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ON THE SOLVABILITY OF ONE CLASS OPERATOR-DIFFERENTIAL EQUATIONS IN WHOLE SPACE

Abstract

Sufficient conditions of solvability for one class operator-differential equations in the whole space, expressed only by the coefficients of the equation are obtained in this paper.

Let H be separable Hilbert space, A is positively defined self-adjoint operator in H, and H_γ is a scale of Hilbert spaces, generated by the operator A, i.e.

$$H_\gamma = D(A^\gamma), (\varphi,\psi)_\gamma = (A^\gamma \varphi, A^\gamma \psi), \ \varphi, \psi \in D(A^\gamma), \ \gamma \geq 0$$

(for $\gamma = 0$ we suppose that $H_0 = H$).

Denote by $R^2 = \{x = (x_1, x_2), -\infty < x_i < \infty, \ i = 1,2\}$ and by $L_2(R^2; H)$ the space of vector-functions $f(x_1, x_2)$ with the values from the space H, summable in square in the range R^2, i.e. measurable vector-functions, for which

$$\|f\|_{L_2(R^2;H)} = \left(\iint_{R^2} \|f(x_1,x_2)\|^2 dx_1 dx_2 \right)^{1/2} < \infty .$$

It is known that $L_2(R^2;H)$ is Hilbert space with the scalar product

$$(f,g)_{L_2(R^2;H)} = \iint_{R^2} (f(x_1,x_2), g(x_1,x_2)) dx_1 dx_2 .$$

Let $D(R^2;H_2)$ be the set of infinitely differentiable vector-functions $u(x_1,x_2)$ with the values from H_2 with compact support in R^2. In this linear set we define the norm

$$\|u\|_{W_2(R^2;H)} = \left(\left\| \frac{\partial^2 u}{\partial x_1^2} \right\|^2_{L_2(R^2;H)} + \left\| \frac{\partial^2 u}{\partial x_2^2} \right\|^2_{L_2(R^2;H)} + \left\| A^2 u \right\|^2_{L_2(R^2;H)} \right)^{1/2} .$$

It is obvious that the set $D(R^2;H_2)$ is pre-Hilbert space relatively the scalar product $(u,\vartheta)_{W_2(R^2;H)}$, generated by the norm $\|u\|_{W_2(R^2;H)}$. Completion of this pre-Hilbert space we denote by $W_2(R^2;H)$.

Consider the following operator-differential equation in the space H:

$$-\frac{\partial^2 u(x_1,x_2)}{\partial x_1^2} - \frac{\partial^2 u(x_1,x_2)}{\partial x_2^2} + A^2 u(x_1,x_2) + A_{1,0} \frac{\partial u(x_1,x_2)}{\partial x_1} +$$

$$+ A_{0,1} \frac{\partial u(x_1,x_2)}{\partial x_2} + i A_{11} \frac{\partial^2 u(x_1,x_2)}{\partial x_1 \partial x_2} + i A_{0,0} u(x_1,x_2) = f(x_1,x_2), \quad x = (x_1,x_2) \in R^2, \quad (1)$$

where $f(x_1,x_2)$, $u(x_1,x_2)$ are the vector-valued functions with the values from H, i is imaginary unit, operators $A_{1,0}$, $A_{0,1}$, A_{11}, A_{00} are linear, generally speaking, unbounded operators in H.

Definition 1. *If there is the vector-function* $u(x_1, x_2) \in W_2(R^2; H)$, *which satisfies the equation (1) almost everywhere in* R^2, *then we'll call it the regular solution of the equation (1).*

Definition 2. *If for any* $f(x_1, x_2) \in L_2(R^2; H)$ *there is the regular solution of the equation (1)* $u(x_1, x_2)$, *which satisfies the inequality*

$$\|u\|_{W_2(R^2;H)} \le const \|f\|_{L_2(R^2;H)}$$

then we'll call the equation (1) regularly solvable.

In this paper we obtain sufficient conditions, expressed only by the coefficients of the equation (1), which provide the regular solvability for the equation (1). Regular solvability of the operator- differential equations with one variable, and the boundary-value problems for them are studied by many authors as well (see, for example, the papers [1-6] and references, which are in these papers). Operator-differential equations with some variables are studied relatively little. In most of these papers at first the solvability of the main part of the equation (1) is studied, but about the perturbated part it is supposed that its relative norm of the main part is sufficiently small (see, for example, the work [7]). In other works the conditions are put on the increase of resolvent for the bunch $P(\xi,\eta)$, $(\xi,\eta) \in R^2$, generated by the equation (1) after Fourier transformation [8]. In author's work [9] the solvability conditions for the equation (1) on the language of smallness for the norms of operators of the equation (1) are found. In the work [10] conditions of such type are given for solvability of the boundary-value problems in semi-space. Introduce the following designations

$$P_0 u = -\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + A^2 u, \quad u \in D(R^2; H_2), \tag{2}$$

$$P_1 u = A_{11} \frac{\partial^2 u}{\partial x_1 \partial x_2} + A_{1,0} \frac{\partial u}{\partial x_1} + A_{0,1} \frac{\partial u}{\partial x_2} + A_{00} u, \quad u \in D(R^2; H_2), \tag{3}$$

$$P u = P_0 u + P_1 u, \quad u \in D(R^2; H_2). \tag{4}$$

It takes place the following

Lemma 1. *Let* A *be the positively-defined self-adjoint operator, i.e.* $A = A^* \ge \ge \mu_0 E \ (\mu_0 > 0)$, $A_{1,0}, A_{0,1}, A_{0,0}, A_{1,1}$ *are symmetric operators in* H, *moreover their domains satisfy the conditions:* $D(A) \subset D(A_{1,0})$, $D(A) \subset D(A_{0,1})$, $D(A^2) \subset D(A_{0,0})$, $D(A_{11}) = H$. *Then the operators* P_0, P_1 *and* P, *defined by the equalities (2), (3) and (4) correspondingly, are extended from the space* $D(R^2; H_2)$ *to the space* $W_2(R^2; H)$ *as continuous operators, acting from* $W_2(R^2; H)$ *to* $L_2(R^2; H)$.

Proof. It is evidently that for $u \in D(R^2; H_2)$

$$\|P_0 u\|^2_{L_2(R^2;H)} = \left\| -\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + A^2 u \right\|^2_{L_2(R^2;H)} \le \left(\left\| \frac{\partial^2 u}{\partial x_1^2} \right\|_{L_2(R^2;H)} + \left\| \frac{\partial^2 u}{\partial x_2^2} \right\|_{L_2(R^2;H)} + \right.$$

$$\left. + \left\| A^2 u \right\|_{L_2(R^2;H)} \right)^2 \le 3 \left(\left\| \frac{\partial^2 u}{\partial x_1^2} \right\|^2_{L_2(R^2;H)} + \left\| \frac{\partial^2 u}{\partial x_2^2} \right\|^2_{L_2(R^2;H)} + \left\| A^2 u \right\|^2_{L_2(R^2;H)} \right) = 3 \|u\|^2_{W_2(R^2;H)}. \tag{5}$$

From the inequality (5) it follows that the operator P_0 is extended from the space $D(R^2; H_2)$ to $W_2(R^2; H)$ as continuous operator, acting from $W_2(R^2; H)$ to $L_2(R^2; H)$.

As the operator $A_{1,1}$ is symmetric and $D(A_{1,1})=H$, then it is bounded operator in H. Then it is evidently that for $u\in D(R^2;H_2)$

$$\left\|A_{1,1}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L_2(R^2;H)}\le\left\|A_{1,1}\right\|\cdot\left\|\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L_2(R^2;H)}. \qquad (6)$$

If we denote by $\hat{u}(\xi,\eta)$ Fourier transformation of the vector-function $u(x_1,x_2)$, i.e.

$$\hat{u}(\xi,\eta)=\frac{1}{2\pi}\iint_{R^2}u(x_1,x_2)e^{-i(\xi x_1+\eta x_2)}dx_1dx_2,$$

then from Plancherel theorem we obtain:

$$\left\|\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|^2_{L_2(R^2;H)}=\left\|\xi\eta\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)}\le\frac{1}{4}\left\|\left(\xi^4+\eta^4\right)\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)}=$$

$$=\frac{1}{4}\left(\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|^2_{L_2(R^2;H)}+\left\|\frac{\partial^2 u}{\partial x_2^2}\right\|^2_{L_2(R^2;H)}\right)\le\frac{1}{4}\|u\|^2_{W_2(R^2;H)}. \qquad (7)$$

From the other side, as $A_{1,0}$, $A_{0,1}$ and $A_{0,0}$ are symmetric operators, then they are closible. Then from the theorem on closed graph it follows that for all $\varphi\in D(A)$ and $\psi\in D(A^2)$ the inequalities (see [11,p.176]):

$$\|A_{1,0}\varphi\|\le c_{1,0}\|A\varphi\|,\ \|A_{0,1}\varphi\|\le c_{0,1}\|A\varphi\|,\ \|A_{0,0}\psi\|\le c_{0,0}\|A^2\psi\|,$$

take place.

Thus, for $u\in D(R^2;H_2)$

$$\left\|A_{1,0}\frac{\partial u}{\partial x_1}\right\|_{L_2(R^2;H)}\le c_{1,0}\left\|A\frac{\partial u}{\partial x_1}\right\|_{L_2(R^2;H)}, \qquad (8)$$

$$\left\|A_{0,1}\frac{\partial u}{\partial x_2}\right\|_{L_2(R^2;H)}\le c_{0,1}\left\|A\frac{\partial u}{\partial x_2}\right\|_{L_2(R^2;H)}, \qquad (9)$$

$$\|A_{0,0}u\|_{L_2(R^2;H)}\le c_{0,0}\|A^2u\|_{L_2(R^2;H)}. \qquad (10)$$

Further, applying the spectral decomposition of the operator A and Plancherel theorem, we obtain:

$$\left\|A\frac{\partial u}{\partial x_1}\right\|^2_{L_2(R^2;H)}=\|\xi A\hat{u}(\xi,\eta)\|^2_{L_2}=\iint_{R^2}\left(\int_{\mu_0}^{\infty}\xi^2\mu^2\left(dE_\mu\hat{u}(\xi,\eta),\hat{u}(\xi,\eta)\right)\right)d\xi d\eta\le\frac{1}{2}\iint_{R^2}\int_{\mu_0}^{\infty}\left(\xi^4+\mu^4\right)\times$$

$$\times\left(dE_\mu\hat{u}(\xi,\eta),\hat{u}(\xi,\eta)\right)d\xi d\eta\le\frac{1}{2}\iint_{R^2}\left(\xi^4\|\hat{u}(\xi,\eta)\|^2+\left\|A^2\hat{u}(\xi,\eta)\right\|^2\right)d\xi d\eta=\frac{1}{2}\left(\left\|\xi^2\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)}+\right.$$

$$\left.+\left\|A^2\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)}\right)=\frac{1}{2}\left(\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|^2_{L_2(R^2;H)}+\left\|A^2u\right\|^2_{L_2(R^2;H)}\right)\le\frac{1}{2}\|u\|^2_{W_2}. \qquad (11)$$

Analogously we obtain, that for $u\in D(R^2;H_2)$

$$\left\|A\frac{\partial u}{\partial x_1}\right\|_{L_2(R^2;H)}\le\frac{1}{\sqrt{2}}\|u\|_{W_2^2(R^2;H)}. \qquad (12)$$

As for $u \in D(R^2; H_2)$

$$\|P_1 u\|_{L_2(R^2;H)} \le \left\|A_{11}\frac{\partial^2 u}{\partial x_1 \partial x_2}\right\|_{L_2(R^2;H)} + \left\|A_{1,0}\frac{\partial u}{\partial x_1}\right\|_{L_2(R^2;H)} + \\ + \left\|A_{0,1}\frac{\partial u}{\partial x_2}\right\|_{L_2(R^2;H)} + \|A_{0,0}u\|_{L_2(R^2;H)}, \qquad (13)$$

then taking into account the inequalities (7), (11), (12) in (13), we obtain:

$$\|P_1 u\|_{L_2(R^2;H)} \le const\|u\|_{W_2(R^2;H)}.$$

Thus, the operators P_0 and P_1 consequently and P are extended on continuity to the space $W_2(R^2;H)$ as bounded operators, acting from $W_2(R^2;H)$ to $L_2(R^2;H)$. Their extensions we also denote by P_0, P_1 and P.

At first we consider the case, when $P_1 = 0$. It takes place

Lemma 2. *Operator* P_0 *maps the space* $W_2(R^2;H)$ *onto* $L_2(R^2;H)$ *isomorphically.*

Proof. From lemma 1 it follows that the operator $P_0 : W_2(R^2;H) \to L_2(R^2;H)$ is bounded. Let $f(x_1,x_2) \in L_2(R^2;H)$ and consider the equation $P_0 u = f$, i.e.

$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + A^2 u = f(x_1,x_2). \qquad (14)$$

After Fourier transformation we obtain:

$$(\xi^2 E + \eta^2 E + A^2)\hat{u}(\xi,\eta) = \hat{f}(\xi,\eta),\ (\xi,\eta) \in R^2.$$

Then we have:

$$\hat{u}(\xi,\eta) = (\xi^2 E + \eta^2 E + A^2)^{-1}\hat{f}(\xi,\eta). \qquad (15)$$

Now, defining vector-function

$$u(x_1,x_2) = \frac{1}{2\pi}\iint_{R^2}(\xi^2 E + \eta^2 E + A^2)^{-1}\hat{f}(\xi,\eta)d\xi d\eta$$

we can see that it satisfies the equation (14) almost everywhere. Let's prove that $u(x_1,x_2) \in W_2(R^2;H)$. According to Plancherel theorem it is sufficiently to prove that $\xi^2\hat{u}(\xi,\eta)$, $\eta^2\hat{u}(\xi,\eta)$ and $A^2\hat{u}(\xi,\eta)$ belong to space $L_2(R^2;H)$.

As $\hat{u}(\xi,\eta)$ is defined by the formula (15), then

$$\|\xi^2\hat{u}(\xi,\eta)\|^2_{L_2(R^2;H)} = \left\|\xi^2(\xi^2 E + \eta^2 E + A^2)^{-1}\hat{f}(\xi,\eta)\right\|^2_{L_2(R^2;H)} \le \\ \le \sup_{(\xi,\eta)\in R^2}\left\|\xi^2(\xi^2 E + \eta^2 E + A^2)^{-1}\right\|^2_{H\to H} \cdot \|\hat{f}(\xi,\eta)\|^2_{L_2(R^2;H)}. \qquad (16)$$

Further, applying the spectral decomposition of the operator A, we obtain that for $(\xi,\eta) \in R^2$ it takes place the inequality

$$\left\|\xi^2(\xi^2 E + \eta^2 E + A^2)^{-1}\right\| = \sup_{\mu\in\sigma(A)}\left\|\xi^2(\xi^2 + \eta^2 + \mu^2)^{-1}\right\| = \frac{1}{\mu_0^2}.$$

Then from (16) we obtain that

$$\left\|\xi^2\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)} \le \frac{1}{\mu_0}\left\|\hat{f}(\xi,\eta)\right\|_{L_2(R^2;H)}. \tag{17}$$

Analogously we find that

$$\left\|\eta^2\hat{u}(\xi,\eta)\right\|_{L_2(R^2;H)} \le \frac{1}{\mu_0}\left\|\hat{f}(\xi,\eta)\right\|_{L_2(R^2;H)}. \tag{18}$$

Further, it is obvious that

$$\left\|A^2\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)} \le \left\|A^2\left(\xi^2E+\eta^2E+A^2E\right)^{-1}\right\|^2 \cdot \left\|\hat{f}(\xi,\eta)\right\|^2_{L_2(R^2;H)}. \tag{19}$$

As

$$\left\|A^2\left(\xi^2E+\eta^2E+A^2E\right)^{-1}\right\| = \sup_{\mu\ge\mu_0}\left\|\mu^2\left(\xi^2+\eta^2+\mu^2\right)^{-1}\right\| \le 1, \tag{20}$$

then, taking into account the inequality (20) in (19), we obtain that

$$\left\|A^2\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)} \le \left\|\hat{f}(\xi,\eta)\right\|^2_{L_2(R^2;H)}. \tag{21}$$

From the inequalities (17), (18) and (21) it follows that $u(x_1,x_2)\in W_2(R^2;H)$. Thus, we have shown that the bounded operator P_0 maps the space $W_2(R^2;H)$ onto $L_2(R^2;H)$ uniquely and because of it, according to Banach theorem on the inverse operator this mapping is isometric. Lemma is proved.

To prove the main theorem we'll prove one important theorem on resolvent estimation.

Theorem 1. *Let the conditions of lemma 1 are satisfied. Then for all* $(\xi,\eta)\in R^2$ *it takes place the estimation*

$$\sum_{j=0}^{2}\left(1+|\xi|+|\eta|\right)^j\left\|A^{2-j}P^{-1}(\xi,\eta)\right\|_{H\to H} \le const. \tag{22}$$

Here

$$P(\xi,\eta)=\left(\xi^2E+\eta^2E+A^2\right)-i\left(\xi\eta A_{1,1}+\xi A_{1,0}+\eta A_{0,1}-A_{0,0}\right). \tag{23}$$

Proof. Denote by

$$P_0(\xi,\eta)=\xi^2E+\eta^2E+A^2 \tag{24}$$

and

$$P_1(\xi,\eta)=-i\left(\xi\eta A_{11}+\xi A_{1,0}+\eta A_{0,1}-A_{0,0}\right). \tag{25}$$

It is obvious that the operator bunch $P(\xi,\eta)$ is invertible for all $(\xi,\eta)\in R^2$.

Really, for any $\psi\in D(A^2)$ and $(\xi,\eta)\in R^2$ from symmetricity of the coefficients $A_{1,1}$, $A_{1,0}$, $A_{0,1}$ and $A_{0,0}$ it follows that

$$\begin{aligned}&\left|(P(\xi,\eta)\psi,\psi)\right| = \left|(P_0(\xi,\eta)\psi,\psi)+(P_1(\xi,\eta)\psi,\psi)\right| \ge \\ &\ge (P_0(\xi,\eta)\psi,\psi) \ge \left(\xi^2+\eta^2\right)\|\psi\|^2+\|A\psi\|^2+\mu_0^2\|\psi\|^2.\end{aligned} \tag{26}$$

From the inequality (26) it follows that

$$\mu_0^2\|\psi\|^2 \le \left|(P(\xi,\eta)\psi,\psi)\right| \le \|P(\xi,\eta)\psi\|\cdot\|\psi\|.$$

Consequently, the inverse operator $P^{-1}(\xi,\eta)$ exists and is bounded in H. Now let's show the truth of inequality (22).

As for $\psi\in D(A^2)$ from the inequality (26) it follows that

$$\left(\xi^2+\eta^2+\mu_0^2\right)\|\psi\|^2\le\left|\left(P(\xi,\eta)\psi,\psi\right)\right|\le\|P(\xi,\eta)\psi\|\cdot\|\psi\|,$$

i.e.

$$\left\|P^{-1}(\xi,\eta)\right\|\le\left(\xi^2+\eta^2+\mu_0^2\right)^{-1}. \tag{27}$$

The inequality (22) for summand with degree $j=2$ follows from here.

Now we'll show that

$$\left\|A^2P^{-1}(\xi,\eta)\right\|\le const.$$

Let $y\in H$, then $A^{-2}y\in D\left(A^2\right)$. Consequently, for any $y\in H$ it takes place the inequality

$$\left|\left(P(\xi,\eta)A^{-2}y,A^{-2}y\right)\right|\ge\left(P_0(\xi,\eta)A^{-2}y,A^{-2}y\right)\ge$$
$$\ge\lambda^2\left\|A^{-2}y\right\|^2+\mu^2\left\|A^{-2}y\right\|^2+\left(y,A^{-2}y\right)\ge\left(y,A^{-2}y\right). \tag{28}$$

From the inequality (28) it follows that for all $y\in H$ with norm $\|y\|=1$ it takes place the inequality

$$\left(y,A^{-2}y\right)\le\left|\left(P(\xi,\eta)A^{-2}y,A^{-2}y\right)\right|\le\sup_{\|y\|=1}\left|\left(P(\xi,\eta)A^{-2}y,A^{-2}y\right)\right|\le$$
$$\le\sup_{\|y\|=1}\left\|P(\xi,\eta)A^{-2}y\right\|\cdot\left\|A^{-2}y\right\|\le\sup_{\|y\|=1}\left\|P(\xi,\eta)A^{-2}y\right\|\cdot\left\|A^{-2}\right\|\cdot\|y\|=\left\|A^{-2}\right\|\cdot\sup_{\|y\|=1}\left\|P(\xi,\eta)A^{-2}y\right\|.$$

From the last inequality we obtain that

$$\left\|A^{-2}\right\|=\sup_{\|y\|=1}\left(y,A^{-2}y\right)\le\left\|A^{-2}\right\|\sup_{\|y\|=1}\left\|P(\xi,\eta)A^{-2}y\right\|,$$

i.e.

$$\sup_{\|y\|=1}\left\|P(\xi,\eta)A^{-2}y\right\|\ge1.$$

Consequently, for $(\xi,\eta)\in R^2$

$$\left\|A^2P^{-1}(\xi,\eta)\right\|\le1, \tag{29}$$

i.e. the inequality (22) for summand with degree $j=0$ is also proved.

We'll show that from the inequalities (27) and (29) it follows that

$$\left(1+|\xi|+|\eta|\right)\left\|AP^{-1}(\xi,\eta)\right\|\le const.$$

Really, as for $y=P^{-1}(\xi,\eta)\in D\left(A^2\right)$ the inequality

$$\|\xi Ay\|^2=(\xi Ay,\xi Ay)=\left(\xi^2y,A^2y\right)\le\xi^2\|y\|\cdot\left\|A^2y\right\|\le\frac{1}{2}\left(\xi^4\|y\|^2+\left\|A^2y\right\|^2\right)$$

takes place, then for any $\psi\in D\left(A^2\right)$ it follows that

$$\left\|\xi AP^{-1}(\xi,\eta)\psi\right\|^2\le\frac{1}{2}\left(\left\|\xi^2P^{-1}(\xi,\eta)\psi\right\|^2+\left\|A^2P_0^{-1}(\xi,\eta)\psi\right\|^2\right). \tag{30}$$

Taking into account the inequalities (27) and (29) in the inequality (30), we obtain that

$$\left\|\xi AP^{-1}(\xi,\eta)\right\|\le const. \tag{31}$$

Analogously it is proved that

$$\left\|\eta AP^{-1}(\xi,\eta)\right\|\le const,\ \left\|AP^{-1}(\xi,\eta)\right\|\le const. \tag{32}$$

The inequality (22) for summand with degree $j=1$ follows from the inequalities (31) and (32). Theorem is proved.

Now we'll prove the main

Theorem 2. *Let the conditions of lemma 1 are satisfied. Then the equation (1) is regularly solvable.*

Proof. For any $f(x_1,x_2)\in L_2(R^2;H)$ define the vector-function

$$\hat{u}(\xi,\eta)=P^{-1}(\xi,\eta)\hat{f}(\xi,\eta),\ (\xi,\eta)\in R^2, \tag{33}$$

where $\hat{f}(\xi,\eta)$ is Fourier transformation of the vector-function $f(x_1,x_2)$.

According to theorem 1 for the vector-function $\hat{u}(\xi,\eta)$ it takes place the inequality

$$\left\|\xi^2\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)}+\left\|\eta^2\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)}+\left\|A^2\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)}\le const\left\|\hat{f}(\xi,\eta)\right\|^2_{L_2(R^2;H)},$$

i.e. the inverse Fourier transformation of the vector-function $\hat{u}(\xi,\eta)$, equal to $u(x_1,x_2)$ belongs to the space $W_2(R^2;H)$ and according to Plancherel theorem the inequlity

$$\|u\|_{W_2(R^2;H)}\le const\|f\|_{L_2(R^2;H)} \tag{34}$$

takes place for it.

It is obvious that the vector-function $u(x_1,x_2)$ satisfies the equation (1) almost everywhere. Consequently, the equation (1) is regularly solvable. Theorem is proved.

From the proof of theorem it follows

Theorem 3. *Let A be a self-adjoint positively defined operator, operators A_1, A_2 and A_0 are symmetric in H, moreover $D(A)\subset D(A_1)$, $D(A^2)\subset D(A_0)$, $D(A_2)=H$.*

Then the operator-differential equation

$$-\frac{d^2u}{dt^2}+A^2u+A_1\frac{du}{dt}+iA_2\frac{d^2u}{dt^2}+iA_0u=f(t),\ t\in R_+ \tag{35}$$

is regularly solvable.

Note that this theorem for $A_2=A_0=0$ is proved in the book [5, p.64] by the other method-method of reducing he equation (35) to the linear equation of the first order.

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