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THE STABILITY OF THE INVERSE PROBLEM OF THE SCATTERING THEORY FOR NON-SELF-ADJOINT OPERATOR

Abstract

In the paper the stability of the inverse problem of the scattering problem for non- self-adjoint operator is studied. The estimates of difference solutions and potentials of two boundary value problems whose scattering data coincide on given interval of spectral parameter alternation, are obtained.

In [1] it shown that boundary value problem of the non-self-adjoint operator can be completely restored on the data of scattering.

Which information on a function $q(x)$ or generally on boundary value problem may be derived if the scattering data are known on some interval of spectral parameter alternation is of great importance for quantum mechanics. The natural from physical point of view the statement of the question is so: how strongly two boundary value problem can differ, whose scattering data differ very little on the given interval of spectral parameter alternation.

By L we'll denote the operator generated in a Hilbert space $L^2(0,\infty)$ with differential expression

$$ly = -y'' + q(x)y \tag{1}$$

and boundary condition

$$y(0) = 0. \tag{2}$$

$q(x)$ is a complex valued continuous function and with some $\varepsilon > 0$, it satisfies the condition

$$\int_0^\infty e^{\varepsilon x}|q(x)|dx < \infty. \tag{3}$$

A set of boundary value problems of which

$$\int_x^\infty e^{\varepsilon t}|q(t)|dt \le C(x), \quad (0 \le x < \infty), \tag{4}$$

where $C(x)$ is a continuous, non-increasing summable on semi-axis $[0,\infty)$ function we denote by $v\{C(x)\}$. At first we'll be interested in exactness of reconstruction of solutions $e(x,\rho)$ of corresponding equations, since they are reconstructed more stably.

The equation $ly = \rho^2 y$ has the solution $e(x,\rho)$, which for every $x \ge 0$ is a holomorphic function from ρ, with $\operatorname{Im}\rho > -\frac{\varepsilon}{2}$ and satisfies the relation:

$$e(x,\rho) = \exp\{ix\rho\} + \int_x^\infty K(x,t)e^{it\rho}dt,$$

where the kernel of transformation $K(x,t)$ has continuous derivatives by x and t and with $0 \le x \le t < \infty$

$$|K(x,t)| < C\exp\left\{-\varepsilon\frac{x+t}{2}\right\}. \tag{5}$$

According to [2] the unreal roots of function $e(0,\rho)=e(\rho)$ located in domain $\operatorname{Im}\rho>0$ are called singular numbers and real roots $e(\rho)$ are called special points of the operator L. The singular numbers denote by $\rho_1,...,\rho_2$.

Multiplicity m_k of the root ρ_k $\left(k=\overline{1,2}\right)$ the multiplicity of the singular number ρ_k.

The next functions $F_k(x)$ and $F_s(x)$ characterizing the operator L on pointed and continuous spectrums were introduced in [1]

Namely

$$F_s(x)=\frac{1}{2\pi}\int_{+}[s(\rho)-1]\exp\{ix\rho\}d\rho,$$

where $s(\rho)=\dfrac{e(-\rho)}{e(\rho)}$ is the scattering function of the operator L, and

$$F_k(x)=P_k\left(\frac{d}{ids}\right)\exp\{ix\rho\}\Big|_{\rho=\rho_k}\qquad\left(k=\overline{1,\alpha}\right),$$

where P_k is a polynomial of degree m_k-1. The functions $P_1,...,P_\alpha$ are called normalized polynomials of the operator L.

The scattering function $S(\rho)$ singular numbers $\rho_1,...,\rho_\alpha$ and normalized polynomials $P_1,...,P_\alpha$ are called scattering data of the operator L.

Let's deduce the formulas giving convenient representations for differences of the solutions by the differences of corresponding scattering datas.

Let's consider two boundary value problems $\{q_1(x)\}$, $\{q_2(x)\}$ from the set $v\{C(x)\}$. Let's construct for the inverse problems the basic integral equations and substruct one from another. In result we receive

$$K_{1,2}(x,y)-\int_x^\infty F_1(t+y)K_{1,2}(x,t)dt=F_{1,2}(x+y)+\int_x^\infty F_{1,2}(t+y)K_2(x,t)dt,$$

Here

$$K_{1,2}(x,y)=K_1(x,y)-K_2(x,y),\ \ F_{1,2}(x)=F_1(x)-F_2(x),$$

$$F_i(x)=F_{s_i}(x)-\sum_{k=1}^{\alpha}F_{ki}(x),\ \ i=1,2.$$

Let $\{s_j(\rho),\rho_k,P_k(j)\}$ $(j=1,2)$ be the scattering data and $e_j(\rho,x)$ be the solutions of considered problems. For the brevity we don't put ρ_k the index j, we simply think $\rho_{kj}=0$ if ρ_k isn't a root of function $e_j(\rho)$. Then

$$F_{1,2}(x)=\sum_{k=1}^{\alpha}\left[P_{k_2}\left(\frac{d}{id\rho}\right)-P_{k_1}\left(\frac{d}{id\rho}\right)\right]\exp\{ix\rho\}\Big|_{\rho=\rho_k}+$$

$$+\frac{1}{2\pi}\int_{+}[s_1(\rho)-s_2(\rho)]e^{ix\rho}d\rho,$$

where the integral is taken along the straight line $\operatorname{Im}\rho=\eta$ $\left(0<\eta<\dfrac{\varepsilon}{2}\right)$ in the direction from the left to right. Using this equality for the differences of solutions we find

$$e_1(\mu,x)-e_2(\mu,x)=\sum_{k=1}^{\alpha}\left[P_{k_2}\left(\frac{d}{id\rho}\right)-P_{k_1}\left(\frac{d}{id\rho}\right)\right]\left[E_{1,2}(\rho,\mu,x)\right]\Big|_{\rho=\rho_k}+$$
$$+\frac{1}{2\pi}\int_{+}[s_1(\rho)-s_2(\rho)]E_{1,2}(\rho,\mu,x)d\rho,$$

where

$$E_{j,i}(\rho,\mu,x)=\frac{e_i(\rho,x)\{e_j'(\rho,x)e_j(\mu,x)-e_j'(\mu,x)e_j(\rho,x)\}}{\rho^2-\mu^2}.$$

Let fulfill the next transformation:

$$\left(\rho^2-\mu^2\right)\{E_{1,2}(\rho,\mu,x)e_2(\mu,x)-E_{2,1}(\rho,\mu,x)e_1(\mu,x)\}=$$
$$=\int_x^{\infty}\{q_1(t)-q_2(t)\}\{e_1(\rho,x)e_2(\rho,x)e_1(\mu,t)e_2(\mu,t)-e_1(\mu,x)e_2(\mu,x)e_1(\rho,t)e_2(\rho,t)\}dt.$$

The next lemma is proved.

Lemma 1. *At all values* μ *from the half-plane* $\operatorname{Im}\mu>\frac{\varepsilon}{2}$ *for which* $\operatorname{Im}\mu\neq\operatorname{Im}\rho_k$, *the next identity is true*

$$\{e_1(\mu,x)-e_2(\mu,x)\}^2=\int_x^{\infty}[q_1(t)-q_2(t)]\{A_{1,2}(\mu,x,t)-A_{1,2}(\mu,t,x)\}dt.$$

where

$$A_{1,2}(\mu,t,x)=e_1(\mu,t)e_2(\mu,t)\sum_{k=1}^{\alpha}\left[P_{k_2}\left(\frac{d}{id\rho}\right)-P_{k_1}\left(\frac{d}{id\rho}\right)\right]\left(\frac{e_1(\rho,x)-e_2(\rho,x)}{\rho^2-\mu^2}\right)_{\rho=\rho_k}+$$
$$+\frac{e_1(\mu,x)e_2(\mu,x)}{2\pi}\int_{+}\frac{[s_1(\rho)-s_2(\rho)]}{\rho^2-\mu^2}e_1(\rho,x)e_2(\rho,x)d\rho. \tag{6}$$

Let given scattering data of two boundary value problems coincide when $\operatorname{Re}\rho^2\in(-\infty,N)$

$$s_1(\rho)=s_2(\rho),\qquad |\operatorname{Re}\rho|<\sqrt{N+\eta^2},\quad 0<\operatorname{Im}\rho=\eta<\frac{\varepsilon}{2}$$

$$\rho_k(1)=\rho_k(2),\qquad P_k(1)=P_k(2)\quad \left(k=\overline{1,2}\right),\ \operatorname{Im}\rho_k>\frac{\varepsilon}{2}.$$

How much strongly these boundary value problems can differ. Let's estimate first of all difference of the solutions of the corresponding differential equations.

Theorem 1. *If given scattering data of two boundary value problems coincide at all* $\operatorname{Re}\rho^2\in(-\infty,N)$, *then at all* $\operatorname{Re}\mu^2\in(-\infty,N)$ *the next inequality is true.*

$$|e_1(\mu,x)-e_2(\mu,x)|\le\frac{8\left(1+C_1e^{-\varepsilon x}\right)^4(C(x))}{\pi N\left(1-\frac{\left|\operatorname{Re}\mu^2\right|+\operatorname{Re}\mu^2}{2N}\right)}. \tag{7}$$

Proof. Using the formula (6) the equality $S_j(\rho)=1+O\left(\frac{1}{\rho}\right)$ where $|\rho|\to\infty$, $\operatorname{Im}\rho<\eta$ and the estimation

$$\left|e_j(\rho,x)\right| \le e^{\frac{\varepsilon}{2}x}\left(1+C_1e^{-\varepsilon x}\right) \tag{7'}$$

for $\{A_{1,2}(\mu,x,t)\}$ we get

$$\left|A_{1,2}(\mu,x,t)\right| \le \frac{e^{\varepsilon x}\left(1+C_1e^{-\varepsilon x}\right)^2\left(1+C_1e^{-\varepsilon t}\right)^2}{2\pi}\int\limits_{\substack{|\mathrm{Re}\,\rho|>\sqrt{N+\varphi^2}\\ \mathrm{Im}\,\rho=\eta}}\frac{\left|s_1(\rho)-s_2(\rho)\right|}{\left|\rho^2-\mu^2\right|}\left|d\rho\right| <$$

$$< \frac{e^{\varepsilon x}\left(1+C_1e^{-\varepsilon x}\right)^2\left(1+C_1e^{-\varepsilon t}\right)^2}{2\pi}\frac{1}{\sqrt{N+\eta^2}}\int\limits_{\substack{|\mathrm{Re}\,\rho|>\sqrt{N+\eta^2}\\ \mathrm{Im}\,\rho=\eta}}\frac{\left|d\rho\right|}{\left|\mathrm{Re}\left(\rho^2-\mu^2\right)\right|} <$$

$$< \frac{e^{\varepsilon x}\left(1+C_1e^{-\varepsilon x}\right)^2\left(1+C_1e^{-\varepsilon x}\right)^2}{\pi N\left(1-\frac{\left|\mathrm{Re}\,\mu^2\right|+\mathrm{Re}\,\mu^2}{2N}\right)}.$$

The inequality (7) follows from the formula (6)

$$\left|e_1(\mu,x)-e_2(\mu,x)\right|^2 = \left|\int\limits_x^\infty [q_1(t)-q_2(t)]\{A_{1,2}(\mu,x,t)-A_{1,2}(\mu,t,x)\}dt\right| \le$$

$$\le \frac{2\left(1+c_1e^{-\varepsilon x}\right)^4}{\pi N\left(1-\frac{\left|\mathrm{Re}\,\mu^2\right|+\mathrm{Re}\,\mu^2}{2N}\right)}\int\limits_x^\infty\left|e^{\varepsilon x}-e^{\varepsilon t}\right|\times\left|q_1(t)-q_2(t)\right|dt < \frac{8\left(1+c_1e^{-\varepsilon x}\right)^4}{\pi N\left(1-\frac{\left|\mathrm{Re}\,\mu^2\right|+\mathrm{Re}\,\mu^2}{2N}\right)}.$$

Theorems is proved.

Remark. According to (7') the estimation of (7) can be non-trivial only in domain, where $C(x) < N$.

Let's pass to the estimation differences of potentials $q_1(x)-q_2(x)$ of considered problems.

For the sufficiently smooth function $g(x)$ we get [3]

$$\frac{1}{2}\left|\int\limits_{x_0}^{x_0+h}\{q_1(t)-q_2(t)\}g(t)dt\right| \le \frac{2}{\pi}\left(\frac{n}{h}\right)^n\frac{\left(N+\eta^2\right)^{-\frac{n-2}{2}}}{n-2}\times$$

$$\times\left\{1+\frac{C(x_0)}{\sqrt{N+\eta^2}}+\frac{C^2(x_0)m_N^2(x_0)}{N+\eta^2}\right\}+$$

$$+\frac{2C(x_0)\beta(h,x_0)m_N^2(x_0)}{\pi\sqrt{N+\eta^2}}\left(4n+9C(x_0)h\left(1+C(x_0)he^{hC(x_0)}\right)\right),$$

where

$$m(\rho,x) = \max_{j=1,2}\left\{\sup_{x<t<x+h}\left|e_j(\rho,t)\right|\right\},\quad m_N(x) = \sup_{|\mathrm{Re}\,\rho|>\sqrt{N+\eta^2}} m(\rho,x)$$

$$\rho(h,x) = \max_{j=1,2}\left\{\sup_{x\le t<x+h}\left\{q_j(t)e^{\frac{\varepsilon_j}{2}}\right\}\right\}.$$

Using this estimation the next theorem is proved.

Theorem 2. *By fulfilling the conditions of the theorem 1 and* $N+\eta^2 \geq 1$ *in domain where*

$$\frac{5\{[Ln(N+\eta^2)]+1\}}{\sqrt{N+\eta^2}}C(x)<1 \qquad \left(C(x)<\sqrt{N+\eta^2}\right)$$

the inequality

$$|q_1(x)-q_2(x)| \leq \frac{2\{[\ln(N+\eta^2)]+3\}}{\sqrt{N+\eta^2}}\{38C(x)\beta(h,x)+5\gamma(h,x)\}+$$

$$+\frac{1}{\sqrt{N+\eta^2}\{3[\ln(N+\eta^2)]+1\}}$$

is true.

Here $h=5(N+\eta^2)^{-\frac{1}{2}}\{[\ln(N+\eta^2)]+1\}$, $\quad \gamma(h,x)=\max\limits_{j=1,2}\left\{\sup\limits_{x<t<x+h}|q_j'(t)|\right\}$.

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