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REGULARITY OF BOUNDARY POINTS WITH RESPECT TO THE FIRST BOUNDARY VALUE PROBLEM FOR DEGENERATE PARABOLIC EQUATIONS

Abstract

The article deals with the first boundary value problem for the class of the second order parabolic equations of non-divergent structure with non-uniform degree degeneration on the boundary. The sufficient regularity condition for boundary point in terms of special parabolic capacities has been established.

Introduction. The aim of this paper is to obtain sufficient regularity condition of boundary point with respect to the first boundary value problem for class of the second order parabolic equations allowing non-uniform degree degeneration in boundary point. We notice that first results on boundary properties of solutions of the second order parabolic equations are related to famous works of I.G. Petrowsky [1] and A.N. Tikhonov [2] in which the regularity conditions of boundary points for heat equation have been established. For parabolic equations with variable coefficients mention the works [3-8]. The results like to main result of present paper has been proved in [9]. We especially mark the significant monography of E.M. Landis [10], which was the source of ideas for us.

1^0. Construction of special L-subparabolic functions.

Consider in bounded domain D with parabolic boundary $\Gamma(D)$ (look [10]), situated in $(n+1)$- dimensional Euclidean space $\mathbf{R}_{n+1}$ of the points $(x,t)=(x_1,...,x_n,t)$, $(0,0)\in\Gamma(D)$, parabolic equation

$$Lu=\sum_{i,j=1}^{n}a_{ij}(x,t)\,u_{x_ix_j}-u_t=0 \qquad (1)$$

under assumption that for all $(x,t)\in D$ and arbitrary n-dimensional vector ξ

$$\mu\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le\mu\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2\,, \qquad (2)$$

where $\mu\in(0,1]$ is constant, $\lambda_i(x,t)=\left(|x|_\alpha+\sqrt{|t|}\right)^{\alpha_i}$, $|x|_\alpha=\sum_{i=1}^{n}|x_i|^{\frac{2}{2+\alpha_i}}$, $\alpha=(\alpha_1,...,\alpha_n)$, $\alpha_i\ge0$, $i=1,...,n$.

Under the solution of equation (1) in domain D we will understand the function $u(x,t)\in C^{2,1}(D)\cap C(\overline{D})$ that transforms (1) into identity. Function $u(x,t)\in C^{2,1}(D)$, that satisfies in D the inequality $Lu\ge0$ is called L- subparabolic in D. Function $u(x,t)$ is called L-superparabolic in D, if $-u(x,t)$ is L- subparabolic in D.

Point $(x^0,t^0)\in\Gamma(D)$ is called regular with respect to the first boundary value problem in D

$$\left.\begin{aligned}&Lu=0,\ (x,t)\in D;\\&u\big|_{\Gamma(D)}=\varphi,\ \varphi\in C[\Gamma(D)]\end{aligned}\right\} \qquad (3)$$

if for any function $\varphi(x,t)$ which is continuous on $\Gamma(D)$ following equality is implemented

$$\lim_{(x,t)\to(x^0,t^0)} u(x,t)=\varphi\left(x^0,t^0\right).$$

Let $\alpha^+=\max\{\alpha_1,...,\alpha_n\}$, $\alpha^-=\min\{\alpha_1,...,\alpha_n\}$. Let further for $R>0$, $k>0$, n – dimensional vector x^0

$$E_R^{x_0}(k)=\left\{x:\sum_{i=1}^n\frac{\left(x_i-x_i^0\right)^2}{R^{\alpha_i}}<(kR)^2\right\},$$

is ellipsoid for $t_1<t_2$ $C_{x^0;R;k}^{t_1,t_2}=E_R^{x_0}(k)\times(t_1,t_2)$.

For $s>0$, $\beta>0$, $R>0$ we denote

$$G_{s,\beta}^{(R)}(x,t)=\begin{cases} t^{-s}\exp\left[-\sum_{i=1}^n\frac{(x_i-y_i)^2}{R^{\alpha_i}}\Big/4\beta\, t\right], & \text{if } t>0,\\ 0, & \text{if } t\le 0.\end{cases}$$

Let $b=\min\left\{\frac{1}{16\beta s},31\right\}$;;

$$\mathbf{C}_1=C_{0;R;17}^{-2bR^2,0},\quad \mathbf{C}_2=C_{0;R;1}^{-\frac{3}{16}bR^2,0},\quad \mathbf{C}_3=C_{0;R;9}^{-\frac{3}{8}bR^2,0},\quad \mathbf{C}_4=C_{0;R;10}^{-\frac{11}{8}bR^2,-\frac{b}{2}R^2},$$

$$\mathbf{C}_5=C_{0;R;8}^{-\frac{7}{8}bR^2,-\frac{b}{2}R^2},\quad A_1=\mathbf{C}_1\setminus\mathbf{C}_2,\quad A_2=\mathbf{C}_4\setminus\mathbf{C}_5.$$

Lemma 1. *There are constants* s *and* β, *which depend only on* μ, n *and* α *such that*

$$L_{(x,t)}G^{(R)}(x-y,t-\tau)\ge 0 \quad \textit{for} \quad (x,t)\in A_1\setminus\{y,\tau\}.$$

Proof. We fix (y,τ) and let $t>\tau$. We denote $G_{s,\beta}^R$ through G and have

$$LG=G\left\{\sum_{i,j=1}^n a_{ij}(x,t)\frac{(x_i-y_i)(x_j-y_j)}{4\beta^2(t-\tau)^2R^{\alpha_i+\alpha_j}}-\frac{1}{2\beta(t-\tau)}\sum_{i=1}^n\frac{a_{ii}(x,t)}{R^{\alpha_i}}+\right.$$

$$\left.-\frac{1}{2\beta(t-\tau)}\sum_{i=1}^n\frac{a_{ii}(x,t)}{R^{\alpha_i}}+\frac{s}{t-\tau}-\frac{\sum_{i=1}^n\frac{(x_i-y_i)^2}{R^{\alpha_i}}}{4\beta(t-\tau)^2}\right\}. \qquad (4)$$

According to condition (2)

$$\sum_{i,j=1}^n a_{ij}(x,t)\frac{(x_i-y_i)(x_j-y_j)}{4\beta^2(t-\tau)^2R^{\alpha_i+\alpha_j}}\ge\mu\sum_{i=1}^n\frac{\lambda_i(x,t)}{R^{\alpha_i}}\frac{(x_i-y_i)^2}{4\beta^2(t-\tau)^2R^{\alpha_i}}.$$

But for $(x,t)\in A_1$

$$\sum_{i=1}^n\frac{x_i^2}{R^{\alpha_i}}\ge R^2$$

and there is number i_0, $1\le i_0\le n$, such that

$$\frac{x_{i_0}^2}{R^{\alpha_{i_0}}}\ge\frac{R^2}{n}.$$

Now

$$|x_{i_0}|\geq\frac{R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}},$$

$$|x|_\alpha\geq|x_{i_0}|^{\frac{2}{2+\alpha_{i_0}}}\geq\frac{R^{\left(1+\frac{\alpha_{i_0}}{2}\right)\frac{2}{\alpha_{i_0}+2}}}{n^{\frac{1}{2+\alpha_{i_0}}}}=C_1(\alpha,n)R,$$

$$\lambda_i(x,t)=\left(|x|_\alpha+\sqrt{|t|}\right)^{\alpha_i}\geq\left(|x|_\alpha\right)^{\alpha_i}\geq C_2(\alpha,n)R^{\alpha_i},\quad \frac{\lambda_i(x,t)}{R^{\alpha_i}}\geq C_2.$$

So that

$$\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{(x_i-y_i)(x_j-y_j)}{4\beta^2(t-\tau)^2R^{\alpha_i+\alpha_j}}\geq\frac{\mu C_2}{4\beta^2(t-\tau)^2}\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}.$$

Further for $i=1,\ldots,n$ we have

$$\frac{a_{ii}(x,t)}{R^{\alpha_i}}\leq\frac{\mu^{-1}\lambda_i(x,t)}{R^{\alpha_i}}.$$

But for $(x,t)\in A_1$

$$\sum_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}<(17R)^2 \text{ and } |x_i|<17R^{1+\frac{\alpha_i}{2}}.$$

So that

$$|x|_\alpha=\sum_{i=1}^{n}|x_i|^{\frac{2}{2+\alpha_i}}\leq 17nR,$$

$$\sqrt{|t|}\leq\sqrt{\frac{2}{16\beta s}R^2}=\frac{1}{\sqrt{8\beta s}}R,$$

$$\lambda_i(x,t)\leq R^{\alpha_i}\left(17n+\frac{1}{\sqrt{8\beta s}}\right)^{\alpha_i}.$$

Then

$$\frac{\lambda_i(x,t)}{R^{\alpha_i}}\leq\left(17n+\frac{1}{\sqrt{8\beta s}}\right)^{\alpha^*},$$

and from (4) we have

$$LG\geq G\left\{\frac{\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}}{4\beta(t-\tau)^2}\left[\frac{\mu C_2}{\beta}-1\right]+\frac{1}{t-\tau}\left[s-\frac{n}{2\beta}\left(17n+\frac{1}{\sqrt{8\beta s}}\right)^{\alpha^*}\right]\right\}.$$

We choose and fix $\beta=\mu C_2$, s from condition

$$s\geq\frac{n}{2\mu C_2}\left(17n+\frac{1}{\sqrt{8\mu C_2 s}}\right)^{\alpha^*}.$$

Then $LG\geq 0$ and lemma is proved.

Further we consider numbers s and β fixed according to lemma 1.

2^0. Theorem on increasing of positive solutions.

Definition. *Let H be B-set in A_1. We call measure μ admissible on H, if*

$$\int_H G_{s,\beta}^{(R)}(x-y,t-\tau)d\mu(y,\tau)\le 1,$$

when $(x,t)\notin H$.

Number $p_{s,\beta}^{(R)}(H)=\sup\mu(H)$, where the least upper bound is taken by all admissible measures is called parabolic $(s;\beta;R)$-capacity of set H.

Lemma 2. *Let in cylinder $\mathbf{C}_1$ is situated domain D having limit points at $\Gamma(\mathbf{C}_1)$ and intersecting $\mathbf{C}_2$. Let further in D positive solution $u(x,t)$ of the equation $Lu=0$, continuous in $\overline{D}$ and vanishing at that part $\Gamma(D)$, which lies strictly in $\mathbf{C}_1$ is defined. Then there is positive constant $\eta>0$ which depends only on coefficients of operator L and n such that*

$$\sup_D u\ge\left(1+\eta R^{-2s}p_{s,\beta}^{(R)}(H)\right)\sup_{D\cap\mathbf{C}_2} u,$$

where $H=\mathbf{C}_3\setminus D$.

Proof. At first we will receive some estimates of function $G_{s,\beta}^{R}(x,t)$. Let $(y,\tau)\in H$, (x,t) is an arbitrary fixed point at the lateral surface S of cylinder $\mathbf{C}_1$. We estimate

$$\sup_{\substack{(x,t)\in S\\(y,\tau)\in H}} G^{(R)}(x-y,t-\tau).$$

Without loss of generality we can suppose, that $t>\tau$ (otherwise $G^{(R)}=0$). We find value $t>\tau$ for which function $G^{(R)}(x-y,t-\tau)$ achieves its maximum (x and (y,τ) are fixed). We have

$$\frac{\partial}{\partial t}G^{R}(x-y,t-\tau)=0,$$

$$t-\tau=\frac{\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}}{4\beta s}.$$

But $x\in\partial E_R^{x'}(8)$, i.e.

$$\sum_{i=1}^{n}\frac{(x_i-x_i')^2}{R^{\alpha_i}}=(8R)^2.$$

Otherwise $y\in E_R^{x'}(1)$, i.e.

$$\sum_{i=1}^{n}\frac{(y_i-x_i')^2}{R^{\alpha_i}}\le R^2.$$

Further we have

$$\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}=\sum_{i=1}^{n}\frac{\left[(x_i-x_i')-(y_i-x_i')\right]^2}{R^{\alpha_i}}=\sum_{i=1}^{n}\frac{(x_i-x_i')^2}{R^{\alpha_i}}+$$

$$+\sum_{i=1}^{n}\frac{(y_i-x_i')^2}{R^{\alpha_i}}-2\sum_{i=1}^{n}\frac{(x_i-x_i')(y_i-x_i')}{R^{\alpha_i}}. \qquad (5)$$

For any $\varepsilon>0$

$$2\left|\sum_{i=1}^{n}\frac{(x_i-x_i')(y_i-x_i')}{R^{\alpha_i}}\right|\le\varepsilon\sum_{i=1}^{n}\frac{(x_i-x_i')^2}{R^{\alpha_i}}+\frac{1}{\varepsilon}\sum_{i=1}^{n}\frac{(y_i-x_i')^2}{R^{\alpha_i}}.$$

From (5) we receive

$$\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}\ge(1-\varepsilon)\sum_{i=1}^{n}\frac{(x_i-x_i')^2}{R^{\alpha_i}}+\left(\frac{1}{\varepsilon}-1\right)\sum_{i=1}^{n}\frac{(y_i-x_i')^2}{R^{\alpha_i}}.$$

Consider $\varepsilon\le 1$ we have

$$\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}\ge(1-\varepsilon)64R^2-\left(\frac{1}{\varepsilon}-1\right)R^2=R^2\left(64(1-\varepsilon)+1-\frac{1}{\varepsilon}\right).$$

We suppose $\varepsilon=\frac{1}{2}$. Now

$$\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}\ge R^2\left[\frac{64}{2}-1\right]=31R^2\ge bR^2.$$

Taking into account that $G^{(R)}(x-y,t-\tau)$ is monotone by t up to first maximum we receive

$$\sup_{\substack{(x,t)\in S\\(y,\tau)\in H}}G^{(R)}(x-y,t-\tau)\le\left(bR^2\right)^{-s}\exp\left[-\frac{31R^2}{4\beta\, bR^2}\right]\le$$

$$\le\left(bR^2\right)^{-s}\exp\left[-\frac{31}{4\beta\, b}\right]\le\left(bR^2\right)^{-s}\exp\left[-\frac{7}{b\beta}\right]. \tag{6}$$

Now we estimate

$$\inf_{\substack{(x,t)\in C_2\\(y,\tau)\in H}}G^{(R)}(x-y,t-\tau).$$

We have

$$\frac{bR^2}{4}\le t-\tau\le bR^2,$$

$$\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{R^{\alpha_i}}=\sum_{i=1}^{n}\frac{\left[(x_i-x_i')+(x_i'-y_i)\right]^2}{R^{\alpha_i}}\le$$

$$\le 2\left(\sum_{i=1}^{n}\frac{(x_i-x_i')^2}{R^{\alpha_i}}+\sum_{i+1}^{n}\frac{(y_i-x_i')^2}{R^{\alpha_i}}\right)\le 2\left(R^2+R^2\right)=4R^2.$$

Thus

$$\inf_{\substack{(x,t)\in C_2\\(y,\tau)\in H}}G^{(R)}(x-y,t-\tau)\ge\left(bR^2\right)^{-s}\exp\left[-\frac{4R^2}{4\beta\cdot\frac{bR^2}{4}}\right]=\left(bR^2\right)^{-s}\exp\left[-\frac{4}{b\beta}\right]. \tag{7}$$

Now we fix arbitrary $\varepsilon>0$ and suppose that $p_{s,\beta}^{(R)}(H)>0$. Otherwise statement of lemma is obvious. Let $0<\varepsilon<p_{s,\beta}^{(R)}(H)$. We consider in D function

$$\vartheta(x,t)=M\left[1-U(x,t)+\left(bR^2\right)^{-s}\exp\left[-\frac{7}{b\beta}\right]p_{s,\beta}^{(R)}(H)\right],$$

where $M=\sup\limits_{D}u$ and $U(x,t)=\int\limits_{H}G_{s,\beta}^{(R)}(x-y,t-\tau)d\mu(y,\tau)$.

Then the measure μ at H such that

$$U(x,t)\le 1,\quad \text{when}\quad (x,t)\notin H, \tag{8}$$

$$\mu(H)\ge p_{s,\beta}^{(R)}(H)-\varepsilon. \tag{9}$$

Now we estimate $\inf\limits_S \vartheta(x,t)$. We have

$$\inf_S \vartheta(x,t)\ge M\left[1-\sup_S U(x,t)+\left(bR^2\right)^{-s}\exp\left[-\frac{7}{b\beta}\right]p_{s,\beta}^{(R)}(H)\right].$$

But taking into account (6), we obtain

$$\inf_S U(x,t)\le \sup_{\substack{(x,t)\in S\\(y,\tau)\in H}} G^{(R)}(x-y,t-\tau)\mu(H)\le\left(bR^2\right)^{-s}\exp\left[-\frac{7}{b\beta}\right]p_{\varepsilon,\beta}^{(R)}(H).$$

Thus

$$\inf_S \vartheta(x,t)\ge M\ge u(x,t)$$

and

$$\vartheta\big|_S\ge u\big|_S\,. \tag{10}$$

Let Γ_1 be that part $\Gamma(D)$ which is situated at the lower base of cylinder $\mathbf{C}_1$. When $(x,t)\in\Gamma_1$ $G^{(R)}(x-y,t-\tau)=0$, i.e. $U(x,t)=0$. Then

$$\vartheta\big|_{\Gamma_1}\ge M\ge u(x,t).$$

That's why

$$\vartheta\big|_{\Gamma_1}\ge u\big|_{\Gamma_1}\,. \tag{11}$$

Let Γ_2 be that part $\Gamma(D)$ which is situated strictly in $\mathbf{C}_1$. We have

$$u\big|_{\Gamma_2}=0$$

according to condition of theorem. But from (6) follows

$$u\Big|_{\Gamma_2}\ge M\left[\left(bR^2\right)^{-s}\exp\left[-\frac{7}{b\beta}\right]p_{s,\beta}^{(R)}(H)\right]>0.$$

Now

$$\vartheta\big|_{\Gamma_2}\ge u\big|_{\Gamma_2}\,. \tag{12}$$

From (10)-(12) we deduce

$$(\vartheta-u)_{\Gamma(D)}\ge 0\,. \tag{13}$$

But at the same time function $\vartheta(x,t)$ is L – superparabolic according to previous lemma. Then from (13) we receive that $\vartheta-u\ge 0$ in D and particularly

$$\sup_{D\cap\mathbf{C}_2} u\le M\left[1-\inf_{\mathbf{C}_2} U(x,t)+\left(bR^2\right)^{-s}\exp\left[-\frac{7}{b\beta}\right]p_{(s,\beta)}^{(R)}(H)\right].$$

But because of (7) and (9)

$$\inf_{\mathbf{C}_2} U(x,t)\ge \inf_{\substack{(x,t)\in\mathbf{C}_2\\(y,\tau)\in H}} G^{R}(x-y,t-\tau)\mu(H)\ge\left(bR^2\right)^{-s}\exp\left[-\frac{4}{b\beta}\right]\left(p_{s,\beta}^{(R)}(H)-\varepsilon\right).$$

So that

$$\sup_{D\cap\mathbf{C}_2} u\le M\left[1-R^{-2s}b^{-s}\left(\exp\left[-\frac{4}{b\beta}\right]-\exp\left[-\frac{7}{b\beta}\right]\right)p_{s,\beta}^{(R)}(H)+\left(bR^2\right)^{-s}\exp\left[-\frac{4}{b\beta}\right]\varepsilon\right]. \tag{14}$$

We denote $b^{-s}\left(\exp\left[-\frac{4}{b\beta}\right]-\exp\left[-\frac{7}{b\beta}\right]\right)$ through η. From (14) and arbitrariness of ε we conclude

$$\sup_{D\cap\mathbf{C}_2} u \le M\left[1-\eta R^{-2s} p_{s,\beta}^{(R)}(H)\right] \le M - \sup_{D\cap\mathbf{C}_2} u\cdot\eta\, R^{-2s} p_{s,\beta}^{(R)}(H).$$

The statement of lemma is followed.

Lemma 3. *Let in A_1 be domain D intersect $\mathbf{C}_3$ and having limit point at $\Gamma(A_1)$. Let further in D there is L-subparabolic function $u(x,t)$ which is non-negative and continuous in $\overline{D}$ and vanishes at part Γ_1 of boundary $\Gamma(D)$ that lies strictly in A_1. Then if $H = A_2 \setminus D$,*

$$\sup_D u \ge \left[1+\eta R^{-2s} p_{s,\beta}^{(R)}(H)\right] \sup_{D\cap\Gamma(\mathbf{C}_3)} u.$$

Here constant $\eta > 0$ depends only on s,β,n and coefficients of operator L.

Proof. Without loss of generality we suppose that $\sup\limits_{D\cap\Gamma(\mathbf{C}_3)} u = 1$. So we have to show that

$$\sup_D u \ge 1+\eta\, R^{-2s} p_{s,\beta}^{(R)}(H). \qquad (15)$$

We choose at $\Gamma(\mathbf{C}_3)$ minimal number of points $(x^1,t^1),\ldots,(x^m,t^m)$ in order to

i) $A_2 \le \bigcup\limits_{n=1}^{m} C_{x^i;R;1}^{t^i-bR^2,\,t^i-\frac{bR^2}{2}}$;

ii) for any i_0 and any point $(x^*,t^*)\in\Gamma(\mathbf{C}_3)$ there exists the chain $(x^{i_1},t^{i_1})..(x^{i_k},t^{i_k})$ so that $(x^*,t^*)\in \mathbf{C}_{x^{i_k};R;1}^{t^{i_k}-\frac{b}{4}R^2,\,t^{i_k}}$ and in intersection $\mathbf{C}_{x^{i_l};R;1}^{t^{i_l}-bR^2,\,t^{i_l}-\frac{b}{2}R^2} \cap \mathbf{C}_{x^{i_{l+1}};R;1}^{t^{i_{l+1}}-bR^2,\,t^{i_{l+1}}-\frac{b}{2}R^2}$ contains cylinder $\mathbf{C}_{\tilde{x}^l;R;\frac{1}{8}}^{\tilde{t}^l-\frac{b}{R^2}64,\,\tilde{t}^l}$, $l=0,1,\ldots,k-1$.

It is clear that $m = m(n)$.

According to capacity condition among cylinders $\mathbf{C}_{x^i;R;1}^{t^i-bR^i,\,t^i-\frac{b}{2}R^2}$, $i=1,\ldots,m$ there is $\mathbf{C}_{x^{i_0};R;1}^{t^{i_0}-bR^2,\,t^{i_0}-\frac{b}{2}R^2}$ such that

$$p_{s,\beta}^{(R)}\left(H\cap\mathbf{C}_{x^{i_0};R;1}^{t^{i_0}-bR^2,\,t^{i_0}-\frac{b}{2}R^2}\right) \ge \frac{p_{s,\beta}^{(R)}(H)}{m}. \qquad (16)$$

Let (x^*,t^*) be point at $\Gamma(C_3)$ in which $u(x^*,t^*)=1$, $(x^{i_1},t^{i_1}),\ldots,(x^{i_k},t^{i_k})$ is the chain from condition ii). Let

$$\delta = \frac{\eta\, p_{s,\beta}^{(R)}(H)R^{-2s}}{2m\left[1+\frac{\eta p_{s,\beta}^{(R)}(\mathbf{C}_1)}{R^{2s}}\right]}.$$

We denote $H_0 = H\cap\mathbf{C}_{x^{i_0};R;1}^{t^{i_0}-bR^2,\,t^{i_0}-\frac{b}{2}R^2}$. From (16) it follows

$$p_{s,\beta}^{(R)}(H_0)\geq \frac{p_{s,\beta}^{(R)}(H)}{m}.$$

j) We suppose

$$\sup_{D\cap C_{x^{i_0};R;1}^{t^{i_0}-\frac{b}{4}R^2,t^{i_0}}} \geq 1-\delta .$$

Then

$$\sup_D u \geq \sup_{D\cap C^{t^{i_0}-bR^2,t^{i_0}}} u \geq (1-\delta)\left(\frac{1+\eta\, p_{s,\beta}^{(R)}(H)}{R^{2s}}\right) = 1+\frac{\eta}{m}\frac{p_{s,\beta}^{(R)}(H)}{R^{2s}} -$$

$$-\frac{\eta}{2m}\frac{p_{s,\beta}^{(R)}(H)}{R^{2s}} = 1+\frac{\eta}{2m}\frac{p_{s,\beta}^{(R)}(H)}{R^{2s}}$$

and lemma is proved.

jj) We suppose that

$$\sup_{D\cap C_{x^{i_0};R;1}^{t^{i_0}-\frac{b}{4}R^2,t^{i_0}}} u < 1-\delta .$$

We put $\vartheta_1(x,t)=u(x,t)-1+\delta$. It is clear, that $L\vartheta_1\geq 0$ in D. We denote through D_1 set of points $(x,t)\in D$, where $\vartheta_1(x,t)>0$. According to our admission cylinder $C_{x^{i_0};R;1}^{t^{i_0}-\frac{b}{4}R^2,t^{i_0}}$ belongs to complement of domain D_1. Intersection of cylinder $C_{x^{i_1};R;1}^{t^{i_1}-bR^2,t^{i_1}-\frac{b}{2}R^2}$ with complement D contains cylinder $C_{\widetilde{x}^1;R;\frac{1}{8}}^{\widetilde{t}^1-\frac{b}{64}R^2,\widetilde{t}^1}=\widetilde{\mathbf{C}}$. Thus

$$p_{s,\beta}^{(R)}(\widetilde{\mathbf{C}})\geq \eta_1(s,\beta)R^{2s}.$$

Let

$$\sigma=\frac{\eta\eta_1}{2(1+\eta\eta_1)}.$$

If $\sup_{(x,t)\in D\cap C_{x^{i_1},R}^{t^{i_2}-\frac{b}{4}R^2,t^{i_1}}} \vartheta_1 \geq \delta(1-\delta)$, i.e.

$$\sup_{D\cap C_{x^{i_1};R;1}^{t^{i_1}-\frac{b}{4}R^2,t^{i_1}}} u \geq \delta(1-\sigma)+1-\delta = 1-\sigma\delta ,$$

then

$$\sup_D \vartheta_1 \geq \sup_{D\cap C_{x^{i_1},R;8}^{t^{i_1}-bR^2,t^{i_1}}} \vartheta_1 \geq (1+\eta\eta_1)\sup_{C_{x^{i_1};R;1}^{t^{i_1}-\frac{b}{4}R^2,t^{i_1}}} \vartheta$$

or

$$\sup_D u \geq 1-\delta+(1+\eta\eta_1)\delta(1-\sigma)=1+\delta\,\frac{\eta\eta_1}{2}\geq 1+\eta_2(L,n)\frac{p_{s,\beta}^{(R)}(H)}{R^{2s}}$$

and in this case lemma is proved.

Let

$$\sup_{D\cap C_{x^{i_1};R;1}^{t^{i_1}-\frac{bR^2}{4},t^{i_1}}} u \leq 1-\delta\sigma .$$

Then we consider function

$$\vartheta_2(x,t)=u(x,t)-1+\delta\sigma .$$

By analogy with last case we obtain if

$$\sup_{D\cap C_{x^{i_2};R;1}^{t^{i_2}-\frac{bR^2}{4},t^{i_2}}} \vartheta_2 \le \delta\sigma(1-\sigma),$$

i.e.

$$\sup_{D\cap C_{x^{i_2};R;1}^{t^{i_2}-\frac{bR^2}{4},t^{i_2}}} \vartheta \le 1-\sigma^2\delta ,$$

then lemma is proved.

If

$$\sup_{D\cap C_{x^{i_2};R;1}^{t^{i_2}-\frac{bR^2}{4},t^{i_2}}} \vartheta \le 1-\sigma^2\delta ,$$

we put $\vartheta_3(x,t)=u(x,t)-1+\delta\sigma^2$ and continue by analogy. Cylinder $C_{x^{i_k};R;1}^{t^{i_k}-\frac{bR^2}{4},t^{i_k}}$ has point (x^*,t^*), where $u(x^*,t^*)=1$ and no later than on k -th step we obtain alternative

$$\sup_{D\cap C_{x^{i_k};R;1}^{t^{i_k}-\frac{bR^2}{4},t^{i_k}}} u \ge 1-\delta\sigma^2$$

and because of $k \le m$ finally we get

$$\sup_D u \ge 1+\min\left\{\frac{\eta}{2m},\delta(1+\eta\eta_1)\sigma^{m^{-1}}\right\}\frac{p_{s,\beta}^{(R)}(H)}{R^{2s}} \ge 1+\eta_3(L,\eta)\frac{p_{s,\beta}^{(R)}(H)}{R^{2s}}$$

and lemma is proved.

Let for natural m

$$\mathbf{C}^m = C_{0;4^{-m};17}^{-2h4^{-2m},0}, A^m = C^{-\frac{1}{18}h\cdot 4^{-2m},-\frac{h}{2}4^{-2m}} \setminus C_{0;4^{-m};8}^{-\frac{7}{8}h4^{-2m},-\frac{h}{2}4^{-2m}} ;\ H^m = A^m \setminus D,\ p_{s,\beta}^{(4-m)}(H^m)=\gamma_m .$$

Theorem 1. *Let in $\mathbf{C}^m$ there is domain D which intersects $\mathbf{C}^{m+1}$ and has limit points at $\Gamma(\mathbf{C}^m)$. Let further in D $L-$subparabolic function $u(x,t)$, which is non-negative and continuous in $\overline{D}$ and vanishes at $\Gamma(D)\cap\mathbf{C}^m$ is defined. Then*

$$\sup_D u \ge \left(1+\eta 4^{4ms}\gamma_m\right) \sup_{D\cap\mathbf{C}^{m+1}} u .$$

Proof. Let for natural m $\mathbf{C}_1^m = C_{0;4^{-m};9}^{-\frac{3}{8}h4^{-2m},0}$. We prove that $\mathbf{C}^{m+1} \subset \mathbf{C}^m$. It is enough to show that for $m=1,2,\ldots$

$$17\cdot 4^{-(m+1)\left(1+\frac{\alpha_i}{2}\right)} \le 9\cdot 4^{-m\left(1+\frac{\alpha_i}{2}\right)};\ i=1,\ldots,n ,$$

$$2\cdot 4^{-2(m+1)} \le \frac{3}{8}4^{-2m} .$$

These two conditions are equivalent to next

$$\frac{1}{4} \le \left(\frac{9}{17}\right)^{\frac{2}{2+\alpha_i}},\quad i=1,\ldots,n ,$$

$$\frac{1}{16} \le \frac{6}{16},$$

which are obviously fulfilled.

Now we apply lemma 3 and obtain

$$\sup_D u \geq \left(1+\eta\, 4^{4ms}\gamma_m\right) \sup_{\Gamma(D)\cap \mathbf{C}^{m-1}} u .$$

Now it is enough to take into account that according to maximum principle

$$\sup_{\Gamma(D)\cap \mathbf{C}^{m+1}} u = \sup_{D\cap \mathbf{C}^{m+1}} u$$

and theorem has been proved.

3^0. Regularity of boundary point.

Theorem 2. *Let in domain* D *coefficients of operator* L *satisfying to condition (2) are defined. Then for regularity of point* $(0,0)$ *with the respect to the first boundary value problem it is sufficient that*

$$\sum_{m=1}^{\infty} 4^{4ms}\gamma_m = \infty .$$

Proof. We have to show that for any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there is $\delta > 0$ such that for any domain $D' \subset D$ which completely lies in half-space $t<0$, for any uniformly parabolic operator L defined in D' for any L-subparabolic for this operator function $u'(x,t)$ which not greater than 1 in D and not greater than zero at intersection of $\Gamma(D')$ with ε_1-neighbourhood of point $(0,0)$ in intersection D' with δ-neighbourhood of point $(0,0)$ (if it is nonempty) the inequality $u'(x,t) < \varepsilon_2$ is realized.

We denote through m_0 the least natural number that $\mathbf{C}^{m_0} \subset O_{\varepsilon_1}(0,0)$.

Let number $m > m_0$ is such that there is point $(x,t) \in D \cap \mathbf{C}^m$ for which $u'(x',t') \geq \varepsilon_2$.

Our aim is to show that then number m is necessary less than some constant m^*, which depends on ε_1, ε_2, L.

For every $i, i = m_0, m_0+1, ..., m$ we consider cylinders $\mathbf{C}^i$, $\mathbf{C}^{i-1}$. We examine set of points $(x,t) \in D' \cap \mathbf{C}^{i-1}$, where $u(x,t) > 0$ and in this set we choose component D' which contains the point where function u' reaches to value $M_i = \sup_{D' \cap \mathbf{C}^i} u'$. Then

$$p_{s,\beta}^{(4^{-i})}\left(A^{i-1} \setminus D'\right) \geq p_{s,\beta}^{(4^{-i})}\left(A^{i-1} \setminus D\right) = \gamma_{i-1} .$$

Now we use theorem 1 and obtain

$$M_{i-1} \geq \left(1+\eta\, 4^{4(i-1)s}\gamma_{i-1}\right) M_i .$$

Thus

$$1 \geq M_{m_0} \geq M_m \prod_{i=m_0+1}^{n} \left(1+\eta\, 4^{2(i-1)s}\gamma_{i-1}\right) \geq \varepsilon_2 \prod_{i=m_0+1}^{m} \left(1+\eta\, 4^{2(i-1)s}\gamma_{i-1}\right).$$

Hence

$$\prod_{i=m_0}^{m-1} \left(1+\eta\, 4^{2is}\gamma_i\right) \leq \ln\frac{1}{\varepsilon_2}$$

and therefore

$$\sum_{i=m_0}^{m-1} \ln\left(1+\eta\, 4^{2is}\gamma_i\right) \leq \ln\frac{1}{\varepsilon_2} .$$

Since

$$\ln\left(1+\eta\, 4^{2is}\gamma_i\right) \geq a \cdot 4^{2is}\gamma_i ,$$

where $a > 0$ is constant which depends on η, then

$$\sum_{i=m_0}^{m-1} 4^{2is}\gamma_i \le \frac{1}{a}\ln\frac{1}{\varepsilon}. \qquad (17)$$

Because of divergence of series $\sum_{i=1}^{\infty} 4^{2is}\gamma_i$ there exists natural number m^* that inequality (17) could be fulfilled only when $m < m^*$. This m^* depends on ε_1 and ε_2. Thus theorem is proved.

Corollary. *Let in the neighborhood of point* $(0,0)$ *boundary* ∂D *of domain* D *is given by equation*

$$|x|^2 = -t\alpha(-t), \quad -d < t < 0,$$

where d *is positive constant,* $\alpha(z)$ *- positive non-decreasing function and*

$$\alpha(z) \le q\, z^{\frac{\alpha^+}{2}}, \qquad q < \frac{10}{\left(\frac{11}{8}b\right)^{\frac{\alpha^-}{4}}}.$$

Then point $(0,0)$ is regular with respect to the first boundary problem.

In order to prove we notice that in this case for sufficiently great m

$$\alpha\left(\sqrt{\frac{11}{8}b}\,4^{-m}\right) \le q_1 \frac{10}{\left(\frac{11}{8}b\right)^{\frac{\alpha^+}{4}}} \cdot 4^{-\frac{m\alpha_i}{2}}, \quad i = 1,\dots,n$$

with constant $q_1 \in (0,1)$. Hence set H_m constants cylinder $C^{t^*-h_1 4^{-2m}, t^*}_{x^*; h_2 4^{-2m}; 1}$, where h_1 and h_2 are positive constants. Therefore for sufficiently great m

$$\gamma_m \ge h_3 4^{-4ms}, \quad h_3 > 0$$

and the series $\sum_{m=1}^{\infty} 4^{4ms}\gamma_m$ diverges.

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