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ON THE COMPLETENESS OF A PART OF EIGEN AND ADJOINT VECTORS OF A SECOND ORDER OPERATOR BOUNDLESS CLASS

Abstract

In the paper sufficient conditions ensuring the completeness of the part of eigen and adjoint vectors responding to eigen values from some sectors in the left half plane of one class of second order operational bundles, basic part of which contains a normal operator, are found.

In the separable Hilbert space H consider the polynomial operator bundle

$$P(\lambda) = -\lambda^2 E + \lambda A_1 + A_2 + A^2, \tag{1}$$

where E is a unit operator and the linear operators A, A_1, A_2 satisfy the following conditions:

1^0. The operator A is normal and has completely continuous inverse A^{-1} with a spectrum lying on the finite number rays from the sector

$$S_\varepsilon = \{\lambda : |\arg\lambda| \le \varepsilon\},\quad 0 \le \varepsilon < \pi/2 .$$

2^0. The operators $B_1 = A_1 A^{-1}$, $B_2 = A_2 A^{-2}$ are bounded in H, $E + B_2$ is invertible in H.

By satisfying the condition 1^0 for any $\gamma \ge 0$ the operator A^γ is represented in the form of

$$A^\gamma \cdot = \sum_{n=1}^{\infty} \lambda_n^\gamma (\cdot, e_n) e_n ,$$

where $|\lambda_1| \le |\lambda_2| \le ... |\lambda_n| \le ...,$ λ_n are eigenvalues of the operator A and e_n are orthonormalized eigenvectors i.e

$$Ae_n = \lambda_n e_n ,\ n = 1,2,...,$$

and

$$\lambda_n^\gamma = |\lambda_n|^\gamma e^{i\gamma \arg\lambda_n},\quad -\varepsilon \le \arg\lambda_n \le \varepsilon .$$

The domain of definition of the operator A^γ

$$D(A^\gamma) = \left\{ x : x \in H,\ \sum_{n=1}^{\infty} |\lambda_n|^{2\gamma} |(x, e_n)|^2 < \infty \right\}$$

becomes the Hilbert space H_γ with respect to the scalar product

$$(x, y)_\gamma = (A^\gamma x, A^\gamma y),\quad x, y \in D(A^\gamma).$$

Let's determine the class $H_2(\alpha : H)$ consisting of vector functions $f(z)$ holomorphic in the sector

$$S_\alpha = \{z : |\arg z| < \alpha\},\quad 0 < \alpha < \pi/2$$

satisfying the condition

$$\sup_{|\varphi|<\alpha} \left\| f(te^{i\varphi}) \right\|^2_{L_2(R_+ ; H)} \equiv \sup_{|\varphi|<\alpha} \int_0^\infty \left\| f(te^{i\varphi}) \right\|^2 dt < \infty .$$

As it is known the functions from the class $H_2(\alpha:H)$ have the boundary values $f(te^{i\alpha})\in L_2(R_+:H)$, $f(te^{-i\alpha})\in L_2(R_+:H)$ and this class is a Hilbert space with respect to the norm (see [1])

$$\|f\|_{2,\alpha}=\frac{1}{\sqrt{2}}\left(\left\|f\left(te^{i\alpha}\right)\right\|_{L_2}^2+\left\|f\left(te^{-i\alpha}\right)\right\|_{L_2}^2\right)^{1/2}.$$

Further denote by

$$W_2^2(\alpha:H)=\{u(z):u''(z)\in H_2(\alpha:H);\ A^2u\in H_2(\alpha:H)\},$$

$$\overset{\circ}{W}{}_2^2(\alpha:H)=\{u(z):u(z)\in W_2^2(\alpha:H);\ u(0)=0\}$$

Hilbert spaces with the norm

$$\|u\|_{2,\alpha}=\left(\|u''\|_{2,\alpha}^2+\|A^2u\|_{2,\alpha}^2\right)^{1/2}.$$

Here and further, the derivatives are regarded in sense of a complex analysis in abstract Hilbert spaces.

Note that for the further functions from the space $W_2^2(\alpha:H)$ the theorem of intermediate derivatives and the theorem of traces holds:

If $u\in W_2^2(\alpha:H)$, then

$$\left\|A^{2-j}u^{(j)}\right\|_{2,\alpha}\le C_j\|u\|_{2,\alpha},\quad j=0,1,2,$$

$$\left\|A^{2-j-1/2}u^{(j)}(0)\right\|_H\le\widetilde{C}_j\|u\|_{2,\alpha},\quad j=0,1.$$

Let's bind the bundle $P(\lambda)$ with the boundary value problem

$$P(d/dz)u(z)=0,\quad z\in S_\alpha, \tag{2}$$

$$u(0)=\psi,\quad \psi\in H_{3/2}. \tag{3}$$

Definition 1. *We will call the function $u(z)\in W_2^2(\alpha:H)$ a holomorphic regular solution of the boundary value problems (2), (3), if the vector function $u(z)\in W_2^2(\alpha:H)$ satisfies the equation (2) identically in S_α and the boundary condition (3) is fulfilled in the sense of*

$$\lim_{\substack{|z|\to 0\\ |\arg z|<\alpha}}\|(u(z)-\varphi)\|_{3/2}=0.$$

Definition 2. *Let the vectors $\varphi_0\neq 0,\ \varphi_1,\dots,\varphi_{m_0}\in H_2$ satisfy the equations*

$$P(\lambda_0)\varphi_0=0,\quad P(\lambda_0)\varphi_1+\left.\frac{\partial P(\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_0}\varphi_0=0,$$

$$P(\lambda_0)\varphi_k+\left.\frac{\partial P(\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_0}\varphi_{k-1}+\left.\frac{\partial^2 P(\lambda)}{\partial\lambda^2}\right|_{\lambda=\lambda_0}\varphi_{k-2}=0,\quad k=2,3,\dots,m_0$$

then λ_0 is called an eigenvalue and φ_0 is an eigenvector, $\varphi_1,\dots,\varphi_{m_0}$ are adjoint vectors responding to the eigenvalue λ_0.

Denote by

$$\widetilde{S}_\alpha=\{\lambda:|\arg\lambda|>\pi/2+\alpha\}$$

and by $K(\widetilde{S}_\alpha)$ - a system of all eigen and adjoint vectors responding to the eigenvalues of the sector $\widetilde{S}_\alpha$.

Further denote by σ_p $(0 < p < \infty)$ a class of completely continuous operators B, for which

$$\sum_{n=1}^{\infty}\left(\lambda_n\left(B^*B\right)^{1/2}\right)^p < \infty .$$

In the present work we'll prove a theorem of the completeness of the system $K(\widetilde{S}_\alpha)$, i.e. the parts of eigen and adjoint vectors of the bundle (1) responding to eigenvalues of the sector $\widetilde{S}_\alpha$ in the space $H_{3/2}$.

Analogies of this theorem in different situations are investigated for example in [1-5]. At first we will prove some auxiliary statements

Lemma 1. *Let* $\varphi \in H_{3/2}$. *Then for* $z \in S_\alpha$

$$e^{-zA}\varphi \in W_2^2(\alpha : H).$$

Proof. Really for $z \in S_\alpha$ and $\varphi \in H_{3/2}$ we have

$$\left\|e^{-zA}\varphi\right\|_{2,\alpha} = 2\left\|A^2e^{-zA}\varphi\right\|_{2,\alpha}^2 = \left\|A^2e^{-te^{i\alpha}}\varphi\right\|_{L_2}^2 + \left\|A^2e^{-te^{-i\alpha}}\varphi\right\|_{L_2}^2 .$$

Since

$$\left\|A^2e^{-te^{\pm i\alpha}}\right\|_{L_2(R_+:H)}^2 = \int_0^\infty\left\|\sum_{n=1}^{\infty}\lambda_n^2e^{-\lambda_n te^{\pm i\alpha}}(\varphi,e_n)\right\|^2 dt =$$

$$= \int_0^\infty\sum_{n=1}^{\infty}|\lambda_n|^4e^{-2t|\lambda_n|\cos(\pm\alpha+\arg\lambda_n)}|(\varphi,e_n)|^2dt \le$$

$$\le \int_0^\infty\sum_{n=1}^{\infty}|\lambda_n|^4e^{-2t\cos(\alpha+\varepsilon)}|(\varphi,e_n)|^2dt =$$

$$= \sum_{n=1}^{\infty}|\lambda_n|^3[2\cos(\alpha+\varepsilon)]^{-1}|(\varphi,e_n)|^2 = [2\cos(\alpha+\varepsilon)]^{-1}\|\varphi\|_{3/2}^2 ,$$

i.e.

$$\left\|e^{-zA}\varphi\right\|_{2,\alpha} \le \sqrt{2}[\cos(\alpha+\varepsilon)]^{-1}\|\varphi\|_{3/2} .$$

The lemma is proved.

Lemma 2. *Let the conditions* 1^0 *and* 2^0 *be satisfied. Then on the rays* $\Gamma_{\pm(\pi/2)+\alpha} = \{\lambda : \arg\lambda = \pm(\pi/2+\alpha)\}$ *the estimations*

$$\left\|\lambda A\left(-\lambda^2E+A^2\right)^{-1}\right\| \le C_1(\varepsilon), \qquad (4)$$

$$\left\|A^2\left(-\lambda^2E+A^2\right)^{-1}\right\| \le C_2(\varepsilon), \qquad (5)$$

$$\left\|\lambda^2\left(-\lambda^2E+A^2\right)^{-1}\right\| \le C_2(\varepsilon), \qquad (6)$$

where

$$C_1(\varepsilon) = (2\cos(\alpha+\varepsilon))^{-1}, \quad C_2(\varepsilon) = \begin{cases} 1, & 0 < \alpha+\varepsilon < \pi/2 \\ \left(\sqrt{2}\cos(\alpha+\varepsilon)\right)^{-1}, & \pi/4 \le \alpha+\varepsilon < \pi/2 \end{cases}$$

are valid.

Proof. Let $\lambda = re^{\pm(\pi/2+\alpha)}$, $r>0$. Then from a spectral expansion of the operator A we obtain

$$\left\|\lambda A\left(-\lambda^2 E + A^2\right)^{-1}\right\| = \sup_n |\lambda_n|\, r \left|r^2 e^{\pm 2i\alpha} + |\lambda_n|^2 e^{2i\arg\lambda_n}\right|^{-1} =$$
$$= \sup_n |\lambda_n| r \left(r^4 + |\lambda_n|^4 + 2r^2|\lambda_n|^2 \cos 2(\pm\alpha + \arg\lambda_n)\right)^{-1/2} \le$$
$$\le \sup_n |\lambda_n| r \left(r^4 + |\lambda_n|^4 + 2|\lambda_n|^2 r^2 \cos 2(\alpha+\varepsilon)\right)^{-1/2} \le$$
$$\le \sup_n |\lambda_n| r \left[2r^2|\lambda_n|^2 + 2|\lambda_n|^2 r^2 \cos 2(\alpha+\varepsilon)\right]^{-1/2} \le$$
$$\le \left(2\cos(\alpha+\varepsilon)\right)^{-1}.$$

The inequality (4) is proved. Now we'll prove the inequality (5). Again with regard to $\lambda = re^{\pm(\pi/2+\alpha)}$, $r>0$ analogously previous one we obtain

$$\left\|A^2\left(-\lambda^2 E + A^2\right)^{-1}\right\| \le \sup_n |\lambda_n|^2 \left[r^4 + |\lambda_n|^4 + 2r^2|\lambda_n|^2\cos(\alpha+\varepsilon)\right]^{-1/2}.$$

Hence for $0<\alpha+\varepsilon\le\pi/4$ we obtain that

$$\left\|A^2\left(-\lambda^2 E + A^2\right)^{-1}\right\| \le \sup_n |\lambda_n|^2\left(r^4 + |\lambda_n|^4\right)^{-1/2} \le 1,$$

and for $\pi/4\le\alpha+\varepsilon<\pi/2$ we get

$$\left\|A^2\left(-\lambda^2 E + A^2\right)^{-1}\right\| \le \sup_n |\lambda_n|^2\left[r^4 + |\lambda_n|^4 + \left(r^4 + |\lambda_n|^4\right)\cos 2(\alpha+\varepsilon)\right]^{-1/2} \le$$
$$\le \sup_n\left[|\lambda_n|^2\left(|\lambda_n|^4 + r^4\right)^{-1/2}\right]\left(\sqrt{2}\cos(\alpha+\varepsilon)\right)^{-1} \le \left(\sqrt{2}\cos(\alpha+\varepsilon)\right)^{-1}.$$

The inequality (6) is proved analogously to the inequality (5).

Theorem 1. *Let the conditions 1^0 and 2^0 be satisfied and the inequality*

$$K(\varepsilon;\alpha) = C_1(\varepsilon;\alpha)\|B_1\| + C_2(\varepsilon;\alpha)\|B_2\| < 1, \qquad (7)$$

where the coefficients $C_1(\varepsilon;\alpha)$ and $C_2(\varepsilon;\alpha)$ are determined from lemma 2, is valid. Then the problem (2), (3) has unique regular holomorphic solution $u(z)$ for any $\psi\in H_{3/2}$, where

$$\|u\|_{2,\alpha} \le const\|\psi\|_{3/2}.$$

Proof. After the substitution $u(z) = v(z) - e^{zA}\psi$ from the boundary value problem (2), (3) we find that

$$P(d/dz)v(z) = g(z), \quad z\in S_\alpha, \qquad (8)$$
$$v(0) = 0, \qquad (9)$$

where the vector function

$$g(z) = (A_1 A + A_2)e^{-zA}\psi = (B_1 + B_2)A^2 e^{-zA}\psi$$

belongs to the space $H_2(\alpha:H)$, since B_1 and B_2 are bounded operators, and for $\psi\in H_{3/2}$ by lemma 1 $A^2 e^{-zA}\psi\in H_2(\alpha:H)$. Further note that from lemma 1 it follows that

$$\|g(z)\|_{H_2(\alpha:H)} \le (\|B_1\| + \|B_2\|)\|A^2 e^{-zA}\psi\|_{H_2(\alpha:H)} \le const\|\psi\|_{3/2}. \qquad (10)$$

From the basic theorem of work [8] it follows that the problem (8), (9) has a regular solution $v(z) \in W_2^2(\alpha : H)$ which satisfies the inequality

$$\|v\|_{2,\alpha} \le const\|g(z)\|_{H_2(\alpha:H)}. \qquad (11)$$

Then $u(z) = v(z) - e^{zA}\psi$ is a regular solution of the problem (2), (3), where by he inequality (10) with regard to lemma 1 the inequality

$$\|u\|_{2,\alpha} \le \|v\|_{2,\alpha} + \|e^{zA}\psi\|_{2,\alpha} \le const\|\psi\|_{3/2}$$

holds.

The theorem is proved.

Theorem 2. *Let the conditions 1^0, 2^0, the inequality (7) and one of the next conditions be satisfied*

1) $A^{-1} \in \sigma_p \quad \left(0 < p \le \dfrac{\pi}{\pi - 2\alpha}\right)$

2) $A^{-1} \in \sigma_p \quad (0 < p < \infty), \qquad B_1 = A_1 A^{-1},\ B_2 = A_2 A^{-2}$

be completely continuous operators in H.

Then the system $K(\widetilde{S}_\alpha)$ is complete in the space $H_{3/2}$.

Proof. From the condition of theorem 2 by theorem 1 it follows that for any $\psi \in H_{3/2}$ there exists a regular holomorphic solution $u(z) \in W_2^2(\alpha : H)$ representable in the form of

$$u(z) = \frac{1}{2\pi i}\int_{\Gamma_{(\pi/2-\alpha)}} \hat{u}(\lambda) e^{\lambda z} d\lambda - \frac{1}{2\pi i}\int_{\Gamma_{-(\pi/2+\alpha)}} \hat{u}(\lambda) e^{\lambda z} d\lambda, \qquad (12)$$

where

$$\hat{u}(\lambda) = P^{-1}(\lambda)((\lambda E - A_1)u(0) + u'(0)) \equiv P^{-1}(\lambda)Q(\lambda). \qquad (13)$$

Since

$$P(\lambda) = -\lambda^2 E + \lambda A_1 + A_2 + A^2 = (E + M(\lambda))(E + B_2)A^2,$$

where $M(\lambda) = \lambda T_1 - \lambda^2 T_2$ and $T_1 = B_1 A^{-1}(E + B_2)^{-1} \in \sigma_p$, $T_2 = B_2 A^{-2}(E + B_2)^{-1} \in \sigma_{p/2}$, then $M(\lambda)$ is completely operator valued function and the operator $E + M(0) = E$ is invertible. Then by Keldysh's lemma [2,5,6] the resolvent

$$P^{-1}(\lambda) = A^{-2}(E + B_2)^{-1}(E + M(\lambda))^{-1}$$

is representable in the form of relation of two entire functions of order p and of minimal type for order p.

Let the condition 1) be satisfied. Then on the rays $\Gamma_{\pm(\pi/2+\alpha)}$,

$$\|P^{-1}(\lambda)\| = \|(P_0(\lambda) + P_1(\lambda))^{-1}\| = \|P_0^{-1}(\lambda)\|\|(E + P_1(\lambda)P_0^{-1}(\lambda))^{-1}\|,$$

but from lemma 2 it follows that

$$\|P_1(\lambda)P_0^{-1}(\lambda)\| = \|(\lambda A_1 + A_2)P_0^{-1}(\lambda)\| \le \|\lambda A_1 P_0^{-1}(\lambda)\| + \|A_2 P^{-1}(\lambda)\| \le$$

$$\le \|B_1\|\|\lambda A(-\lambda^2 E + A^2)^{-1}\| + \|B_2\|\|A^2(-\lambda^2 E + A^2)^{-1}\| \le K(\varepsilon;\alpha) < 1,$$

i.e.

$$\left\|P^{-1}(\lambda)\right\| \le \left\|P_0^{-1}(\lambda)\right\| \frac{1}{1-K(\varepsilon;\alpha)} = \frac{1}{1-K(\varepsilon;\alpha)} \sup_n \left|-\lambda^2+\lambda_n^2\right|^{-1} \le const \frac{1}{1+|\lambda|^2}. \qquad (14)$$

It is obvious that the inequality (14) holds and for some sector $S_{\pm\theta} = \{\lambda : |\arg\lambda \pm (\pi/2+\alpha)| < \theta\}$ for sufficiently small $\theta > 0$.

Analogously proved that in these sectors the inequality

$$\left\|A^{3/2}P^{-1}(\lambda)\right\| \le const(1+|\lambda|)^{-1/2} \qquad (15)$$

holds.

Now assume that there exists a vector $\psi \in H_{3/2}$ which is orthogononal to all vectors from the system $K(\widetilde{S}_\alpha)$. Denote by $u(z)$ a regular holomorphic solution of the problem (2), (3) and for $t > 0$ denote by

$$\xi(t) = (u(t),\psi)_{3/2} = \frac{1}{2\pi i}\int_\Gamma \left(A^{3/2}\hat{u}(\lambda)e^{\lambda t},\psi\right)d\lambda =$$

$$= \frac{1}{2\pi i}\int_\Gamma \left(A^{3/2}P^{-1}(\lambda)Q(\lambda), A^{3/2}\psi\right)e^{\lambda t}d\lambda =$$

$$= \frac{1}{2\pi i}\int_\Gamma \left(Q(\lambda), \left(A^{3/2}P^{-1}(\lambda)\right)^* A^{3/2}\psi e^{\lambda t}\right)d\lambda =$$

$$= \frac{1}{2\pi i}\int_\Gamma \left(Q(\lambda), R(\bar{\lambda})\right)e^{\lambda t}d\lambda,$$

where $\Gamma = \Gamma_{(\pi/2+\alpha)} \cup \Gamma_{-(\pi/2+\alpha)}$ and $R(\lambda) = \left(A^{3/2}P^{-1}(\bar{\lambda})\right)^* A^{3/2}\psi$.

From Keldysh's lemma [2,5] it follows that if ψ is orthogonal to the system $K(\widetilde{S}_\alpha)$, then the vector-function $R(\lambda)$ is holomorphic in the sector $\widetilde{S}_\alpha = \{\lambda : |\arg\lambda| > \pi/2+\alpha\}$ and by the inequality (15) on the rays $\Gamma_{\pm(\pi/2+\alpha)}$ the estimation

$$|g(\lambda)| = \left|Q(\lambda), R(\bar{\lambda})\right| \le const(1+|\lambda|) \qquad (16)$$

is valid.

On the other hand in the sector $S_{\pi/2+\alpha} = \{\lambda : |\arg\lambda| < \pi/2+\alpha\}$.

$g(\lambda)$, as Laplace transformation of the function from $H_2(\alpha : H)$ is holomorphic and decreasing [7]. Thus $g(\lambda)$ is an entire function. Since the angle between $\Gamma_{\pi/2+\alpha}$ and $\Gamma_{\pi/2-\alpha}$ is equal to $\pi - 2\alpha$, and $p \le \pi/\pi - 2\alpha$, so by Fregman-Lindelöf theorem [7] we obtain that $g(\lambda) = a_0 + \lambda a_1$, where $a_0, a_1 \in H$. Then for $t > 0$

$$\xi(t) = (u(t),\psi)_{3/2} = \frac{1}{2\pi i}\int_\Gamma (a_0+\lambda a_1)e^{\lambda t}dt.$$

But the last integral is equal to zero. Consequently, for $t > 0$ $\xi(t) = 0$. Tending $t \to +0$, we obtain that $(\psi,\psi)_{3/2} = 0$, i.e. $\psi = 0$. In case 1) the theorem is proved.

In case 2) by Kedysh's lemma [2,6] we have a system of the sectors

$$S_\delta = \mathbf{C} \setminus \bigcup_{i=1}^{s}\{\lambda : |\arg\lambda - \omega_i| < \delta\} \bigcup_{i=1}^{s}\{\lambda : |\arg\lambda - (\pi+\omega_i)| < \delta\},$$

outside of which the estimation (16) is valid. Here $\arg\lambda = \omega_i$ are those rays on which the spectrum of the operator A from the sector S_ε is arranged.

Again applying Fregman-Lindelöf theorem in this case analogously to the first case we complete the proof of the theorem.

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References

[1]. Гасымов М.Г. *О разрешимости краевых задач для одного класса операторно-дифференциальных уравнений.* ДАН СССР, 1977, т.235, №3, сю505-508ю
[2]. Келдыш М.В. *О полноте собственных функций некоторых несамосопряженных линейных уравнений.* УМН, 1971, т.27, №4, с.15-41.
[3]. Мирзоев С.С. *Кратная полнота части корневых векторов полиномиальных операторных пучков.* В кн. Spectral Theory, Banach Center Publications, v.8, PWW-Polish. public., Warsow , 1982.
[4]. Мирзоев С.С. *О кратной полноте корневых векторов полиномиальных операторных пучков, отвечающих краевым задачам на полуоси.* Функциональный анализ и его приложения, 1983, т.17, вып.2, с.84-85.
[5]. Гасымов М.Г., Мирзоев С.С. *О разрешимости краевых задач для операторно-дифференциальных уравнений эллиптического типа второго порядка.* Дифференциальные уравнения, 1992, т.28, №2, с.651-661.
[6]. Радзиевский Г.В. *Задача о полноте корневых векторов в спектральной теории оператор-функций.* УМН, 1982, т.37, вып.2, с.81-145.
[7]. Евграфов М.А. *Аналитические функции.* М.: Наука, 1991, с.447.
[8]. Мирзоев С.С., Амирова Л.И. Известия АН Азерб., сер. физ.-тех. и матем. наук, т.XVIII, математика и механика, №2, 1998, с.61-66.

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