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INVESTIGATION OF ELASTIC EQUILIBRIUM OF A TRANSVERSALLY-ISOTROPIC HOLLOW CYLINDER WITH A FIXED LATERAL SURFACE

Abstract

On the basis of homogeneous solutions method the asymptotic solution of elasticity theory problem for a transversally isotropic hollow cylinder for rigid sealing of lateral surface is investigated.

With the help of variational principle the boundary value problem is led to the solution of infinite system of linear algebraic equations.

In papers [1,2] the asymptotic behavior of the first basic problem of elasticity theory for transversally isotropic hollow cylinder is studied.

In the present paper on the basis of the homogeneous solutions method the asymptotic solution of the elasticity theory problem for a transversally-isotropic hollow cylinder for rigid sealing of a lateral surface is investigated.

1. Consider an axially symmetric problem of elasticity theory for a transversally isotropic hollow cylinder.

Assume that lateral surface of the cylinder is rigidly fixed, i.e.

$$u_r = 0,\ u_z = 0 \ \text{for} \ R = R_n \ \ (n = 1,2). \tag{1.1}$$

For the present we will not refine the character of boundary conditions on end-walls of the cylinder. But we'll assume them such that an envelope is in equilibrium.

In displacements the equilibrium equations have the form [1,2]

$$\begin{gathered} b_{11}\left(\Delta_0 U_\rho - \frac{U_\rho}{\rho}\right) + \frac{\partial^2 U_\rho}{\partial \xi^2} + (1 + b_{13})\frac{\partial^2 U_\xi}{\partial\rho\partial\xi} = 0, \\ (1 + b_{13})\frac{\partial}{\partial\xi}\left(\frac{\partial U_\rho}{\partial\rho} + \frac{U_\rho}{\rho}\right) + \Delta_0 U_\xi + b_{33}\frac{\partial^2 U_\xi}{\partial\xi^2} = 0. \end{gathered} \tag{1.2}$$

Here these designations are the same as in works [1,2]. Using the results of works [1,2] we'll represent the general solution (1.2) in the form of

$$\begin{gathered} U_\rho = \left[(b_{33}\mu^2 - \alpha_1^2)Z_1(\alpha_1\rho) + (b_{33}\mu^2 - \alpha_2^2)Z_1(\alpha_2\rho)\right]m'(\xi), \\ U_\xi = -(1 + b_{13})\mu^2\left[\alpha_1 z_0(\alpha_1\rho) + \alpha_2 z_0(\alpha_2\rho)\right]m(\xi), \end{gathered} \tag{1.3}$$

where the function $m(\xi)$ is subordinated to the condition

$$m''(\xi) - \mu^2 m(\xi) = 0.$$

$Z_k(\rho) = C_1 J_k(\rho) + C_2 Y_k(\rho)$, the functions $J_k(\rho), Y_k(\rho)$ are linear-independent solutions of the Bessel equation, C_1, C_2 are arbitrary constants, $\alpha_n = \sqrt{t_n}$, t_n are roots of the square equation

$$\begin{gathered} t^2 - 2q_1\mu^2 t + q_2\mu^4 = 0, \ t_n = \mu^2 S_n, \\ q_1 = \frac{\nu_1}{\nu_2}(1 - \nu_1\nu_2)^{-1}\cdot(1+\nu)\cdot(G_0 - \nu_2), \\ q_2 = \frac{\nu_1}{\nu_2}(1 - \nu_1\nu_2)^{-1}\cdot(1 - \nu_2), \ S_n = \sqrt{q_1 - (-1)^n\sqrt{q_1^2 - q_2}}. \end{gathered} \tag{1.4}$$

Satisfying the homogeneous boundary conditions (1.1) we'll get the characteristic equation

$$\Delta(\mu,\rho_1,\rho_2)=\frac{2}{\rho_1\rho_2}\left(b_{33}\mu^2-\alpha_1^2\right)\left(b_{33}\mu^2-\alpha_2^2\right)-\alpha_2^2\left(b_{33}\mu^2-\alpha_1^2\right)^2\times$$
$$\times L_{00}(\alpha_2\rho_1,\alpha_2\rho_2)L_{11}(\alpha_1\rho_1,\alpha_1\rho_2)+\alpha_1\alpha_2\left(b_{33}\mu^2-\alpha_1^2\right)\left(b_{33}\mu^2-\alpha_2^2\right)\times \qquad (1.5)$$
$$\times\left[L_{10}(\alpha_1\rho_1,\alpha_1\rho_2)L_{01}(\alpha_2\rho_1,\alpha_2\rho_2)+L_{01}(\alpha_1\rho_1,\alpha_1\rho_2)L_{10}(\alpha_2\rho_1,\alpha_2\rho_2)\right]-$$
$$-\alpha_1^2\left(b_{33}\mu^2-\alpha_2^2\right)^2L_{00}(\alpha_1\rho_1,\alpha_1\rho_2)L_{11}(\alpha_2\rho_1,\alpha_2\rho_2)=0,$$
$$L_{ij}(x)=J_i(x\rho_1)Y_j(x\rho_2)-J_j(x\rho_2)Y_i(x\rho_1)\ \ (i,\,j=0;1).$$

The left hand side of the equation (1.5) as an entire function of the parameter μ has a denumerable set of zeros with a condensation point at infinity. For effective study of its roots, as in [1,2] we'll assume that the envelope is thin-shelled

$$\rho_1=1-\varepsilon\,,\ \rho_2=1+\varepsilon\,,\ \varepsilon=(2R_0)^{-1}(R_2-R_1). \qquad (1.6)$$

We assume that ε is a small parameter. Substituting (1.6) in (1.5) we'll obtain

$$D(\mu,\varepsilon)=\Delta(\mu,\rho_1,\rho_2)=0\,. \qquad (1.7)$$

As in papers [1,2] we can prove that all zeros of the function $D(\mu,\varepsilon)$ unbounded by increase for $\varepsilon\to 0$ and here only the case $\varepsilon\mu_k\to const$ for $\varepsilon\to 0$ is possible.

For constructing the asymptotics of zeros we will search μ_n $(n=1,2,...)$ in the form of

$$\mu_n=\frac{\delta_n}{\varepsilon}+0(\varepsilon). \qquad (1.8)$$

But as was noted in [1,2] depending on the characteristics of material ν,ν_1,ν_2,G_0 the parameters q_1,q_2 admit different values, that emply different records of solutions by the Bessel function. This in its turn leads to different asymptotic representations of the Bessel function.

Consider the following possible cases

1. $q_1>0\,,\ q_1^2-q_2\neq 0,\ \alpha_{1,2}=\pm\mu S_1\,,\ \alpha_{3,4}=\pm\mu S_2\,,$

   $S_{1,2}=\sqrt{q_1\pm\sqrt{q_1^2-q_2}}\,,\ q_1^2>q_2\,,$

   $S_{1,2}=\chi+i\beta=\sqrt{q_1\pm i\sqrt{q_2-q_1^2}}\,,\ q_1^2<q_2\,;$
2. the roots of a characteristic equation are multiple

   $\alpha_{1,2}=\alpha_{3,4}=\pm\mu\cdot P\,,\ q_1>0\,,\ q_1^2-q_2=0\,;$
3. $q_1<0\,,\ q_1^2-q_2\neq 0,\ \ P=\sqrt{q}\,,\ \ \alpha_{1,2}=\pm i\mu S_1\,,\ \alpha_{3,4}=\pm i\mu S_2\,,$

   $S_{1,2}=\sqrt{|q_1|\pm\sqrt{q_1^2-q_2}}\,,\ q_1^2>q_2\,,$

   $S_{1,2}=\sqrt{|q_1|\pm i\sqrt{q_2-q_1^2}}\,,\ q_1^2<q_2\,;$
4. $q_1<0\,,\ q_1^2-q_2=0,\ \ \alpha_{1,2}=\alpha_{3,4}=\pm i\mu P,\ \ P=\sqrt{|q_1|}\,.$

In cases 1, 2 after substitution (1.8) in (1.5) and transformation it with the help of asymptotic expansions $J_\gamma(x),Y_\gamma(x)$ for δ_n we obtain

$$\frac{b_{33}+S_1S_2}{b_{33}-S_1S_2}(S_2-S_1)\sin(S_1+S_2)\delta_n\pm(S_1+S_2)\sin(S_2-S_1)=0\,, \qquad (1.9)$$

$$\frac{b_{13}+3}{b_{13}+1}\sin 2P\delta_n\pm 2p\delta_n=0\,, \qquad (1.10)$$

$$\chi\left(b_{33}-\chi^2-b^2\right)sh2\chi\delta_n \mp \beta\left(b_{33}+\chi^2+\beta^2\right)\sin 2\beta\delta_n = 0. \qquad (1.11)$$

For the cases 3 and 4 the results are obtained from the cases 1 and 2 by the formal substitution S_1, S_2 to iS_1, iS_2.

These equations have a denumerable set of zeros and factually coincide with characteristic equations of the analogous problem for transversally-isotropic elastic bands [3].

2. Assuming that ε is a small parameter, we'll bring asymptotic construction of homogeneous solutions corresponding to different groups of a characteristic equation.

Here the expressions are given only for the components of a displacement vector. We can obtain expressions for the components of stress tensor with the help of generalized Hook's law.

Group 1.

$$U_r = \varepsilon R_0 \sum_{n=1,3,\dots}^{\infty} C_n \Big[S_2\left(b_{33}-S_1^2\right)\sin S_2\delta_n \cdot \cos S_1\delta_n\eta - \qquad (2.1)$$
$$- S_1\left(b_{33}-S_2^2\right)\sin S_1\delta_n \cos S_2\delta_n\eta + 0(\varepsilon)\Big]\frac{dm_n}{d\xi},$$

$$U_z = R_0\left(b_{13}+1\right)S_1S_2 \sum_{n=1,3,\dots}^{\infty} C_n\delta_n\left[\sin S_2\delta_n \sin S_1\delta_n\eta - \sin S_1\delta_n \sin S_2\delta_n\eta + 0(\varepsilon)\right]m_n(\xi).$$

Group 2.

$$U_r = \varepsilon R_0 \sum_{n=1,3,\dots}^{\infty} B_n\left[\left(\frac{b_{13}+3}{b_{13}+1}\sin p\delta_n + p\delta_n\cos p\delta_n\right)\cdot \cos p\delta_n + p\delta_n\eta \sin p\delta_n \times \qquad (2.2)\right.$$
$$\left.\times \sin p\delta_n\eta + 0(\varepsilon)\right]\frac{dm_n}{d\xi},$$

$$U_z = R_0 \sum_{n=1,3,\dots}^{\infty} B_n\delta_n^2\left[\cos p\delta_n \sin p\delta_n\eta - \eta\sin p\delta_n \cdot \cos p\delta_n\eta + 0(\varepsilon)\right]m_n(\xi).$$

Group 3.

$$U_r = R_0\varepsilon \sum_{n=1,3,\dots}^{\infty} D_n\Big\{\left[\chi\left(b_{33}-\chi^2-\beta^2\right)\cos\beta\delta_n sh\chi\delta_n - \beta\left(b_{33}+\chi^2+\beta^2\right)\sin\beta\delta_n ch\chi\delta_n\right]\times$$
$$\times\cos\beta\delta_n\eta ch\chi\delta_n\eta + \left[\chi\left(b_{33}-\chi^2-\beta^2\right)\sin\beta\delta_n ch\chi\delta_n + \beta\left(b_{33}+\chi^2+\beta^2\right)\cos\beta\delta_n sh\chi\delta_n\right]\times$$
$$\times\sin\beta\delta_n\eta\cdot sh\chi\delta_n\eta + 0(\varepsilon)\Big\}\frac{dm_n}{d\xi},$$

$$U_z = R_0\left(b_{13}+1\right)\left(\chi^2+\beta^2\right)\sum_{n=1,3,\dots}^{\infty} D_n\Big[\cos\beta\delta_n \cdot sh\chi\delta_n \cdot \sin\beta\delta_n\eta\cdot ch\chi\delta_n\eta -$$
$$- \sin\beta\delta_n\cdot ch\chi\delta_n\cdot\cos\beta\delta_n\eta sh\chi\delta_n\eta + 0(\varepsilon)\Big]m_n(\xi),$$

$$m_n(\xi) = E_n\exp\left(\frac{\delta_n}{\varepsilon}\xi\right) + N_n\exp\left(-\frac{\delta_n}{\varepsilon}\xi\right), \qquad (2.3)$$

$\eta = \frac{1}{\varepsilon}(\rho-1)$, C_n, B_n, D_n, E_n, N_n are arbitrary constants.

The expressions for $n = 2,4,6,\dots$ are obtained from (2.1), (2.2), (2.3) substituting $\cos x$ by $\sin x$ and $\sin x$ by $-\cos x$, chx by shx and shx by $-chx$ respectively.

In the formulas (2.1), (2.2), (2.3) substituting S_1, S_2, P by iS_1, iS_2, iP we will obtain the solutions of cases 3 and 4 respectively.

As was noted in papers [1,2] in the case of essential anisotropy having for sufficiently large values of G_0 saint Venant's boundary layer damps very weakly and the solutions (2.1), (2.2), (2.3) must be reckoned to penetrating solutions. Therefore in this case stress strain state of transversally-isotropic and isotropic envelopes strongly differ.

3. Consider a problem of satisfaction of boundary conditions on end-walls of the cylinder with the help of a class of homogeneos solutions. For that summing the roots of a characteristic equation the homogeneous solutions may be represented in the form of

$$\begin{aligned} U_r &= R_0 \sum_{k=1}^{\infty} C_k U_k(\rho) \frac{dm_k}{d\xi}, \\ U_z &= R_0 \sum_{k=1}^{\infty} C_k W_k(\rho) m_k(\xi), \\ \sigma_r &= G_1 \sum_{k=1}^{\infty} C_k Q_{rk}(\rho) \frac{dm_k}{d\xi}, \\ \sigma_\varphi &= G_1 \sum_{k=1}^{\infty} C_k Q_{\varphi k}(\rho) \frac{dm_k}{d\xi}, \\ \sigma_z &= G_1 \sum_{k=1}^{\infty} C_k Q_{zk}(\rho) \frac{dm_k}{d\xi}, \\ \tau_{rz} &= G_1 \sum_{k=1}^{\infty} C_k T_k(\rho) m_k(\xi). \end{aligned} \tag{3.1}$$

C_k are arbitrary constants. As is shown in [1, 2] a system of homogeneous solutions independent of boundary conditions on the lateral surface of the cylinder satisfies the generalized orthogonality conditions permitting exactly to solve the elasticity theory problems for mixed boundary conditions on end wells of the cylinder

$$\int_{\rho_1}^{\rho_2} \left[T_p(\rho) U_k(\rho) - Q_{zk}(\rho) W_p(\rho)\right] \rho \, d\rho = 0 \quad k \neq p. \tag{3.2}$$

As an example we'll consider the first variant of mixed pavement conditions, where for simplicity we'll assume that they are symmetric with respect to the plane $\xi = 0$. The skew-symmetric case is considered analogously (in the symmetric case it is possible to assume $m_k = ch\mu_k\xi$, in the skew-symmetric case we are to take $m_k = sh\mu_k\xi$).

Thus, let the following conditions be given

$$\sigma_z = Q(\rho), \; U_r = a(\rho) \; \text{ for } \; \xi = \pm l_0, \tag{3.3}$$

$l_0 = \dfrac{l}{R_0}$, $2l$ is a length of cylinder.

For satisfying the boundary conditions (3.3) it is necessary to fulfill the next expansions

$$Q(\rho) = \sum_{k=1}^{\infty} C_k Q_{zk}(\rho) ch\mu_k l_0, \quad a(\rho) = \sum_{k=1}^{\infty} C_k U_k(\rho) ch\mu_k l_0. \tag{3.4}$$

C_k are arbitrary constants subjected to be determined from pavement conditions.

If we use the relation (3.2) it is possible to find the constants C_k from the equation (3.4). Let's multiply the first equality of (3.4) by $\rho W_p(\rho) sh\mu_\rho l_0$, the second one by $\rho T_p(\rho) sh\mu_\rho l_0$, add the obtained products and integrate over ρ from ρ_1 to ρ_2. By virtue of the relations of generalized orthogonality, the desired constants have the following form

$$C_k = 2^{-1}\Delta_k^{-1} ch^{-1}\mu_k l_0 \int_{\rho_1}^{\rho_2} [a(\rho)T_K(\rho) - Q(\rho)W_k(\rho)]\rho d\rho, \tag{3.5}$$

where Δ_k is a value of the integral (3.2) for $p = k$.

Note that using the asymptotics of solutions corresponding to the various group of the roots (2.1), (2.2), (2.3) it is easy to obtain asymptotic formulas for Δ_k.

As was noted [3], the generalized orthogonality of homogeneous solutions allows exactly to solve the elasticity theory problem for mixed pavement conditions. In all other cases for satisfying the boundary conditions on the end-walls of the cylinder we appeal to different approximate approach.

Therefore we'll consider a question of satisfying the boundary conditions on end-walls of the cylinder with the help of a class of homogeneous solutions.

Let the next conditions be given

$$\sigma_z = Q(\rho),\ \tau_{rz} = T(\rho) \text{ for } \xi = \pm l_0.$$

We'll search a solution in the form of (3.1). to determine the arbitrary constants C_k $(k = 1,2,...)$ the variations of which we consider independent, we use Lagrange's variational principle.

Since the homogeneous solutions satisfy the equilibrium equation and the boundary conditions on cylindrical surface, the variation principle takes the following form

$$\int_{\rho_1}^{\rho_2} [(\sigma_z - Q)\delta W + (\tau_{rz} - T)\delta U]\rho d\rho = 0. \tag{3.6}$$

From (3.6) we'll obtain an infinite system of linear algebraic equations

$$\sum_{k=1}^{\infty} M_{kp} C_k = N_p \quad (p = 1,2,...),$$
$$M_{kp} = \int_{\rho_1}^{\rho_2} (Q_{zk} W_p + T_k U_p)\rho d\rho, \tag{3.7}$$
$$N_p = \int_{\rho_1}^{\rho_2} (QW_p + TU_p)\rho d\rho.$$

Using the smallness of thin shellness parameter of the envelope ε it is possible to construct the asymptotic solution of the system (3.7). This approach is well-known [3, 4], so here we will not stop explicitly.

References

[1]. Maksudov F.G., Mekhtiev M.F., Sadikov P.M. *Construction of homogeneous solutions for a transversally-isotopic hollow cylinder.* Proceedings of Inst.of Math. and Mech. Acad.Sci., 1999, v.X(XVIII), p.199-210.

[2]. Максудов Ф.Г., Мехтиев М.Ф., Садыков П.М. *Асимптотическая теория для трансверсально-изотропного полого цилиндра.* Труды 5-ой Межд.Конф. «Современные Проблемы Механики Сплошной Среды» (12-14 октября 1999г.), Ростов на Дону, 2000г, с.134-139.

[3]. Мехтиев М.Ф. *Асимптотический анализ некоторых пространственных задач теории упругости для полых тел.* Автореф.Док.Дис., Физмат наук, Ленинград, 1989, 30с.
[4]. Космодамианский С.А., Шалдырван В.А. *Толстые многосвязные пластины.* Киев, «Наукова думка», 1978, 239с.

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