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ON PROPERTIES OF THE SOLUTION OF ONE NON-LINEAR PARABOLIC EQUATION

Abstract

In the present paper the question of the smoothness of the solution of one nonlinear degenerate equation has been investigated. Besides for the investigated problem some analogue of the maximum principle has been proved.

In the present paper the question of the smoothness of the solution of the initial boundary-value problem for the equation

$$\frac{\partial u}{\partial t} - D\left(|u|^{p_0} Du + |Du|^{p_1} Du\right) = f(x,t) \tag{1}$$

under some conditions on the parameters p_0 and p_1 is investigated.

Note that the case $p_0, p_1 > 2$ has been considered earlier in [1] and [2], where some smoothness of the solution of the considered problem has been obtained. The problem with a free boundary also has been considered in [2].

Naturally, the question about the possibility of getting such smoothness under another conditions on parameters arises. It is found out that this is also possible in the case of $p_0 \le 2$. But on this addition some condition which defines the dependence of the parameters p_0 and p_1, from each-other, arises §1 of the present paper is devoted to the investigation of the indicated case.

Besides, in the present paper under same conditions on the parameters p_0 and p_1, that in [2], the result of the maximum principle type is given.

Note that for the first time the question on the solvability of the equation (1) was studied in [3].

In the same place the extensive references devoted to the investigations of different problems for the equations of type (1) was given.

§1. Smoothness of the solvability of a mixed problem.

The following problem is investigated:

$$\frac{\partial u}{\partial t} - D\left(|u|^{p_0} Du + |Du|^{p_1} Du\right) = f(x,t), \quad (x,t) \in Q \equiv \Omega \times (0,T), \tag{1}$$

$$u(x,0) = u_0(x), \ x \in \Omega \equiv (a,b), \tag{2}$$

$$u|_{\Gamma} = 0, \quad \Gamma \equiv \partial\Omega \times (0,T), \tag{3}$$

where $f(x,t)$, $u_0(x)$ are some functions, (a,b) is a bounded interval, $D \equiv \dfrac{\partial}{\partial x}$.

As was noted above the question of the solvability of the solution of the problem (1)-(3) has been studied earlier in [1], in case when $p_0, p_1 > 2$.

In the present paper the case $1 \le p_0 \le 2$, $p_1 \ge 4$ is investigated.

The following condition which defines the alteration of the parameter p_0 depending on the parameter p_1 in the segment [1, 2], in case when $p_1 \ge 4$ we call the condition A):

A) For $p_1 \ge 4$ $p_0 \in \left[\frac{2p_1}{p_1+4}, 2\right]$.

Let's introduce the following space intersection of functions $\vartheta : Q \to R$:

$$R_1(Q) \equiv L_\nu\left(0,T;\overset{\circ}{S}{}^{\,}_{1,p_0+p_*,2}(\Omega)\right) \cap L_\infty\left(0,T;\overset{\circ}{W}{}^{1}_{\chi+2}(\Omega)\right) \cap L_\mu\left(0,T;\overset{\circ}{S}{}_{1,p_*,p_1,2}(\Omega)\right) \cap$$
$$\cap L_{\nu_1}\left(0,T;\underset{\circ}{S}{}^{1}_{1,p_1+\chi,2}(\Omega)\right) \cap W_2^1(Q) \cap L_{p_0+2}\left(0,T;S_{2,p_0,2}(\Omega)\right) \cap L_{p_1+2}\left(0,T;S^1_{1,p_1,2}(\Omega)\right) \cap$$
$$\cap \left\{\vartheta \middle| \vartheta(x,0) = u_0(x)\right\},$$ where

$\nu_1 = \chi + p_1 + 2$, $\nu = p_* + p_0 + 2$, $\chi \ge p_1$, $M = p_1 + p_* + 2$, $f_* \ge (4p_0 - 2)(\chi + 2)$ (about spaces S see [3], [4]).

Definition 1. *The solution of the problem (1)-(3) is the function* $u(x,t) \in R_1(Q)$ *satisfying the equation (1) in the sense of the space* $L_2(Q)$, *i.e. for any* $\vartheta \in L_2(Q)$ *the equality*

$$\int_Q \frac{\partial u}{\partial t}\vartheta dxdt - \int_Q D\left(|u|^{p_0} Du + |Du|^{p_1} Du\right)\vartheta\, dxdt = \int_Q f\vartheta dxdt$$

holds.

The following theorem is proved.

Theorem 1. *The problem (1)-(3) is solvable in the sense of determination on* $u_0(x) \in \overset{\circ}{W}{}^{1}_{\chi+2}(\Omega) \cap L_{p_*+2}(\Omega)$, $f(x,t) \in L_2\left(0,T;\overset{\circ}{W}{}^{1}_{\chi+2}(\Omega)\right)$, *where* $\chi \ge \max\left\{p_1, 2\left(\beta - \frac{p_1}{2}\right) - 2\right\}$, $p_* = \max\left\{(4p_0 - 2)(\chi + 2), 2\left(\alpha - \frac{p_0}{2}\right)\right\}$, *and* p_0 *and* p_1 *satisfy the condition A), where* $\alpha = (p_0 - 1)q \ge \frac{p_0}{2}$, $\beta = 2\frac{q}{q-1} \ge \frac{p_1}{2}$, $q \ge \frac{p_0}{2(p_0 - 1)}$.

The proof of the theorem is analogously to the proof of the corresponding fact in the case when $p_0 \ge 2$, $p_1 \ge 2$. Therefore note only some distinctive moments of the investigated case.

For the proof the generalized method of compactness [5] is used. Corresponding a priori estimations are obtained by using the inequality of the generalized coerciveness which in its turn is shown by the help of construction of some operator generating the coercive pair with operator, generated by the considered problem.

Let's state briefly the method of construction of the indicated operator.

Let's introduce the following space:

$$R_2(Q) \equiv R_1(Q) \cap L_2\left(0,T;W_2^2(\Omega)\right) \cap L_{\chi+2}\left(0,T;\underset{\circ}{S}{}^{1}_{1,\chi,2}(\Omega)\right) \cap \left\{\vartheta \middle| |Dv|^{\chi+2} Dv_t \in L_2(Q)\right\} \cap$$
$$\cap \left\{\vartheta \middle| D\vartheta_t \in L_2(Q)\right\},$$
$$R_2^1(Q) \equiv R_1(Q) \cap L_2\left(0,T;W_2^2(\Omega)\right) \cap L_{\chi+2}\left(0,T;\underset{\circ}{S}{}^{1}_{1,\chi,2}(\Omega)\right) \cap \left\{\vartheta \middle| \sqrt{T-t}\,|D\vartheta|^{\chi/2} D\vartheta_t \in L_2(Q)\right\} \cap$$
$$\cap \left\{\vartheta \middle| \sqrt{T-t}\,D\vartheta_t \in L_2(Q)\right\}.$$

Let $E : R_2(Q) \to W_2^1(Q) \in L_2(Q)$ be the restriction on Q of the operator $\widetilde{E} : R_2^1(Q_1) \to W_2^1(Q_1)$ generated by the problem

$$c_1 u_t - |Du|^{\chi} D^2 u - c^2 D^2 u - c_3 |u|^{p_*} u = h(x,t),\ (x,t) \in Q_1 \equiv [a,b) \times [0,T_1), \tag{4}$$

$$u(x,0) = u_0(x), \tag{5}$$

$$u|_{\Gamma_1} = 0 \qquad \Gamma_1 = \partial\Omega \times [0,T_1), \tag{6}$$

where $T_1 > T$; χ, p_*, c_1, c_2, c_3 are positive constants.

Theorem 2. *Let $h(x,t) \in W_2^1(Q_1)$, $Q_1 \equiv \Omega \times [0,T_1)$. Then for $\forall \chi > 2$ and $u_0(x) \in$ $\in W_2^1(\Omega) \cap S^1_{1,\chi,2}(\Omega) \cap L_{p_*+2}(\Omega)$ there exists $u(x,t) \in R_2(Q_1)$, which is the solution of the problem (4)-(6).*

The proof of the theorem is analogously to the proof of the corresponding fact from [6], where the problem (4)-(6) has been studied in the case, when $c_1 = 1$, $c_2 = c_3 = 0$.

Thus the operator $E: R_2(Q) \to \overset{\circ}{W}{}_2^1(Q)$ was defined. Also note the validity of the following two facts, which are necessary for the further reasoning.

Lemma 1. *The operator $E^{-1}: \overset{\circ}{W}{}_2^1(Q) \to R_2(Q)$ is weakly compact (the proof see [7]).*

Lemma 2. *The operator $\psi: R_1(Q) \to L_2(Q)$ generated by the expression $\psi(u) = \frac{\partial u}{\partial t} - D\left(|u|^{p_0} Du + |Du|^{p_1} Du\right)$ is weakly compact (the proof see in [7]).*

The latter two facts in totality with the inequality of the generalized coercitivity, about which we will write below, admits to apply the "general theorem of solvability" (see [3], [5]), to conclude on the solvability of the investigated problem.

As in the case p_0, $p_1 > 2$, the basic inequality of the generalized coercitivity is got step-by-step, i.e. $\psi(u)$ is scalarly multiplied by components of expression $E(u)$, after what the obtained inequalities are added.

Note that, the obtaining of inequalities in the given case is going on without alternations after multiplication by u_t, $|u|^{p_*} u$, $|Du|^{\chi+2} D^2 u$.

For example, the multiplication by $|u|^{p_*} u$ results in the following

Lemma 3. *Let $u(x,t)$ satisfy the condition (3) and $u(x,t) \in R_2(Q)$. Then the following equality is true:*

$$< \psi(u), c_1 |u|^{p_*} u > |_\tau = \frac{c_1}{p_* + 2} \int_\Omega |u|^{p_*+2} dx|_\tau + c_1(p_* + 1) \int_{Q_\tau} |u|^{p_*+p_0} |Du|^2 dxdt +$$

$$+ c_1(p_* + 1) \int_{Q_\tau} |u|^{p_*} |Du|^{p_1+2} dxdt - \frac{c_1}{p_* + 2} \int_\Omega |u_0|^{p_*+2} dx,$$

where $Q_\tau \equiv [0,\tau) \times \Omega$, $p_* = (4p_0 - 2)(\chi + 2)$, $c_1 > 0$.

The proof follows from the integration by parts. However multiplication by $-D^2 u$ some difficulties related with the alterations of the parameters p_1 arise, in connection with what we'll prove the following inequality.

Lemma 4. *Let $u(x,t)$ satisfy the condition (3) and $u(x,t) \in R_2(Q)$. Then the following inequality is true*

$$<\psi(u),-D^2u>|_\tau \ge \frac{1}{2}\int_\Omega |Du|^2 dx|_\tau + M_1\iint_{Q_\tau} |u|^{p_0}|D^2u|^2 dxdt +$$

$$+M_2\iint_{Q_\tau}|Du|^{p_1}|D^2u|^2 dxdt - \frac{1}{2}\int_\Omega |Du_0|^2 dx - \iint_{Q_\tau}|u|^{2\left(\alpha-\frac{p_0}{2}\right)}dxdt - \iint_{Q_\tau}|Du|^{2\left(\beta-\frac{p_1}{2}\right)}dxdt.$$

As a result of the multiplication $\psi(u)$ by $-D^2u$ it becomes obvious, that the whole problem is in the estimation of the integral $\iint_{Q_\tau}|u|^{p_0-2}u|Du|^2 D^2u$. Let's estimate the indicate integral.

Applying the Hölder inequality we obtain

$$\iint_{Q_\tau}|u|^{p_0-1}|Du|^2|D^2u|dxdt \le \iint_{Q_\tau}|u|^\alpha|D^2u|dxdt + \iint_{Q_\tau}|Du|^\beta|D^2u|dxdt,$$

where

$$\alpha=(p_0-1)q \ge \frac{p_0}{2},\ \beta=\frac{2q}{q-1}\ge\frac{p_1}{2}. \tag{7}$$

Show, that such q exists on our conditions.

For the condition $\alpha \ge \frac{p_0}{2}$ to be fulfilled it is necessary the fulfilling of the condition

$$q \ge \frac{p_0}{2(p_0-1)}. \tag{8}$$

On the other hand we've to choose β such that the condition (7) be fulfilled. Hence, it follows that the condition

$$\frac{4}{p_1}\ge 1-\frac{1}{q} \text{ or } \frac{1}{q}\ge 1-\frac{4}{p_1}=\frac{p_1-4}{p_1} \tag{9}$$

has to be fulfilled. From (8) and (9) it follows that

$$\frac{2(p_0-1)}{p_0}\ge\frac{p_1-4}{p_1} \text{ or } \frac{p_1+4}{p_1}\ge\frac{2}{p_0}.$$

The latter inequality is equivalent to the condition A), i.e. $p_0 \ge \frac{2p_1}{p_1+4}$.

Now applying Young's inequality to the integrals in the right hand side of (7) we'll obtain:

$$\iint_{Q_\tau}|u|^{p_0-1}|Du|^2|D^2u|dxdt \le \varepsilon\iint_{Q_\tau}|Du|^{p_1}|D^2u|^2dxdt + c(\varepsilon)\iint_{Q_\tau}|Du|^{2\left(\beta-\frac{p_1}{2}\right)}dxdt +$$

$$+\varepsilon\iint_{Q_\tau}|u|^{p_0}|D^2u|^2dxdt + c(\varepsilon)\iint_{Q_\tau}|u|^{2\left(\alpha-\frac{p_0}{2}\right)}dxdt.$$

The correctness of the inequality of lemma 4 is easily obtained from the latter inequality.

Now, taking into account also the results of the multiplication by $|Du|^{\chi+2}D^2u$ and u_t we obtain the following statement.

Lemma 5. *Let $u(x,t)$ satisfy the condition (3) and $u(x,t)\in R_2(Q)$. Then the following inequality is valid*

$$<\psi(u),Eu>\big|_\tau=M_1\int_\Omega|u(x,\tau)|^{(4p_0-2)(\chi+2)+2}dx+M_2\int_{Q_\tau}|u|^{(4p_0-2)(\chi+2)+p_0}|Du|^2dxdt+$$
$$+M_3\int_{Q_\tau}|u|^{(4p_0-2)(\chi+2)}|Du|^{p_1+2}dxdt+M_4\int_\Omega|Du(x,r)|^2dx+M_5\int_{Q_\tau}|u|^{p_0}|D^2u|^2dxdt+$$
$$+M_6\int_{Q_\tau}|Du|^{p_1}|D^2u|^2dxdt+M_7\int_\Omega|Du(x,t)|^{\chi+2}dx+M_8\int_{Q_\tau}|u|^{p_0}|Du|^{\chi}|D^2u|^2dxdt+$$
$$+M_9\int_{Q_\tau}|Du|^{p_1+\chi}|D^2u|^2dxdt+M_{10}\int_{Q_\tau}|u_t|^2dxdt+M_{11}\int_\Omega|Du(x,\tau)|^{p_1+2}dx+$$
$$+M_{12}\int_\Omega|u(x,\tau)|^{p_0}|Du(x,\tau)|^2dx-M_{13},$$

where M_i are some positive constants independent of $u(x,t)$.

Further part of the proof of the theorem is led according to the scheme of the proof of the "general theorem" from [3], with regard to lemma 1, 2, 5.

For the brevity of the presentation we omit the remaining reasonings.

§2. Maximum principle.

As was said above, the equation (1) was investigated in the case, when $p_0,p_1>2$, in [2]. Particularly the following result was obtained.

Theorem 3 (see [2]). *Let $u_0(x)\in \overset{\circ}{W}{}^1_{\chi+2}(\Omega)$, $f\in \overset{\circ}{W}{}^1_m(Q)$, $m,\chi\ge 0$. Then the solution $u(x,t)$ of the problem (1)-(3), for $p_0,p_1>2$ belongs to the space $W_2^1(Q)\cap C(0,T;L_2(\Omega))$. Furthermore, the inclusion*

$$u(x,t)\in C\left(0,T;C^{1+\alpha}(\Omega)\right),\ 0<\alpha<1$$

is valid. Using the latter fact, prove some analogue of the maximum principle.

Let's consider at first the case $f\equiv 0$.

Multiplying the equation (1) by $u(x,t)$ and integrating by parts we easily obtain the validity of the following statement.

Lemma 6. *Let the condition of the theorem 3 $(f\equiv 0)$ be fulfilled and $u(x,t)$ be the solution of the problem (1)-(3). Then the inequality*

$$\int_\Omega|u(x,t_1)|^2dx\le\int_\Omega|u(x,t_2)|^2dx,$$

where $t_2\ge t_1$ is correct.

Moreover the following theorem is correct.

Theorem 7. *Let the condition of the theorem (3) $f\equiv 0$ be fulfilled and $u(x,t)$ be the solution of the problem (1)-(3). Then the following estimation*

$$k_1\le u(x,t)\le k_2,$$

where $k_1=\min\{\min u_0(x),0\}$, $k_2=\max\{\max u_0(x),0\}$ is correct.

The proof of theorem 4. Assume inversely, i.e. assume that the maximum is admitted in interior of domain. Then taking into account the smoothness of the function $u(x,t)$ (see, theorem 3) there exist one or several subdomains Q with sufficiently smooth boundary, where $u(x,t)>k_2$. For the simplicity assume that there is one such subdomain (note that, if there are several such subdomains then as will be seen below, the further reasonings will not change principally) and $(a_1(t_0), a_2(t_0))$ is the section of the straight line $t = t_0$. Naturally, that $u(x,t_0) > k_2$, for $x \in (a_1(t_0), a_2(t_0))$.

Let's introduce the following function $u_{k_2} = \max\{0, u - k_2\}$. Multiply the equation (1) by u_{k_2} and integrate by $(a_1(t_0), a_2(t_0))$. We will obtain

$$\int_{a_1(t_0)}^{a_2(t_0)} \frac{\partial u}{\partial t} u_{k_2} dx - \int_{a_1(t_0)}^{a_2(t_0)} D\left(|Du|^{p_1} Du + |u|^{p_0} Du\right) u_{k_2} dx = 0 .$$

Taking into account that $u_{k_2} \in C^1([a_1(t_0), a_2(t_0)])$ (by virtue of theorem 3) and $u_{k_2}(a_1(t_0)) = u_{k_2}(a_2(t_0)) = 0$ integrate by parts the second integral of the latter equality:

$$\int_{a_1(t_0)}^{a_2(t_0)} \frac{\partial u}{\partial t} u_{k_2} dx - \int_{a_1(t_0)}^{a_2(t_0)} |Du|^{p_1} DuDu_{k_2} dx + \int_{a_1(t_0)}^{a_2(t_0)} |u|^{p_0} DuDu_{k_2} dx = 0 .$$

That is

$$\int_{a_1(t_0)}^{a_2(t_0)} \frac{\partial u}{\partial t} u_{k_2} dx + \int_{a_1(t_0)}^{a_2(t_0)} |Du|^{p_1} |Du_{k_2}|^2 dx + \int_{a_1(t_0)}^{a_2(t_0)} |u|^{p_0} |Du_{k_2}|^2 dx = 0 .$$

Hence it follows that

$$\int_{a_1(t_0)}^{a_2(t_0)} \frac{\partial u}{\partial t} u_{k_2} dx \le 0 .$$

Thus, using the property of the function u_{k_2} we will get

$$\int_{\Omega} \frac{\partial u}{\partial t} u_{k_2} dx \Big|_{t=t_0} \le 0 .$$

Hence it follows that

$$\int_0^{t_0} \int_{\Omega} \frac{\partial u}{\partial t} u_{k_2} dxdt \le 0$$

or

$$\int_0^{t_0} \int_{\Omega} \frac{\partial u_{k_2}}{\partial t} u_{k_2} dxdt \le 0 . \qquad (10)$$

Let $u_{k_{2n}}$ be the sequence of the functions $\overset{\circ}{C}{}^{\infty}(Q)$ convergent to u_{k_2} strongly in $\overset{\circ}{W}{}_2^1(Q) \cap C(0,T; L_2(\Omega))$.

For the functions $u_{k_{2n}}$ the equality:

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} |u_{k_{2n}}|^2 dx = \int_{\Omega} \frac{\partial u_{k_{2n}}}{\partial t} \cdot u_{k_{2n}} dx$$

is correct.

Consequently

$$\frac{1}{2} \int_0^{t_0} \frac{\partial}{\partial t} \int_{\Omega} |u_{k_{2n}}|^2 dxdt = \int_0^{t_0} \int_{\Omega} \frac{\partial u_{k_{2n}}}{\partial t} \cdot u_{k_{2n}} dxdt$$

or

$$\frac{1}{2} \int_{\Omega} |u_{k_{2n}}|^2 dx \Big|_{t=t_0} - \frac{1}{2} \int_{\Omega} |u_{k_{2n}}|^2 dx \Big|_{t=0} = \int_0^{t_0} \int_{\Omega} \frac{\partial u_{k_{2n}}}{\partial t} \cdot u_{k_{2n}} dxdt .$$

Since $u_{k_{2n}} \to u_{k_2}$ strongly in $C(0,T;L_2(\Omega))$ and $\overset{\circ}{W}{}^1_2(Q)$, then

$$\frac{1}{2}\int_\Omega |u_{k_2}|^2 dx\Big|_{t=t_0} - \frac{1}{2}\int_\Omega |u_{k_2}|^2 dx\Big|_{t=0} = \int_0^{t_0}\int_\Omega \frac{\partial u_{k_2}}{\partial t}\cdot u_{k_2}\,dxdt\,.$$

Hence, by virtue of (10) it follows that

$$\frac{1}{2}\int_\Omega |u_{k_2}|^2 dx\Big|_{t=t_0} - \frac{1}{2}\int_\Omega |u_{k_2}|^2 dx\Big|_{t=0} \le 0\,.$$

Since $u_{k_2}\big|_{t=0} = 0$, then

$$\frac{1}{2}\int_\Omega |u_{k_2}|^2 dx\Big|_{t=t_0} \le 0\,.$$

Consequently $u_{k_2}\big|_{t=t_0} = \max\{0, u-k_2\} = 0$, whence follows that $u(x,t_0) < k_2$, which contradicts to our assumption.

Now, let's prove that $u(x,t) \ge k_1$.

Use standard procedure. Multiply the equation (1) by -1. We will obtain

$$(-u)_t - D\left(\left(|-u|^{p_0} + |D(-u)|^{p_1}\right)D(-u)\right) = 0\,.$$

Besides the conditions:

$$-u|_\Gamma = 0\ ,\quad -u(x,0) = -u_0(x) \le -k_1$$

are fulfilled.

But by virtue of the above given reasoning $-u \le k_1$ on the whole domain Q. That is $u \ge k_1$. So $k_1 \le u \le k_2$.

Theorem 4 has been proved.

Theorem 5. *Let $u(x,t)$ be the solution of the problem (1)-(3) under the conditions of theorem 3. Then*

I) if $f \le 0$, then the estimation $u \le k_2$ is correct;

II) if $f \ge 0$, then the estimation $k_1 \le u$ is correct.

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