{ for } \ \|a\|_B = \|b\|_B = 1.$$

Let $p = (p_1,...,p_n)$, $1 \le p_i \le \infty$ $(i = 1,...,n)$.

By $L_p(G,B)$ we shall denote a space of strongly measurable on $G \subset R^n$ of B-valued functions $f(x)$, for which the norm

$$\|f\|_{L_p(G,B)} = \|f\|_{p,B} = \|f\|_{(p_1,...,p_n),G,B} =$$

$$= \left\{ \int_R \left[\cdots \left\{ \int_R \left(\int_R \|f(x)\|_B^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right\}^{p_3/p_2} \cdots \right]^{p_n/p_{n-1}} dx_n \right\}^{1/p_n}$$

is finite.

Let us agree to write in the sequel $p \ge q$ or $p > q$, where $p = (p_1,...,p_n)$, $q = (q_1,...,q_n)$ if respectively $p_i \ge q_i$ $(i = 1,...,n)$ or $p_i > q_i$ $(i = 1,...,n)$; in particular, $1 \le p \le \infty$ $(1 = (1,...,1),\ \infty = (\infty,...,\infty))$ means that $1 \le p_i \le \infty$ $(i = 1,...,n)$.

Let a vector $a = (a_i,...,a_n)$, $a_i > 0$ $(i = 1,...,n)$ be given, and the function $\rho(x)$ be the a positive solution of the equation

$$\sum_{i=1}^{n} x_i^2 \rho^{-2a_i} = 1 .$$

Let $K : R_0^n \to R$ be an anisotropic singular kernel satisfying the conditions:

1) $K(t^a x) \equiv K(t^{a_1} x_1, \ldots, t^{a_n} x_n) = t^{-|a|} K(x)$

for any $t > 0$, $x \in R_0^n$;

2) $\int_{S^{n-1}} K(x) \sum_{i=1}^{n} a_i x_i^2 d\sigma(x) = 0$;

3) $\int_0^1 \omega_K(t) \frac{dt}{t} < \infty$,

where

$$\omega_K(t) = \sup\{|K(x) - K(y)| : x, y \in S^{n-1}, |x - y| \le t\},$$

S^{n-1} is a unit sphere in R^n, $d\sigma$ is a surface measure on it.

Let $\varepsilon > 0$ be a number. For vector-functions $f : R^n \to B$ we put

$$T_\varepsilon f(x) = \int_{\rho(x-y) > \varepsilon} K(x - y) f(y) dy ,$$

where the integral is understood in Bochner-Lebesgue sense.

Theorem 1 [9]. *Let B_1, B_2 be two Banach spaces, $p_0 \in (1, \infty)$, $M > 0$. Let $A : L_{\mathbf{p}_0}(R^n, B_1) \to L_{\mathbf{p}_0}(R^n, B_2)$ be a linear bounded operator*

1^0. $\|Af\|_{L_{\mathbf{p}_0}(R^n, B_2)} \le M \|f\|_{L_{\mathbf{p}_0}(R^n, B_1)}$, $\forall f \in L_{\mathbf{p}_0}(R^n, B_1)$.

Let there exist a local, summable on $R^n \setminus \{0\}$ function $x \to K(x)$, where $K(x)$ is a linear bounded operator from B_1 to B_2, for which at some $r > 0$

2^0. $\int_{\rho(x) > r\rho(y)} \| K(x) - K(x - y) \|_{B_1 \to B_2} dx \le M$, $\forall y \in R^n$.

Let

$$Af(x) = \int K(x - y) f(y) dy$$

for each function $f \in L_{\mathbf{p}_0}(R_n, B_1)$ with a compact support at almost for all $x \notin \operatorname{supp} f$.

Then for all $p \in (1, \infty)$

$$\|Af\|_{L_{\mathbf{p}}(R^n, B_2)} \le C_{\mathbf{p}} M \|f\|_{L_{\mathbf{p}}(R^n, B_1)}.$$

Consider an anisotropic singular integral operator (a.s.i.o.)

$$Tf(x) = \int_{R^n} K(x - y) f(y) dy := \lim_{\varepsilon \to 0+} T_\varepsilon f(x). \tag{1}$$

Consider the set

$$L_p(R^n) \otimes B = \left\{ \sum_j f_j(x) b_j \text{ (the sum is finite) } f_j \in L_p, b_j \in B \right\}.$$

Theorem 2. *Let B be ζ-convex Banach lattice and the kernel K satisfy the conditions 1)-3). Let $f \in L_{\mathbf{p}}(R^n; B)$, $\mathbf{p} \in (1, \infty)^n$. Then the extension of the operator* T *to B-valued functions (we shall denote this extension by the same symbol) is bounded in*

$L_{\mathbf{p}}(R^n;B)$ *that is, for any* $f \in L_{\mathbf{p}}(R^n;B)$ *it is valid the inequality*

$$\|Tf\|_{L_{\mathbf{p}}(R^n;B)} \le C_p(B)\|f\|_{L_{\mathbf{p}}(R^n;B)},$$

and the operator T *continuously extends in* $L_{\mathbf{p}}(R^n;B)$, *and also it holds the inequality*

$$\|T^* f\|_{L_{\mathbf{p}}(R^n;B)} \le C_p(B)\|f\|_{L_{\mathbf{p}}(R^n;B)},$$

where $C_p(B)$ *doesn't depend on* f.

This theorem has been proved in paper [6] by V.S. Guliev in the case when $p_1 = p_2 = ... = p_n$. From this and theorem 1 the validity of theorem 2 follows.

Theorem 2 in the isotropic case has been proved by a relation method in the paper [5] by A.V. Bukhvalov, when B is a ζ-convex Banach space.

Definition 1 [4]. *Let* $\lambda = (\lambda_1,...,\lambda_n) \in (0,\infty)^n$, $I = [-1,1]^n$. *The open set* $G \subset R^n$ *we call a set with a flexible* λ*-horn condition, if for some* $\theta \in (0,1]$, $T \in (0,\infty)$ *for any* $x \in G$ *there exists the path*

$$\rho(t^\lambda, x) = (\rho_1(t^{\lambda_1}, x),..., \rho_n(t^{\lambda_n}, x)), \quad 0 \le t \le T$$

with the following properties:

1^0. *For all* $i \in \{1,...,n\}$ *the functions* $u \mapsto \rho_i(u,x)$ *are absolutely continuous on* $[0, T^{\lambda_i}]$; $|\rho_i'(u,x)| \le 1$ *for almost all* $u \in [0, T^{\lambda_i}]$, *where* $\rho_i'(u,x) = \frac{\partial}{\partial u}\rho_i(u,x)$;

2^0. $\rho(0,x) = 0$,

$$x + V(\lambda, x, \theta) = x + \bigcup_{0 \le t \le T}\left[\rho(t^\lambda, x) + t^\lambda \theta^\lambda I\right] \subset G.$$

Put $D_j = \frac{\partial}{\partial x_j}$, $D^k = D_1^{k_1}...D_n^{k_n}$. Let $C_0^\infty(G)$ be the totality of all numerical infinitely differentiable finite in G functions. Let f and $D^k f$ be locally summable by Bochner functions on an open set $G \subset R^n$. If for any function $\varphi \in C_0^\infty(G)$ it is fulfilled the equality

$$\int_G \varphi(x) D^k f(x) dx = (-1)^{|k|} \int_G D^k \varphi(x) f(x) dx,$$

where $k = (k_1,...,k_n)$ ($k_i \ge 0$ are entire), then $D^k f$ is called a generalized derivative of the function f of order k in G.

Lemma 1. *Let in the domain* G *be given the function* $f \in L_{\mathbf{p}}^{loc}(G,B)$ *and the sequence of* B*-valued functions* $f_j \in L_{\mathbf{p}}^{loc}(G,B)$ $(j = 1,...)$, *having generalized derivatives* $f_j^{(k)} \in L_{\mathbf{p}}^{loc}(G,B)$ $(j=1,...)$, *where* $\mathbf{p} \in [1,\infty]^n$, $\mathbf{q} \in [1,\infty]^n$.

If $f_j \to f$ $(j \to \infty)$ *in the sense* $L_{\mathbf{p}}^{loc}(G,B)$ *and* $(f_j^{(k)} - f_i^{(k)}) \to 0$ $(j,i \to \infty)$ *in the sense* $L_{\mathbf{q}}^{loc}(G,B)$, *then* B*-valued function* f *have on* G *generalized derivative* $f^{(k)} \in L_{\mathbf{q}}^{loc}(G,B)$ *and* $f_j^{(k)} \to f^{(k)}$ $(j \to \infty)$ *in the sense* $L_{\mathbf{q}}^{loc}(G,B)$.

Let $l=(l_1,...,l_n)$, where $l_i \geq 0$ are entire numbers, $1 \leq \mathbf{p} \leq \infty$. By $W_{\mathbf{p}}^l(G,B)$ we denote a space of locally summable by Bochner on G functions f, having on G generalized derivatives $D_i^{l_i} f$ with a finite norm

$$\left\| f, W_{\mathbf{p}}^l(G,B) \right\| = \|f\|_{L_{\mathbf{p}}(G,B)} + \sum_{i=1}^{n} \left\| D_i^{l_i} f \right\|_{L_{\mathbf{p}}(G;B)}.$$

For the space $W_{\mathbf{p}}^l(G;B)$ we can construct an imbedding theory similar to scalar valued one in the case if B is ζ-convex Banach lattice.

Let $L \in C_0^\infty(G)$, $\int_G L(x)dx = 1$. At $\forall f \in L_1^{loc}(G;B)$ as in [4] it is determined a mean function

$$f_{\vartheta^\lambda}(x) = \vartheta^{-|\lambda|} \int_G L\left(y\vartheta^{-\lambda}, \vartheta^{-\lambda}\rho\left(\vartheta^\lambda, x\right)\right) f(x+y)dy,$$

$\lambda=(\lambda_1,...,\lambda_n)$, $T \in (0,\infty)$, $\vartheta^{-\lambda}y = \left(\vartheta^{-\lambda_1}y_1,...,\vartheta^{-\lambda_n}y_n\right)$, $\vartheta > 0$, $\lambda_i > 0$. For each $x \in G$ we consider the path $\rho\left(\vartheta^\lambda, x\right) = \left(\rho_1\left(\vartheta^{\lambda_1}, x\right),...,\rho_n\left(\vartheta^{\lambda_n}, x\right)\right)$, $0 \leq \vartheta \leq T$.

For the function $f \in W_{\mathbf{p}}^1(G;B)$ we determine on G a mean function $f_{\vartheta^\lambda}^k$ by the formula

$$f_{\vartheta^\lambda}(x) = \vartheta^{-|\lambda|} \int_{R^n} L\left(y\vartheta^{-\lambda}, \vartheta^{-\lambda}\rho\left(\vartheta^\lambda, x\right)\right) f(x+y)dy, \ \lambda_i > 0, \ i=1,...,n,$$

$$L_k \in C_0^\infty(G), \quad \int_G L_k(x)dx = 1.$$

Similar to [4] almost for all $x \in G$ we get

$$f_\varepsilon(x) = f_{h^\lambda}(x) +$$

$$+\sum_{i=1}^{n} \lambda_i \int_\varepsilon^h \vartheta^{-1-|\lambda|+\lambda_i l_i} d\vartheta \int_G L_i\left(\vartheta^{-\lambda}y, \vartheta^{-\lambda}\rho\left(\vartheta^\lambda, x\right), \rho'\left(\vartheta^\lambda, x\right)\right) D_i^{l_i} f(x+y)dy, \qquad (2)$$

where $0 < \vartheta < h$, $f_{h^\lambda}(x)$ is a mean function, determined above, and the function $L_i \in C_0^\infty(G)$.

This shows that from the functions from spaces $W_p^l(G;B)$ it holds Il'in-Besov type integral representation.

Consider the next integral operator playing an important role in imbedding theorems obtained on the basis of integral representation (2)

$$f_{\varepsilon r}(x) = \int_\varepsilon^r \chi\left(\frac{\omega(x)}{t}\right) \int t^{-1-|\lambda|} M\left(yt^{-\lambda}\right) f(x+y)dydt,$$

where ω is a function measurable in R^n.

Theorem 3. *Let* $M \in C_0^\infty(R^n)$, $\int_{R^n} M(x)dx = 0$, ω *is measurable in* R^n. *Let for some* $p_0 \in (1,\infty)$

$$\|f_{\varepsilon r}\|_{L_{p_0}(R^n;B)} \leq C_p \|f\|_{L_{p_0}(R^n;B)}, \quad 0 < \varepsilon < r \leq +\infty.$$

Then for $f \in L_{\mathbf{p}}(R^n;B)$, $p \in (1,\infty)^n$, $f_{\varepsilon r} \in L_{\mathbf{p}}(R^n;B)$ *and for* $\varepsilon \to 0$ $f_{\varepsilon r}$ *converges in* $L_p(R^n;B)$ *to some function* f_{0r}, *and*

$$\|f_{\varepsilon r}\|_{L_{\mathbf{p}}(R^n;B)} \le C_{\mathbf{p}} \|f\|_{L_{\mathbf{p}}(R^n;B)}, \ 0 \le \varepsilon < r \le +\infty, \tag{3}$$

where $C_{\mathbf{p}}$ doesn't depend on f, ε.

We give the proof as in [4].

Proof. Consider

$$(A_{\varepsilon r} f)(x) = \int K_{\varepsilon r}(x,y) f(x+y) dy,$$

$$K_{\varepsilon r}(x,y) = \int_{\varepsilon}^{r} \chi\left(\frac{\omega(x)}{t}\right) M\left(yt^{-\lambda}\right) t^{-1-|\lambda|} dt.$$

As is shown in [4], for $m = 0,1,...,n-1$ with some $N > 0$ for all $h > 0$

$$\int_{\pi[x] > Nh} \int \sup_z \left| K_{\varepsilon r}(z, x-y) - K_{\varepsilon r}(z,x) \right| dx \le C, \ \text{if} \ \pi[y] < h,$$

where $\pi[x] = \max_{m+1 \le i \le n} |x_i|^{1/\lambda_i}$. Then by virtue of theorem 1 we get estimate (3) for $0 < \varepsilon < r \le +\infty$. The convergence $f_{\varepsilon r}$ in $L_{\mathbf{p}}(R^n;B)$ for $\varepsilon \to 0$ and the estimate (3) for $\varepsilon = 0$ has been obtained in a standard way (see [4], p.59).

On this basis and by means of theorem 3 arguing as in [4] we get

Theorem 4. *Let an open set $G \subset R^n$ satisfy the condition of a flexible λ-horn, $\frac{1}{\lambda} = l \in N^n$, $1 \le \mathbf{p} \le \mathbf{q} \le \infty$, $\wp = 1 - \left| \left(\alpha + \frac{1}{\mathbf{p}} - \frac{1}{\mathbf{q}} \right) : l \right| > 0$. Then the continuous imbedding*

$$D^{\alpha} W_{\mathbf{p}}^{l}(G;B) \subset L_q(G;B)$$

holds.

It furthermore B is ζ-convex Banach lattice, the imbedding is valid also for $1 < \mathbf{p} = \mathbf{q} < \infty$, $\wp = 0$.

Saying exactly, for $f \in W_{\mathbf{p}}^{l}(G;B)$ on G there exists a generalized derivative $D^{\alpha} f \in L_{\mathbf{p}}(G;B)$ and exist such numbers $h_0 > 0$ $C_1 > 0$ $C_2 > 0$ that

$$\|D^{\alpha} f\|_{L_{\mathbf{q}}(G;B)} \le C_1 h^{\wp} \sum_{i=1}^{n} \|D_i^{l_i} f\|_{L_{\mathbf{p}}(G;B)} + C_2 h^{\wp - 1} \|f\|_{L_{\mathbf{p}}(G;B)},$$

where the constants C_1, C_2 do not depend on f and $h \in (0, h_0)$. In particular, for $\alpha = 0$, $W_{\mathbf{p}}^{l}(G;B) \subset L_{\mathbf{q}}(G;B)$.

Proof. Applying the differentiation operation D^{α} to the both sides of (2) we find

$$D^{\alpha} f_{\varepsilon^{\lambda}}(x) = D^{\alpha} f_{h^{\lambda}}(x) + \sum_{i=1}^{n} \int_{\varepsilon}^{h} \int_{R^n} \vartheta^{-|\lambda| - (\alpha,\lambda)} \times$$

$$\times M_i^{(\alpha)}\left(\vartheta^{-\lambda} y, \vartheta^{-\lambda} \rho(\vartheta^{\lambda}, x), \rho'(\vartheta^{\lambda}, x)\right) D_i^{l_i} f(x+y) dy d\upsilon. \tag{4}$$

By virtue of Minkovsky's inequality, and Young's inequality from (4) we obtain

$$\left\| D^{\alpha} f_{\varepsilon^{\sigma}}(x) - D^{\alpha} f_{h^{\sigma}}(x) \right\|_{L_{\mathbf{q}}(G;B)} \le \sum_{i=1}^{n} \int_{\varepsilon}^{h} \upsilon^{\wp - 1} d\upsilon \|M_i\|_{L_r(R^n)} \|D_i^{l_i} f\|_{L_{\mathbf{p}}(G;B)} \le$$

$$\le C h^{\wp} \sum_{i=1}^{n} \|D_i^{l_i} f\|_{L_{\mathbf{p}}(G;B)}.$$

Thus, for $\varepsilon \to 0$ $D^\alpha f_{\varepsilon^\sigma}$ converges in $L_{\mathbf{q}}(G;B)$. On the other hand, for $\varepsilon \to 0$ $f_{\varepsilon^\sigma} \to f$ in $L_{\mathbf{p}}(G;B)$, as is shown in [4]. On the basis of lemma 1 hence we conclude that on G there exists a generalized derivative

$$D^\alpha f \in L_{\mathbf{q}}(G;B),\ \left\| D^\alpha f - D^\alpha f_{\varepsilon^\sigma} \right\|_{L_{\mathbf{q}}(G;B)} (\varepsilon \to 0).$$

Then

$$\left\| D^\alpha f \right\|_{L_{\mathbf{q}}(G;B)} \le \left\| D^\alpha f_{h^\sigma} \right\|_{L_{\mathbf{q}}(G;B)} + C_1 h^{\wp} \sum_{i=1}^{n} \left\| D_i^{l_i} f \right\|_{L_{\mathbf{p}}(G;B)}.$$

Estimating by means of 2(18) from [4] the first summand of the right hand-side of (4):

$$\left\| D^\alpha f_{h^\sigma} \right\|_{L_{\mathbf{q}}(G;B)} \le C_2 h^{\wp-1} \left\| f \right\|_{L_{\mathbf{p}}(G;B)}$$

we complete the proof of inequality (1) for $\wp > 0$.

In case when $\wp = 0$ $1 < \mathbf{p} = \mathbf{q} < \infty$ the proof of theorem is obtained from theorem 3.

Corollary. *Let an open set* $G \subset R^n$ *satisfy the condition of a flexible* λ*-horn,* $\frac{1}{\lambda} = l \in N^n$, $1 \le \mathbf{p} \le \mathbf{q} \le \infty$, $\wp = 1 - \left| \left(\alpha + \frac{1}{\mathbf{p}} - \frac{1}{\mathbf{q}} \right) : l \right| > 0$. *Then the continuous imbedding*

$$D^k W_{\mathbf{p}}^l (G; l_\theta) \subset L_{\mathbf{q}}(G; l_\theta)$$

holds.

If furthermore $1 < \theta < \infty$ *the imbedding is valid also for* $\wp = 0$. *And in this case* $1 < \mathbf{p} = \mathbf{q} < \infty$.

Moreover, for $f \in W_{\mathbf{p}}^l(G;l_\theta)$ *on* G *there exist a generalized derivative* $D^\alpha f \in L_{\mathbf{q}}(G;l_\theta)$ $h_0 > 0$ $C > 0$ *such that and numbers*

$$\left\| D^\alpha f \right\|_{L_{\mathbf{q}}(G;l_\theta)} \le C h^{\wp} \sum_{i=1}^{n} \left\| D_i^{l_i} f \right\|_{L_{\mathbf{p}}(G;l_\theta)} + C h^{\wp-1} \left\| f \right\|_{L_{\mathbf{q}}(G;l_\theta)},$$

where a constant C *doesn't depend on* f *and* $h \in (0, h_0)$. *In particular, for* $\alpha = 0$, $W_{\mathbf{p}}^l(G;l_\theta) \subset L_{\mathbf{q}}(G;l_\theta)$.

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Received July 7, 2000; Revised December 20, 2000.
Translated by Aliyeva E.T.