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ASYMPTOTIC BEHAVIOR OF EIGENVALUES FOR THE DISCONTINUOUS PROBLEM WITH FUNCTIONALS IN THE BOUNDARY CONDITIONS

Abstract

In this study, the boundary-value problem with eigenvalue parameter generated by the differential equation with discontinuous coefficients and boundary conditions which contains not only endpoints of the considered interval, but also point of discontinuity and abstract linear functionals is investigated. So, our problem is not pure boundary-value.

We single out a class of linear functionals and find simple algebraic conditions on coefficients, which grantee the existence of infinite number eigenvalues. Also the asymptotic formulas for eigenvalues are found.

Key words: Asymptotic behavior of eigenvalues, boundary-value problems, functional-conditions, discontinuous coefficients

1. Introduction.

The investigation of boundary-value problems for which the eigenvalue parameter appears in both the equation and boundary conditions originates from the works of G.D.Birkhoff [1,2]. There are many papers and books, where the spectral properties of such problems are investigated (see, for example [5,8,10,11,12] and corresponding bibliography).

Usually, in many monographs and papers, the theory of boundary-value problems for ordinary differential equations is considered for equations with a constant coefficient at the highest derivative and for boundary conditions which contain only endpoints of the considered interval.

But this paper deals with one nonstandard boundary-value problem for second order ordinary differential equation with discontinuous coefficients and boundary conditions containing not only endpoints of the considered interval, but also point of discontinuity, and abstract linear functionals. Moreover, the eigenvalue parameter appear both in differential equation and boundary conditions.

Some spectral properties of such problems and its applications to the corresponding initial-boundary-value problems for parabolic equations was investigated by M.L.Rasulov in monographs [8,9].

One must note that in the series of S.Y.Yakubov works, appearing in the recent years, have been construct an abstract theory of boundary-value problems with a parameter in boundary conditions (see [11, 12,13] and corresponding bibliography). In these works, in particular, the main spectral properties of the more general problems, but with continuous coefficients are investigated.

2. Statement of the problem.

Let us consider a differential equation

$$a(x)y'' + c(x)y = \lambda^2 y,\ x \in [-1,0) \cup (0,1] \qquad (1)$$

with the functional-manypoint boundary conditions

$$L_i(y)=\sum_{k=0}^{1}\lambda^{1-k}\left(\alpha_{ik}y^{(k)}(-1)+\delta_{ik}y^{(k)}(-0)+\right.$$
$$\left.+\gamma_{ik}y^{(k)}(+0)+\beta_{ik}y^{(k)}(1)+T_{ik}y\right)=0\,,\quad i=1,2,3,4\,, \tag{2}$$

where $a(x)$ and $c(x)$ are complex-valued functions; $a(x)=a_1$ at $x\in[-1,0)$, $a(x)=a_2$ at $x\in(0,1]$, α_{ik}, δ_{ik}, γ_{ik}, β_{ik}, are complex coefficients; T_{ik} are abstract linear functionals; $y^{(k)}(\pm 0)$ denotes $\lim\limits_{x\to\pm 0}y^{(k)}(x)$.

Below by $W_q^k(-1,0)+W_q^k(0,1)$, $q\in(1,\infty)$, $k=0,1,2,...$ we denote the Banach Space of complex-valued functions $y=y(x)$ defining on $[-1,0)\cup(0,1]$, which belongs to $W_q^k(-1,0)$ and $W_q^k(0,1)$ on intervals $(-1,0)$ and $(0,1)$ respectively, with the norm $\|y\|_{q,k}=\left(\|y\|^q_{W_q^k(-1,0)}+\|y\|^q_{W_q^k(0,1)}\right)^{1/q}$, where $W_q^k(-1,0)$ and $W_q^k(0,1)$ are usual Sobolev Spaces.

Note, that, without loss of generality we consider the equation (1) instead of more general equation

$$a(x)y''+b(x)y'+c(x)y=\lambda^2 y\,. \tag{3}$$

Since, by using the substitution $y=\widetilde{y}e^{\varphi(x)}$, where

$$\varphi(x)=\begin{cases}-\dfrac{1}{2a_1}\displaystyle\int_{-1}^{x}b(t)y(t)dt & at \quad x\in[-1,0)\,,\\ -\dfrac{1}{2a_2}\displaystyle\int_{0}^{x}b(t)y(t)dt & at \quad x\in(0,1]\,.\end{cases}$$

We find that equation (3) takes the form (3) with the same eigenvalue parameter λ. Also, it is easy to verify that under this substitution the form of boundary conditions (2) has not changed. Besides, for simplicity we replace the general domain $[a,c)\cup(c,b]$, $a<c<b$, by $[-1,0)\cup(0,1]$, for we can return to the general case by making the substitution

$$x=\begin{cases}c+(c-a)\,t & at \quad t\in[-1,0)\,,\\ c+(b-c)\,t & at \quad t\in(0,1]\,.\end{cases}$$

Some special cases of the problem (1)-(2) arise in a varied physical transfer problems, in particular, in the heat and mass transfer problems [3].

3. Eigenvalues of the problem.

As usual, these values of parameter λ for which the considered boundary-value problem (1)-(2) has a nontrivial solutions. We called the eigenvalues of the problem (1)-(2).

Let $y_{10}(x,\lambda)$, $y_{20}(x,\lambda)$ and $y_{30}(x,\lambda)$, $y_{40}(x,\lambda)$ denote a some fundamental systems of solutions of the differential equation (1) on $[-1,0)$ and $(0,1]$ consequently.

Defining

$$y_j(x,\lambda)=\begin{cases}y_{j0}(x,\lambda), & x\in I_n\,,\\ 0 & ,\ x\notin I_n\,.\end{cases} \tag{3'}$$

Where $I_1 = I_2 = [-1,0)$, $I_3 = I_4 = (0,1]$ the general solution of the equation (1) can be represented by the form

$$y(x,\lambda) = \sum_{j=1}^{4} c_j y_j(x,\lambda). \tag{4}$$

Substituting (4) into the boundary conditions (2) we obtain a system of linear homogeneous equations

$$\sum_{j=1}^{4} c_j L_\nu(y_j) = 0, \quad \nu = 1,2,3,4. \tag{5}$$

For the determination the constants $c_j, j = \overline{1,4}$. Consequently eigenvalues of the problem (1)-(2) consists of the zeros of characteristic determinant

$$\Delta(\lambda) = \det(L_\nu y_j)_{\nu,j=\overline{1,4}}. \tag{6}$$

At first, according to the considered problem, we shall divide the complex λ-plane into specific sectors, in which by turns we shall find the asymptotic expressions for solutions of the differential equation, for boundary functional and boundary-value forms. Then by substituting these obtained asymptotic expressions into the equation $\Delta(\lambda) = 0$ we shall find the corresponding asymptotic formulas for the eigenvalues. Note that, such formulas not only of interest in themselves, but they also may be used or the establishing the completeness and basis properties of the system of eigen-and associated functions of the considered problem.

The cases $\arg a_1 \neq \arg a_2$ and $\arg a_1 = \arg a_2$ shall be investigated separately.

4. Asymptotic behavior of eigenvalues for the case $\arg a_1 \neq \arg a_2$.

4.1. Separation of the complex λ-plane into specific sectors

Throughout the paper we used the notations: $\omega_1 = \left(\sqrt{a_1}\right)^{-1}$, $\omega_2 = -\left(\sqrt{a_1}\right)^{-1}$, $\omega_3 = \left(\sqrt{a_2}\right)^{-1}$, $\omega_4 = -\left(\sqrt{a_2}\right)^{-1}$, where, $\sqrt{z} := |z| e^{i\frac{\arg z}{2}}$, $-\pi < \arg z \leq \pi$. Divide the complex λ-plane into four sectors $S_\nu, \nu = \overline{1,4}$, by the rays $l_k = \{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda\omega_k = 0, (-1)^k \operatorname{Im}\lambda\omega_k \leq 0\}$. On each of these sectors each of the real-valued functions $\operatorname{Re}\lambda\omega_k$ is of a single sign since this functions can vanish only of boundaries of the sectors S_ν. Let us consider one of the sectors (S_ν) with fixed index ν. Using the same considerations as in [7] it is easy to verify that corresponding to the equation (1) there exists a fundamental system of particular solutions $y_{10}(x,\lambda)$, $y_{20}(x,\lambda)$ on $[-1,0)$ and $y_{30}(x,\lambda)$, $y_{40}(x,\lambda)$ on $(0,1]$ respectively, which are regular analytic functions of $\lambda \in S_\nu$ and for sufficiently large $|\lambda|$, and which with their derivatives, can be expressed in the asymptotic form

$$y_{k0} = e^{\lambda\omega_k x}\left(1 + O\left(\frac{1}{\lambda}\right)\right), \tag{7}$$

$$y'_{k0} = \lambda\omega_k e^{\lambda\omega_k x}\left(1 + O\left(\frac{1}{\lambda}\right)\right).$$

Here, as usual, the expression $O\left(\frac{1}{\lambda}\right)$ denotes any function of the form $f(x,\lambda)/\lambda$ where $|f(x,\lambda)|$ for $x \in I_k$ and sufficiently large $|\lambda|$ always remains less than a constant.

Now let $l'_j\left(j=\overline{1,4}\right)$ are arbitrary rays, which originated from the point $\lambda = 0$, distinct from the rays l_j and situated so as to the from the sequence

$$l_1, l'_1, l_3, l'_3, l_2, l'_2, l_4, l'_4. \tag{8}$$

The rays l'_j divide each sector S_v into two subsectors. Thus we have the eight sector which we shall signify as Ω_j, $j=\overline{1,8}$.

As it seems from the construction, the sectors $\Omega = \{\Omega_1, \Omega_2, ..., \Omega_8\}$ can be distributed into two groups of $\Omega^{(1)} = \{\Omega_1^{(1)}, ..., \Omega_4^{(1)}\}$ and $\Omega^{(2)} = \{\Omega_1^{(2)}, ..., \Omega_4^{(2)}\}$ such that, the group $\Omega^{(k)}$ includes those sectors Ω_j in which

$$\operatorname{Re}\lambda\left(\sqrt{a_k}\right)^{-1} \to \infty \text{ as } \lambda \to \infty.$$

4.2. Asymptotic expressions for the characteristic determinant $\Delta(\lambda)$ in the Ω sectors for large values of $|\lambda|$.

Each of the real-valued functions $\operatorname{Re}\lambda\omega_k$ is not charged sign also in each of the sectors Ω_j, since each of they is a subsector of certain sector S_v.

Let $y_k = y_k(x,\lambda)$, $k=\overline{1,4}$ are functions, defined as (3') for the solutions $y_{k0}(x,\lambda)$ of the equation in I_k, for which satisfied the asymptotic expressions (7).

First we need asymptotically estimate the expressions $T_{vk}y_n(.,\lambda)$ as $\lambda \to \infty$ in the sectors Ω_j. Since the linear functionals T_{vk} acts from $W_q^k(-1,0)+W_q^k(0,1)$ into complex plane $\mathbb{C}$ continuously, in virtue of general representation form of the continuous linear functionals in the $L_q(a,b)$ spaces and using the well-known methods of real analysis it may be shown that there exists a functions $U_{vks} \in W_p^k(-1,0)+W_p^k(0,1)$ such that for every $y \in W_q^k(-1,0)+W_q^k(0,1)$ the equations

$$T_{vk}(y) = \sum_{s=0}^{k}\left(\int_{-1}^{0} y^{(s)}(x)U_{vks}(x)dx + \int_{0}^{1} y^{(s)}(x)U_{vks}(x)dx\right), \quad v=\overline{1,4},\ k=\overline{0,1} \tag{9}$$

are holds, where $\frac{1}{p}+\frac{1}{q}=1$[4].

Only in one of the sectors of the group $\Omega^{(1)}$ the conditions $\operatorname{Re}\lambda\omega_3 \ge 0$ and $\operatorname{Re}\lambda\omega_1 \to +\infty$ as $\lambda \to \infty$ are holds, as well only in one of the sectors of the group $\Omega^{(2)}$, the conditions $\operatorname{Re}\lambda\omega_1 \ge 0$ and $\operatorname{Re}\lambda\omega_3 \to +\infty$ a $\lambda \to \infty$, are holds. These sectors denote as $\Omega_0^{(1)}$ and $\Omega_0^{(2)}$ accordingly. We shall calculate asymptotic approximation expressions for $T_{vk}y_j$ only for these sectors of the group $\Omega^{(1)}$ and $\Omega^{(2)}$, since calculations for others sectors can be made analogously. For convenience throughout below by [M], $M \notin \mathbb{C}$, we shall denote any sum of the from $M + f(\lambda)$, when $f(\lambda) \to 0$ as $\lambda \to \infty$.

First let λ vary in $\Omega_0^{(1)}$ substituting (7) into (9), having in view the equations $\omega_2 = -\omega_1$, $\omega_4 = -\omega_3$ and applying the well-known.

Reimann-Lebesque Lemma [cf. 8, p.117, Lemma 7] we get

$$T_{ik}(y_1) = \sum_{s=0}^{k} \int_{-1}^{0} y_{10}^{(s)}(x,\lambda) U_{\nu ks}(x)dx = \sum_{s=0}^{k} \int_{-1}^{0} (\lambda\omega_1)^s e^{\lambda\omega_1 x}\left(1 + O\left(\frac{1}{\lambda}\right)\right) U_{\nu ks}(x)dx =$$
$$= (\lambda\omega_1)^k \sum_{s=0}^{1} (\lambda\omega_1)^{s-k} \int_0^1 e^{-\lambda\omega_1 x}\left\{U_{\nu ks}(-x)\left(1 + O\left(\frac{1}{\lambda}\right)\right)\right\}dx = \lambda^k [0], \tag{10}$$

$$T_{\nu k}(y_2) = \sum_{s=0}^{k} \int_{-1}^{0} y_{20}^{(s)}(x,\lambda) U_{\nu ks}(x)dx = \sum_{s=0}^{k} (\lambda\omega_2)^s e^{-\lambda\omega_2} \int_{-1}^{0} e^{\lambda\omega_2(1+x)}\left(1 + O\left(\frac{1}{\lambda}\right)\right) U_{\nu ks}(x)dx =$$

$$= (\lambda\omega_2)^k e^{-\lambda\omega_2} \sum_{s=0}^{k} (\lambda\omega_2)^{s-k} \int_0^1 e^{-\lambda\omega_2 x}\left\{U_{\nu ks}(x-1)\left(1 + O\left(\frac{1}{\lambda}\right)\right)\right\}dx = \lambda^k e^{-\lambda\omega_2}[0], \tag{11}$$

$$T_{\nu k}(y_3) = \sum_{s=0}^{k} \int_0^1 y_{30}^{(s)}(x,\lambda) U_{\nu ks}(x)dx = \sum_{s=0}^{k} (\lambda\omega_3)^s e^{\lambda\omega_3} \int_0^1 e^{-\lambda\omega_3(1-x)}\left(1 + O\left(\frac{1}{\lambda}\right)\right) U_{\nu ks}(x)dx =$$
$$= (\lambda\omega_3)^k e^{\lambda\omega_3} \sum_{s=0}^{k} (\lambda\omega_3)^{s-k} \int_0^1 e^{-\lambda\omega_3 x}\left\{U_{\nu ks}(1-x)\left(1 + O\left(\frac{1}{\lambda}\right)\right)\right\}dx = \lambda^k e^{\lambda\omega_3}[0], \tag{12}$$

$$T_{\nu k}(y_4) = \sum_{s=0}^{k} \int_0^1 y_{40}^{(s)}(x,\lambda) U_{\nu ks}(x)dx = \sum_{s=0}^{k} \int_0^1 (\lambda\omega_4)^s e^{\lambda\omega_4 x}\left(1 + O\left(\frac{1}{\lambda}\right)\right) U_{\nu ks}(x)dx =$$
$$= (\lambda\omega_4)^k \sum_{s=0}^{k} (\lambda\omega_4)^{s-k} \int_0^1 e^{-\lambda\omega_3 x}\left\{U_{\nu ks}(x)\left(1 + O\left(\frac{1}{\lambda}\right)\right)\right\}dx = \lambda^k [0]. \tag{13}$$

Using this formulas we find the next asymptotic expressions for $L_\nu j_j$ in the $\Omega_0^{(1)}$, as $\lambda \to \infty$.

$$L_i(y_1) = \sum_{k=0}^{1} \lambda^{1-k}\left\{\alpha_{ik}(\lambda\omega_1)^k e^{-\lambda\omega_1}[1] + \delta_{ik}(\lambda\omega_1)^k[1] + (\lambda\omega_1)^k[o]\right\} = \lambda[\delta_{i0} + \omega_1\delta_{i1}], \tag{14}$$

$$L_i(y_2) = \sum_{k=0}^{1} \lambda^{1-k}\left\{\alpha_{ik}(\lambda\omega_2)^k e^{-\lambda\omega_2}[1] + \delta_{ik}(\lambda\omega_2)^k[1] + (\lambda\omega_2)^k e^{-\lambda\omega_2}[o]\right\} =$$
$$= \lambda e^{-\lambda\omega_2} \sum_{k=0}^{1}\left\{\left[\alpha_{ik}\omega_2^k\right] + \left[\delta_{ik}\omega_2^k\right]e^{\lambda\omega_2}\right\} = \lambda e^{-\lambda\omega_2}[\alpha_{i0} + \omega_2\alpha_{i1}], \tag{15}$$

$$L_i(y_3) = \sum_{k=0}^{1} \lambda^{1-k}\left\{\gamma_{ik}(\lambda\omega_3)^k[1] + \beta_{ik}(\lambda\omega_3)^k e^{\lambda\omega_3}[1] + (\lambda\omega_3)^k e^{-\lambda\omega_3}[o]\right\} =$$
$$= \lambda\left\{[\gamma_{i0} + \omega_3\gamma_{i1}] + e^{\lambda\omega_3}[\beta_{i0} + \omega_3\beta_{i1}]\right\} \tag{16}$$

and analogously

$$L_i(y_4) = \lambda\left\{[\gamma_{i0} + \omega_4\gamma_{i1}] + e^{\lambda\omega_4}[\beta_{i0} + \omega_4\beta_{i1}]\right\} \qquad for \qquad i = 1,2,3,4. \tag{17}$$

All calculations for the sector $\Omega_0^{(2)}$ are carried out analogously. After needed calculations we get the next asymptotic expressions for $T_{\nu k}(y_i)$ and $L_i(y_j)$ as well

$$T_{vk}(y_1)=\lambda^k[o],\ T_{vk}(y_2)=\lambda^k e^{-\lambda\omega_2}[o],\ T_{vk}(y_3)=\lambda^k e^{\lambda\omega_3}[o],\ T_{vk}(y_4)=\lambda^k[o] \qquad (18)$$

and

$$\begin{aligned} L(y_j)&=\lambda\left\{[\alpha_{i0}+\omega_j\alpha_{i1}]e^{-\lambda\omega_j}+[\delta_{i0}+\omega_j\delta_{i1}]\right\},\ j=1,2,\\ L_i(y_3)&=\lambda e^{\lambda\omega_3}[\beta_{i0}+\omega_3\beta_{i1}],\\ L_i(y_4)&=\lambda[\gamma_{i0}+\omega_4\gamma_{i1}]. \end{aligned} \qquad (19)$$

By substituting the asymptotic expressions (14)-(17) for $L_i(y_j)$ into the determinant $\Delta(\lambda)$, taking the common factors λ of each column and the common factor $e^{-\lambda\omega_2}$ of second column outside the determinant and taking into account $\omega_2=-\omega_1$, $\omega_4=-\omega_3$ we may represent this determinant as the next asymptotic form;

$$\Delta(\lambda)=\lambda^4 e^{\lambda\omega_1}\left([A_1^{(1)}]e^{m_1\lambda\omega_3}+[A_2^{(1)}]e^{m_2\lambda\omega_3}+\ldots+[A_{\sigma_1}^{(1)}]e^{m_{\sigma_1}\lambda\omega_3}\right). \qquad (20)$$

Where

$-1=m_1<m_2<\ldots<m_{\sigma_1}=1$ and $A_j^{(k)}$ some complex numbers.

Furthermore, it is easy to see that

$$A_1^{(1)}=\begin{vmatrix} \delta_{10}+\omega_1\delta_{11} & \alpha_{10}+\omega_2\alpha_{11} & \gamma_{10}+\omega_3\gamma_{11} & \beta_{10}+\omega_4\beta_{11}\\ \delta_{20}+\omega_1\delta_{21} & \alpha_{20}+\omega_2\alpha_{21} & \gamma_{20}+\omega_3\gamma_{21} & \beta_{20}+\omega_4\beta_{21}\\ \delta_{30}+\omega_1\delta_{31} & \alpha_{30}+\omega_2\alpha_{31} & \gamma_{30}+\omega_3\gamma_{31} & \beta_{30}+\omega_4\beta_{31}\\ \delta_{40}+\omega_1\delta_{41} & \alpha_{40}+\omega_2\alpha_{41} & \gamma_{40}+\omega_3\gamma_{41} & \beta_{40}+\omega_4\beta_{41} \end{vmatrix},$$

$$A_{\sigma_1}^{(1)}=\begin{vmatrix} \delta_{10}+\omega_1\delta_{11} & \alpha_{10}+\omega_2\alpha_{11} & \beta_{10}+\omega_3\beta_{11} & \gamma_{10}+\omega_4\gamma_{11}\\ \delta_{20}+\omega_1\delta_{21} & \alpha_{20}+\omega_2\alpha_{21} & \beta_{20}+\omega_3\beta_{21} & \gamma_{20}+\omega_4\gamma_{21}\\ \delta_{30}+\omega_1\delta_{31} & \alpha_{30}+\omega_2\alpha_{31} & \beta_{30}+\omega_3\beta_{31} & \gamma_{30}+\omega_4\gamma_{31}\\ \delta_{40}+\omega_1\delta_{41} & \alpha_{40}+\omega_2\alpha_{41} & \beta_{40}+\omega_3\beta_{41} & \gamma_{40}+\omega_4\gamma_{41} \end{vmatrix}.$$

It can be shown analogously that, the characteristic determinant $\Delta(\lambda)$ in the sector $\Omega_0^{(2)}$ has the next asymptotic representation;

$$\Delta(\lambda)=\lambda^4 e^{\lambda\omega_3}\left([A_1^{(2)}]e^{n_1\lambda\omega_1}+[A_2^{(2)}]e^{n_2\lambda\omega_1}+\ldots+[A_{\sigma_2}^{(2)}]e^{n_{\sigma_2}\lambda\omega_1}\right). \qquad (21)$$

Where $-1=n_1<n_2<\ldots<n_{\sigma_2}=1$ and

$$A_1^{(2)}=\begin{vmatrix} \alpha_{10}+\omega_1\alpha_{11} & \delta_{10}+\omega_2\delta_{11} & \beta_{10}+\omega_3\beta_{11} & \gamma_{10}+\omega_4\gamma_{11}\\ \alpha_{20}+\omega_1\alpha_{21} & \delta_{20}+\omega_2\delta_{21} & \beta_{20}+\omega_3\beta_{21} & \gamma_{20}+\omega_4\gamma_{21}\\ \alpha_{30}+\omega_1\alpha_{31} & \delta_{30}+\omega_2\delta_{31} & \beta_{30}+\omega_3\beta_{31} & \gamma_{30}+\omega_4\gamma_{31}\\ \alpha_{40}+\omega_1\alpha_{41} & \delta_{40}+\omega_2\delta_{41} & \beta_{40}+\omega_3\beta_{41} & \gamma_{40}+\omega_4\gamma_{41} \end{vmatrix},$$

$$A_{\sigma_2}^{(2)}=\begin{vmatrix} \delta_{10}+\omega_1\delta_{11} & \alpha_{10}+\omega_2\alpha_{11} & \beta_{10}+\omega_4\beta_{11} & \gamma_{10}+\omega_3\gamma_{11}\\ \delta_{20}+\omega_1\delta_{21} & \alpha_{20}+\omega_2\alpha_{21} & \beta_{20}+\omega_4\beta_{21} & \gamma_{20}+\omega_3\gamma_{21}\\ \delta_{30}+\omega_1\delta_{31} & \alpha_{30}+\omega_2\alpha_{31} & \beta_{30}+\omega_4\beta_{31} & \gamma_{30}+\omega_3\gamma_{31}\\ \delta_{40}+\omega_1\delta_{41} & \alpha_{40}+\omega_2\alpha_{41} & \beta_{40}+\omega_4\beta_{41} & \gamma_{30}+\omega_3\gamma_{31} \end{vmatrix}.$$

4.3. Asymptotic behavior of the eigenvalues.

Now we can find the needed asymptotic formulas for the eigenvalues of the considered problem. For the case $\arg a_1 \neq \arg a_2$.

Theorem 1. *Let the following conditions be satisfied.*

1) $\arg a_1 \neq \arg a_2$,

2) $c(\cdot) \in L_q(-1,1),\ q>1$,

3) $\theta_{ij} =$

$$=\begin{vmatrix} \alpha_{10}\sqrt{a_1}+(-1)^i\alpha_{11} & \delta_{10}\sqrt{a_1}+(-1)^{i+1}\delta_{11} & \beta_{10}\sqrt{a_2}+(-1)^j\beta_{11} & \gamma_{10}\sqrt{a_2}+(-1)^{j+1}\gamma_{11} \\ \alpha_{20}\sqrt{a_1}+(-1)^i\alpha_{21} & \delta_{20}\sqrt{a_1}+(-1)^{i+1}\delta_{21} & \beta_{20}\sqrt{a_2}+(-1)^j\beta_{21} & \gamma_{20}\sqrt{a_2}+(-1)^{j+1}\gamma_{21} \\ \alpha_{30}\sqrt{a_1}+(-1)^i\alpha_{31} & \delta_{30}\sqrt{a_1}+(-1)^{i+1}\delta_{31} & \beta_{30}\sqrt{a_2}+(-1)^j\beta_{31} & \gamma_{30}\sqrt{a_2}+(-1)^{j+1}\gamma_{31} \\ \alpha_{40}\sqrt{a_1}+(-1)^i\alpha_{41} & \delta_{40}\sqrt{a_1}+(-1)^{i+1}\delta_{41} & \beta_{40}\sqrt{a_2}+(-1)^j\beta_{41} & \gamma_{40}\sqrt{a_2}+(-1)^{j+1}\gamma_{41} \end{vmatrix} \neq 0$$

$i=\overline{1,2},\ j=\overline{1,2}$

4) The linear functionals $T_{\nu k}$ *in the spaces* $W_q^k(-1,0)+W_q^k(0,1)$ *are continuous.*

Then the boundary-value problem (1)-(2), has in each sector S_ν *an precisely numerable number eigenvalues, whose asymptotic behavior may be expressed by the following formulas.*

$$\lambda_n^{(1)} = \sqrt{a_1}\pi n i\left(1+O\left(\frac{1}{n}\right)\right), \tag{22}$$

$$\lambda_n^{(2)} = -\sqrt{a_1}\pi n i\left(1+O\left(\frac{1}{n}\right)\right), \tag{23}$$

$$\lambda_n^{(3)} = \sqrt{a_2}\pi n i\left(1+O\left(\frac{1}{n}\right)\right), \tag{24}$$

$$\lambda_n^{(4)} = -\sqrt{a_2}\pi n i\left(1+O\left(\frac{1}{n}\right)\right) \tag{25}$$

$$n=1,2,\ldots.$$

Proof. By the rays l_j' divided the complex λ-plane into four sectors $R_j,\ j=\overline{1,4}$.

Let R_j that sector, which contains the ray l_j. This sector we shall distribute into two groups $R^{(1)}=\{R_1,R_2\}$ and $R^{(2)}=\{R_3,R_4\}$. Obviously that each sector of the group $R^{(k)}$ constants of two sectors of the groups $\Omega^{(k)}$. By $R_0^{(k)}$ denote that sector of the group $R^{(k)}$ which contain $\Omega_0^{(k)}(k=1,2)$.

As seems from the consideration in subsections in subsections 4.1 and 4.2 the asymptotic expressing (20) and (21) are holds also in the sectors $R_0^{(1)}$ and $R_0^{(2)}$ respectively. Let $R_1^{(1)}$ and $R_1^{(2)}$ are the other sectors of the groups $R^{(1)}$ and $R^{(2)}$ respectively.

By the similar way as in subsections 4.1 and 4.2, one can prove that the characteristic determinant $\Delta(\lambda)$ has the asymptotic representations given by

$$\Delta(\lambda)=\lambda^4 e^{-\lambda\omega_3}\left(\left[B_1^{(1)}\right]e^{s_1\lambda\omega_3}+\left[B_2^{(1)}\right]e^{s_2\lambda\omega_3}+\ldots+\left[B_{\tau_1}^{(1)}\right]e^{s_{\tau_1}\lambda\omega_3}\right) \tag{26}$$

and

$$\Delta(\lambda)=\lambda^4 e^{-\lambda\omega_1}\left(\left[B_1^{(2)}\right]e^{t_1\lambda\omega_1}+\left[B_2^{(2)}\right]e^{t_2\lambda\omega_1}+\ldots+\left[B_{\tau_2}^{(2)}\right]e^{t_{\tau_2}\lambda\omega_1}\right), \tag{27}$$

in the sectors $R_1^{(1)}$ and $R_1^{(2)}$ respectively, where $-1=s_1<s_2<\ldots<s_{\tau_1}=1$, $-1=t_1<t_2<\ldots<t_{\tau_2}=1$,

$$B_1^{(1)} = \begin{vmatrix} \alpha_{10}+\omega_1\alpha_{11} & \delta_{10}+\omega_2\delta_{11} & \gamma_{10}+\omega_3\gamma_{11} & \beta_{10}+\omega_4\beta_{11} \\ \alpha_{20}+\omega_1\alpha_{21} & \delta_{20}+\omega_2\delta_{21} & \gamma_{20}+\omega_3\gamma_{21} & \beta_{20}+\omega_4\beta_{21} \\ \alpha_{30}+\omega_1\alpha_{31} & \delta_{30}+\omega_2\delta_{31} & \gamma_{30}+\omega_3\gamma_{31} & \beta_{30}+\omega_4\beta_{31} \\ \alpha_{40}+\omega_1\alpha_{41} & \delta_{40}+\omega_2\delta_{41} & \gamma_{40}+\omega_3\gamma_{41} & \beta_{40}+\omega_4\beta_{41} \end{vmatrix},$$

$$B_{\tau_1}^{(1)} = \begin{vmatrix} \alpha_{10}+\omega_1\alpha_{11} & \delta_{10}+\omega_2\delta_{11} & \beta_{10}+\omega_3\beta_{11} & \gamma_{10}+\omega_4\gamma_{11} \\ \alpha_{20}+\omega_1\alpha_{21} & \delta_{20}+\omega_2\delta_{21} & \beta_{20}+\omega_3\beta_{21} & \gamma_{20}+\omega_4\gamma_{21} \\ \alpha_{30}+\omega_1\alpha_{31} & \delta_{30}+\omega_2\delta_{31} & \beta_{30}+\omega_3\beta_{31} & \gamma_{30}+\omega_4\gamma_{31} \\ \alpha_{40}+\omega_1\alpha_{41} & \delta_{40}+\omega_2\delta_{41} & \beta_{40}+\omega_3\beta_{41} & \gamma_{40}+\omega_4\gamma_{41} \end{vmatrix},$$

$$B_1^{(2)} = \begin{vmatrix} \alpha_{10}+\omega_1\alpha_{11} & \delta_{10}+\omega_2\delta_{11} & \gamma_{10}+\omega_3\gamma_{11} & \beta_{10}+\omega_4\beta_{11} \\ \alpha_{20}+\omega_1\alpha_{21} & \delta_{20}+\omega_2\delta_{21} & \gamma_{20}+\omega_3\gamma_{21} & \beta_{20}+\omega_4\beta_{21} \\ \alpha_{30}+\omega_1\alpha_{31} & \delta_{30}+\omega_2\delta_{31} & \gamma_{30}+\omega_3\gamma_{31} & \beta_{30}+\omega_4\beta_{31} \\ \alpha_{40}+\omega_1\alpha_{41} & \delta_{40}+\omega_2\delta_{41} & \gamma_{40}+\omega_3\gamma_{41} & \beta_{40}+\omega_4\beta_{41} \end{vmatrix},$$

$$B_{\sigma_2}^{(2)} = \begin{vmatrix} \delta_{10}+\omega_1\delta_{11} & \alpha_{10}+\omega_2\alpha_{11} & \gamma_{10}+\omega_3\gamma_{11} & \beta_{10}+\omega_4\beta_{11} \\ \delta_{20}+\omega_1\delta_{21} & \alpha_{20}+\omega_2\alpha_{21} & \gamma_{20}+\omega_3\gamma_{21} & \beta_{20}+\omega_4\beta_{21} \\ \delta_{30}+\omega_1\delta_{31} & \alpha_{30}+\omega_2\alpha_{31} & \gamma_{30}+\omega_3\gamma_{31} & \beta_{30}+\omega_4\beta_{31} \\ \delta_{40}+\omega_1\delta_{41} & \alpha_{40}+\omega_2\alpha_{41} & \gamma_{40}+\omega_3\gamma_{41} & \beta_{40}+\omega_4\beta_{41} \end{vmatrix}.$$

According to the condition 3 of the Theorem, the principal terms of first and last coefficients of the asymptotic quasipolinoms (20), (21), (26) and (27), i.e. the numbers $A_1^{(1)}, A_{\sigma_1}^{(1)}, A_1^{(2)}, A_{\sigma_2}^{(2)}, B_1^{(1)}, B_{\tau_1}^{(1)}, B_1^{(2)}, B_{\tau_2}^{(2)}$ are different from zero.

Since $\Delta(\lambda)=\Delta_i^{(j)}(\lambda)$ when λ vary in sector $R_i^{(j)}$ and all quasipolynoms $\Delta_i^{(j)}(\lambda)$ has the same form, it is enough to investigate only one of them. Namely, we shall investigate the equation $\Delta(\lambda)=0$ only in the sector $R_0^{(1)}$, i.e. the equation

$$\left[A_1^{(1)}\right]e^{m_1\lambda\omega_3}+...+\left[A_{\sigma_1}^{(1)}\right]e^{m_{\sigma_2}\lambda\omega_3}=0. \tag{28}$$

By virtue of the [8, p.100, lemma 1] the equation (28) has in the sector $R_0^{(1)}$ an infinite number of roots λ_n which are contained in a strip

$$\Pi_0^{(1)}=\left\{\lambda\in\mathbb{C}\left\|\operatorname{Re}\lambda\omega_3\right|<\frac{h}{2}\right\}$$

of finite width $h>0$ and have the asymptotic expression

$$|\lambda_n\omega_3|=\left|\pi n\left(1+O\left(\frac{1}{n}\right)\right)\right|. \tag{29}$$

Taking into account $\lambda_n\in\Pi_0^{(1)}$ and $\lambda_n\in R_0^{(1)}$ from (29) we get the needed asymptotic formula

$$\lambda_n=\sqrt{a_2}\,\pi ni\left(1+O\left(\frac{1}{n}\right)\right),\quad n=\pm1,\pm2,..., \tag{30}$$

where there is only one choice possible for the sign of the integer n.

By the same consideration as used for sector $R_0^{(1)}$ it can be shown that the asymptotic behavior of the eigenvalues, which are contain, in the sector $R_1^{(1)}$ also has the same form as (30), but the integer n has the apposite sign. The other formulas (22) and (23) can be obtained by the same procedure, which we used in proving the formula (30).

5. Asymptotic behavior of eigenvalues for the case $\arg a_1 = \arg a_2$.

5.1. Separation of the complex λ-plane into half-planes.

Since, in the case $\arg a_1 = \arg a_2$ the lines $l_1 = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda\omega_1 = 0\}$ and $l_2 = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda\omega_3 = 0\}$ coincided, then the line $l = l_1 = l_2$ is divide the complex λ-plane into two half-plane in each of these half-planes the real-valued functions $\operatorname{Re}(\lambda\omega_1)$ and $\operatorname{Re}(\lambda\omega_3)$ is not changed their signs and moreover, have the same sings. These half-planes we notice as

$$\sum{}_1 = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda\omega_1 \geq 0,\ \operatorname{Re} \lambda\omega_3 \geq 0\} \quad \text{and}$$
$$\sum{}_2 = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda\omega_1 \leq 0,\ \operatorname{Re} \lambda\omega_3 \leq 0\}.$$

5.2. Asymptotic expressions for the characteristic determinant $\Delta(\lambda)$ in the half-planes $\sum_1$ and $\sum_2$.

First we find a suitable asymptotic representation of the $T_{vk} y_n$, when $\lambda \to \infty$ in the half-planes $\sum_1$ and $\sum_2$, where $y_n = y_n(x,\lambda)$ are the same solutions as in section 4.2.

By the similar considerations as used in the case $\arg a_1 \neq \arg a_2$ it can be shown that for the $T_{ik}(y_j)$ takes place the asymptotic representations

$$T_{ik}(y_1) = \lambda^k[0],\ T_{ik}(y_2) = \lambda^k e^{-\lambda\omega_2}[0],\ T_{ik}(y_3) = \lambda^k e^{\lambda\omega_3}[0],\ T_{ik}(y_4) = \lambda^k[0], \tag{31}$$

when $\lambda \in \sum_1$, $\lambda \to \infty$ and

$$T_{ik}(y_k) = \lambda^k e^{-\lambda\omega_1}[0],\ T_{ik}(y_2) = \lambda^k[0],\ T_{ik}(y_3) = \lambda^k[0],\ T_{ik}(y_4) = \lambda^k e^{\lambda\omega_4}[0], \tag{32}$$

when $\lambda \in \sum_2$, $\lambda \to \infty$ taking advantage of this expressions for $T_{ik}(y_j)$, we obtain the asymptotic representations

$$L_i(y_j) = \lambda\left([\alpha_{i0} + \omega_j\alpha_{i1}]e^{-\lambda\omega_j} + [\delta_{i0} + \omega_j\delta_{i1}]\right), \quad i = \overline{1,4},\ j = \overline{1,2},$$
$$L_i(y_j) = \lambda\left([\gamma_{i0} + \omega_j\gamma_{i1}] + [\beta_{i0} + \omega_j\beta_{i1}]e^{\lambda\omega_j}\right), \quad i = \overline{1,4},\ j = \overline{1,2},$$

which are valid both in $\sum_1$ and in $\sum_2$. Substituting these formulas into the characteristic determinant $\Delta(\lambda) = \det(L_i y_j)$ we obtain, after some simple rearrangements the asymptotic representation

$$\Delta(\lambda) = \lambda^4\left([P_1]e^{k_1\lambda(\omega_1+\omega_3)} + [P_2]e^{k_2\lambda(\omega_1+\omega_3)} + \ldots + [P_l]e^{k_l\lambda(\omega_1+\omega_3)}\right), \tag{33}$$

where $-1 = k_1 < k_2 < \ldots < k_l = 1$

$$p_1 = \begin{vmatrix} \alpha_{10} + \omega_1\alpha_{11} & \delta_{10} + \omega_2\delta_{11} & \gamma_{10} + \omega_3\gamma_{11} & \beta_{10} + \omega_4\beta_{11} \\ \alpha_{20} + \omega_1\alpha_{21} & \delta_{20} + \omega_2\delta_{21} & \gamma_{20} + \omega_3\gamma_{21} & \beta_{20} + \omega_4\beta_{21} \\ \alpha_{30} + \omega_1\alpha_{31} & \delta_{30} + \omega_2\delta_{31} & \gamma_{30} + \omega_3\gamma_{31} & \beta_{30} + \omega_4\beta_{31} \\ \alpha_{40} + \omega_1\alpha_{41} & \delta_{40} + \omega_2\delta_{41} & \gamma_{40} + \omega_3\gamma_{41} & \beta_{40} + \omega_4\beta_{41} \end{vmatrix},$$

$$p_l=\begin{vmatrix}\delta_{10}+\omega_1\delta_{11} & \alpha_{10}+\omega_2\alpha_{11} & \beta_{10}+\omega_3\beta_{11} & \gamma_{10}+\omega_4\gamma_{11}\\ \delta_{20}+\omega_1\delta_{21} & \alpha_{20}+\omega_2\alpha_{21} & \beta_{20}+\omega_3\beta_{21} & \gamma_{20}+\omega_4\gamma_{21}\\ \delta_{30}+\omega_1\delta_{31} & \alpha_{30}+\omega_2\alpha_{31} & \beta_{30}+\omega_3\beta_{31} & \gamma_{30}+\omega_4\gamma_{31}\\ \delta_{40}+\omega_1\delta_{41} & \alpha_{40}+\omega_2\alpha_{41} & \beta_{40}+\omega_3\beta_{41} & \gamma_{40}+\omega_4\gamma_{41}\end{vmatrix}.$$

5.3. Asymptotic behavior of the eigenvalues.

Now we can prove the next theorem

Theorem 2. *Let the following conditions be satisfied.*

1) $\arg a_1 = \arg a_2$,

2) $c(\cdot)\in L_q(-1,1),\ \ q>1$,

3) $\theta_{ij}\neq 0,\ \ \theta_{22}\neq 0$

(θ_{11} *and* θ_{22} *are the same determinant as in Theorem 1),*

4) The linear functionals T_{ik} *in the spaces* $W_q^k(-1,0)+W_q^k(0,1)$ *are continuous* $\left(i=\overline{1,4},\ \ k=\overline{0,1}\right)$.

Then the boundary value problem (1)-(2) has in each half-plane $\sum_1$ *and* $\sum_2$ *an precisely numerable number eigenvalues, whose asymptotic behavior may be expressed by the following formulas,*

$$\lambda_n^{(1)}=\frac{\sqrt{a_1}\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}\pi ni\left(1+O\left(\frac{1}{n}\right)\right),\quad n=1,2,...,$$

$$\lambda_n^{(2)}=-\frac{\sqrt{a_1}\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}\pi ni\left(1+O\left(\frac{1}{n}\right)\right),\quad n=1,2,....$$

Proof. According to the condition 3 of this Theorem, the principal terms of the first and last coefficients of the asymptotic quasipolynom (33), i.e. the numbers P_1 and P_l, are different from zero. Again, by virtue of the [8,p.100, lemma 1], the quasipolynom (33) in the half-planes $\sum_1$ and $\sum_2$ has an infinite number of roots $\{\lambda_n^{(1)}\}$ and $\{\lambda_n^{(2)}\}$ respectively, which are contained in a strip

$$\Pi=\left\{\lambda\in\mathbb{C}\,\middle|\,\operatorname{Re}\lambda(\omega_1+\omega_3)<\frac{h}{2}\right\}$$

of finite width $h>0$ and the asymptotic representations

$$\left|\lambda_n^{(s)}(\omega_1+\omega_3)\right|=\left|\pi n\left(1+O\left(\frac{1}{n}\right)\right)\right|,\ \ s=1,2. \tag{34}$$

Taking into account that the eigenvalues $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ belongs to the strip Π from (34) we get the needed asymptotic formulas

$$\lambda_n^{(s)}=\frac{\sqrt{a_1}\sqrt{a_2}}{\sqrt{a_1}+\sqrt{a_2}}\pi ni\left(1+O\left(\frac{1}{n}\right)\right),\ \ n=\pm1,\pm2,...,\quad s=1,2,$$

where there is only one choice possible for the sign of integer n, but opposite sings for the half-planes $\sum_1$ and $\sum_2$ which completes the proof.

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Received July 4, 2000; Revised December 2, 2000.
Translated by authors.