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STRONG SOLVABILITY OF THE DIRICHLET PROBLEM FOR NON-UNIFORMLY DEGENERATE SECOND ORDER ELLIPTIC EQUATIONS

Abstract

The first boundary value problem is considered for a class of second order non-divergence structure elliptic equations with non-uniform power degeneration. The unique strong (almost everywhere) solvability of the problem in corresponding Sobolev weighted spaces is proved.

Introduction. Let $\mathbf{E}_n$ be a n-dimensional Euclidean space of the points $x=(x_1,...,x_n)$, $n\geq 3$, D be a bounded domain in $\mathbf{E}_n$ with the boundary $\partial D\in C^2$, $0\in D$. On D consider the first boundary value problem

$$\mathcal{L}u=\sum_{i,j=1}^{n}a_{ij}(x)\,u_{ij}=f(x),\ x\in D, \tag{1}$$

$$u\big|_{\partial D}=0\,, \tag{2}$$

where $\|a_{ij}(x)\|$ is a real symmetric matrix, moreover for $x\in D, \xi\in\mathbf{E}_n$

$$\gamma\sum_{i=1}^{n}\lambda_i(x)\xi_i^2\leq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leq\gamma^{-1}\sum_{i=1}^{n}\lambda_i(x)\xi_i^2. \tag{3}$$

Here $\gamma\in(0,1]$ is a constant, $\lambda_i(x)=(|x|_\alpha)^{\alpha_i}$, $|x|_\alpha=\sum_{i=1}^{n}|x_i|^{\frac{2}{2+\alpha_i}}$, $\alpha=(\alpha_1,...,\alpha_n)$, $\alpha_i\geq 0$; $u_i=$ $=\dfrac{\partial u}{\partial x_i}$, $u_{ij}=\dfrac{\partial^2 u}{\partial x_i\partial x_j}$; $i,j=1,...,n$.

For $p\in(1,\infty)$ denote by $W^p_{2,\alpha}(D)$ the Banach space of functions $u(x)$ given on D, with a finite norm

$$\|u\|_{W^p_{2,\alpha}(D)}=\left(\int_D\left(|u|^p+\sum_{i=1}^{n}(\lambda_i(x))^{p/2}|u_i|^p+\sum_{i,j=1}^{n}(\lambda_i(x)\lambda_j(x))^{p/2}|u_{ij}|^p\right)dx\right)^{1/p},$$

and let $\dot{W}^p_{2,\alpha}(D)$ be a subspace of $W^p_{2,\alpha}(D)$ in which the totality of all infinitely differentiable in $\overline{D}$ functions vanishing in ∂D is a dense set.

Denote $\min\{\alpha_1,...,\alpha_n\}$ and $\max\{\alpha_1,...,\alpha_n\}$ by α^- and α^+ respectively. We'll assume that

$$\alpha^+<2 \tag{4}$$

and

$$b_{ij}(x)=\frac{a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}}\in C(\overline{D});\ i,j=1,...,n. \tag{5}$$

We understand the last condition in the sense that the functions $b_{ij}(x)$ may be predetermined in $\partial D\cup\{0\}$ such that the continued functions were continuous in $\overline{D}$.

The main result of the paper is in the following: if the conditions (3)-(5) are fulfilled with respect to the coefficients of the operator $\mathcal{L}$, then for any $f(x)\in L_q(D)$

$\left(q\in\left(\frac{n(2+\alpha^{+})}{2-\alpha^{-}},\infty\right)\right)$ the first boundary value problem (1)-(2) is uniquely strongly (almost everywhere) solvable in the space $\dot{W}^{p}_{2,\alpha}(D)$ for $p\in\left(1,\frac{n}{2}\right]$.

Note that for uniformly elliptic second order equations with continuous coefficients the corresponding results have been obtained in [1-2], and for the equations with discontinuous coefficients- in [3]. We refer also to papers [4-6], where solvability questions of boundary value problems were studied for non-linear second order elliptic equations. As to analogous results for second order parabolic equations we note papers [7-10].

1^{0}. Some auxiliary statements.

Let for $R\in(0,1]$, $k>0$, $x^{0}\in E_n$ $\mathcal{E}_R^{x^0}(k)$ be an ellipsoid $\left\{x:\sum_{i=1}^{n}\frac{(x_i-x_i^0)^2}{R^{\alpha_i}}<(kR)^2\right\}$, $r\in\left(0,\frac{1}{n-1}\right)$ be a number that will be chosen later, $\mathcal{B}_R=\mathcal{E}_R^0(1+r)\setminus\overline{\mathcal{E}_R^0(1)}$.

Lemma 1. *Let* $x'\in\partial\mathcal{E}_R^0\left(1+\frac{r}{2}\right)$. *Then* $\mathcal{E}_R^{x'}\left(\frac{r}{2}\right)\subset\mathcal{B}_R$.

Proof. Let $x\in\mathcal{E}_R^{x'}\left(\frac{r}{2}\right)$. According to Minkowsky inequality

$$\sqrt{\sum_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}}\le\sqrt{\sum_{i=1}^{n}\frac{(x_i-x_i')^2}{R^{\alpha_i}}}+\sqrt{\sum_{i=1}^{n}\frac{(x_i')^2}{R^{\alpha_i}}}<\frac{r}{2}R+\left(1+\frac{r}{2}\right)R=(1+r)R.$$

On the other hand

$$\sqrt{\sum_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}}\ge\sqrt{\sum_{i=1}^{n}\frac{(x_i')^2}{R^{\alpha_i}}}-\sqrt{\sum_{i=1}^{n}\frac{(x_i-x_i')^2}{R^{\alpha_i}}}>\left(1+\frac{r}{2}\right)R-\frac{r}{2}R=R,$$

and the lemma is proved.

Lemma 2. *Let* $A_R=\mathcal{E}_R^0\left(1+\frac{r}{2}+\frac{r^2}{64}\right)\setminus\overline{\mathcal{E}_R^0\left(1+\frac{r}{2}-\frac{r^2}{64}\right)}$. *Then a closed layer* $\overline{A}_R$ *may be covered by a denumerable system of ellipsoids* $\left\{\mathcal{E}_R^{x^\nu}\left(\frac{r}{4}\right)\right\}$, $x^{\nu}\in\partial\mathcal{E}_R^0\left(1+\frac{r}{2}\right)$, $\nu=1,2,\ldots$.

Proof. Let $x\in\overline{A}_R$. Without losing of generality we can consider that $x_1\neq0$. Let for the definiteness $x_1>0$. Choose $\beta\ge0$ so that the point $x^{\nu}=(\beta,x_2,\ldots,x_n)$ belongs to $\partial\mathcal{E}_R^0\left(1+\frac{r}{2}\right)$. Since and

$$\sqrt{\sum_{i=2}^{n}\frac{x_i^2}{R^{\alpha_i}}+\frac{\beta^2}{R^{\alpha_1}}}=\left(1+\frac{r}{2}\right)R$$

and

$$\left(1+\frac{r}{2}-\frac{r^2}{64}\right)R\le\sqrt{\sum_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}}\le\left(1+\frac{r}{2}+\frac{r^2}{64}\right)R,$$

it is clear that such β exists. For the definiteness assume that $\beta\le x_1$ we have:

$$\frac{x_1^2-\beta^2}{R^{\alpha_1}}\le R^2\left[\left(1+\frac{r}{2}+\frac{r^2}{64}\right)^2-\left(1+\frac{r}{2}\right)^2\right]=\frac{R^2r^2}{64}\left(2+r+\frac{r^2}{64}\right)\le\frac{R^2r^2}{16}.$$

But for $\beta\le x_1$ $x_1^2-\beta^2\ge(x_1-\beta)^2$. Therefore

$$\sqrt{\sum_{i=1}^{n}\frac{\left(x_i-x_i^{\nu}\right)^2}{R^{\alpha_i}}}=\sqrt{\frac{(x_1-\beta)^2}{R^{\alpha_1}}}\le\frac{Rr}{4}$$

and the lemma is proved.

Corollary. *From the covering of $\overline{A}_R$ by the ellipsoid system $\left\{\mathcal{E}_R^{x^{\nu}}\left(\frac{r}{4}\right)\right\}$ we can choose a finite subcovering $\mathcal{E}_R^{x^1}\left(\frac{r}{4}\right),\ldots,\mathcal{E}_R^{x^N}\left(\frac{r}{4}\right)$, where the number N depends only on r and n, but doesn't depend on R.*

Lemma 3. *Let R_0 be such that $\overline{\mathcal{E}_{R_0}^0\left(1+\frac{r}{2}+\frac{r^4}{64}\right)}\subset D$. Then, if*

$$a=\frac{1}{2}\left(\frac{1+\frac{r}{2}-\frac{r^2}{64}}{1+\frac{r}{2}+\frac{r^2}{64}}\right)^{\frac{2}{2+\alpha^+}}$$ *and for $m=1,2,\ldots,$ $R_m=R_0a^m$, then*

$$\mathcal{E}_{R_0}^0\left(1+\frac{r}{2}+\frac{r^2}{64}\right)\setminus\{0\}\subset\bigcup_{m=0}^{\infty}A_{R_m}.$$

Proof. It is sufficiently to show that for any natural m

$$\mathcal{E}_{R_{m+1}}^0\left(1+\frac{r}{2}+\frac{r^2}{64}\right)\supset\overline{\mathcal{E}}_{R_m}^0\left(1+\frac{r}{2}-\frac{r^2}{64}\right). \tag{6}$$

The inclusion (6) is equivalent to one that for natural m and $i=1,\ldots,n$

$$\left(1+\frac{r}{2}+\frac{r^2}{64}\right)(R_{m+1})^{1+\frac{\alpha_i}{2}}>\left(1+\frac{r}{2}+\frac{r^2}{64}\right)(R_m)^{1+\frac{\alpha_i}{2}}.$$

The last inequality will be fulfilled if only

$$\frac{R_m}{R_{m+1}}<\left(\frac{1+\frac{r}{2}+\frac{r^2}{64}}{1+\frac{r}{2}-\frac{r^2}{64}}\right)^{\frac{2}{2+\alpha^+}}=\frac{2}{a}.$$

Compete the proof it is sufficient to consider that $\frac{R_m}{R_{m+1}}=\frac{1}{a}$.

Remark. It is valid the inclusion

$$\bigcup_{m=0}^{\infty}\mathcal{E}_{R_m}\subset\mathcal{E}_{R_0}^{0}(1+r),$$

where the covering by a layer system $\{\mathcal{E}_{R_m}\}$ has finite multiplicity N_1 depending only on n.

Lemma 4. *Let* $x\in\mathcal{E}_R$, $R\in(0,1]$. *Then for* $i=1,\dots,n$

$$C_1(n,\alpha)R^{\alpha_i}\le\lambda_i(x)\le C_2(n,\alpha)R^{\alpha_i}. \qquad (7)$$

Here and later record $C(\cdots)$ means that the positive constant C depends only on the content of brackets.

Proof. For $x\in\mathcal{E}_R$ and $i=1,\dots,n$

$$\sum_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}<(1+r)^2R^2\ ,\ \text{i.e.}\quad |x_i|\le 2R^{1+\frac{\alpha_i}{2}}.$$

Therefore $|x|_\alpha\le 2nR$ and

$$\lambda_i(x)\le(2n)^{\alpha^+}R^{\alpha_i}\ ;\ i=1,\dots,n. \qquad (8)$$

On other hand, if $x\in\mathcal{E}_R$, then there exists such i_0, $1\le i_0\le n$, that

$$\frac{x_{i_0}}{R^{\alpha_{i_0}}}\ge\frac{R^2}{n}.$$

Thus, $|x_{i_0}|\ge\frac{R^{1+\frac{\alpha_{i_0}}{2}}}{\sqrt{n}}$, $|x|_\alpha\ge\frac{R}{n^{\frac{1}{2+\alpha^-}}}$ and consequently

$$\lambda_i(x)\ge\frac{R^{\alpha_i}}{n^{\frac{1}{2+\alpha^-}}};\quad i=1,\dots,n. \qquad (9)$$

Now the required estimate (7) follows from (8) and (9).

2^0. Integral estimates.

In the sequel, everywhere we'll assume that $|\alpha|=\alpha_1+\dots+\alpha_n>0$. On the contrary the equation (1) is not degenerated and the main result of the paper follows from [1]. Let $x'\in\partial\mathcal{E}_R^0\left(1+\frac{r}{2}\right)$, $E=\mathcal{E}_R^{x'}\left(\frac{r}{2}\right)$, $R\in(0,1]$

$$\mathcal{L}_0^{x'}=\sum_{i=1}^{n}\lambda_i(x')\frac{\partial^2}{\partial x_i^2}.$$

Lemma 5. *For any function* $u(x)\in C_0^\infty(E)$ *for any* $p\in\left(1,\frac{n}{2}\right]$ *the estimate*

$$\int_E\sum_{i,j=1}^{n}(\lambda_i(x)\lambda_j(x))^{\frac{p}{2}}|u_{ij}|^p\,dx\le C_3(n,p,\alpha)\int_E|\mathcal{L}_0^{x'}u|^p\,dx \qquad (10)$$

is valid.

Proof. Transform the coordinates $y_i = R^{-\frac{\alpha_i}{2}} x_i; i = 1,...,n$. Let $\widetilde{E}$, y' and $\widetilde{u}(y)$ be the images of the ellipsoid E, point x' and the function $u(x)$, respectively. It is clear that $\widetilde{u}(y) \in C_0^\infty(\widetilde{E})$, and the operator $\mathcal{L}_0^{x'}$ will turn into the operator

$$\widetilde{\mathcal{L}} = \sum_{i=1}^{n} \frac{\lambda_i(x')}{R^{\alpha_i}} \frac{\partial^2}{\partial y_i^2}.$$

Make one more transformation $z_i = \sqrt{\frac{R^{\alpha_i}}{\lambda_i(x')}} y_i \; i = 1,...,n$. Let E' and $u'(z)$ be the images of the ball $\widetilde{E}$ and the functions $\widetilde{u}(y)$, respectively. It is clear that $u'(z) \in C_0^\infty(E')$ and the Laplace operator Δ will be the image $\widetilde{\mathcal{L}}$. According to [1]

$$\int_{E'} \sum_{i,j=1}^{n} \left| \frac{\partial^2 u'}{\partial z_i \partial z_j} \right|^p dz \le C_4(n,p) \int_{E'} |\Delta u'|^p dz. \tag{11}$$

Coming back to variables y and denoting $\left(\prod_{i=1}^{n} \frac{R^{\alpha_i}}{\lambda_i(x')} \right)^{1/2}$ by I, we get from (11)

$$\int_{\widetilde{E}} \sum_{i,j=1}^{n} \left(\frac{\lambda_i(x)\lambda_j(x')}{R^{\alpha_i+\alpha_j}} \right)^{\frac{p}{2}} \left| \frac{\partial^2 \widetilde{u}}{\partial y_i \partial y_j} \right|^p I dy \le C_4 \int_{\widetilde{E}} |\widetilde{\mathcal{L}} \widetilde{u}|^p I \, dy. \tag{12}$$

Now return to the variables x. Denoting $\left(\prod_{i=1}^{n} R^{-\alpha_i} \right)^{1/2}$ by J, we have from (12)

$$\int_{E} \sum_{i,j=1}^{n} \left(\lambda_i(x')\lambda_j(x') \right)^{\frac{p}{2}} |u_{ij}|^p J \, dx \le C_4 \int_{E} |\mathcal{L}_0^{x'} u|^p J \, dx. \tag{13}$$

Now it is sufficient to consider that by lemma 4

$$\lambda_i(x') \ge \frac{C_1}{C_2} \lambda_i(x); \; x \in E; \; i = 1,...,n,$$

and the required estimate follows from (13). The lemma is proved.

Corollary. *Let*

$$\mathcal{L}^{x'} = \sum_{i,j=1}^{n} \sqrt{\lambda_i(x')\lambda_j(x')} b_{ij}(x') \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then by fulfilling the condition (3) for any function $u(x) \in C_0^\infty(E)$ *and any* $p \in \left(1, \frac{n}{2}\right]$ *the estimate*

$$\int_{E} \sum_{i,j=1}^{n} \left(\lambda_i(x)\lambda_j(x) \right)^{\frac{p}{2}} |u_{ij}|^p dx \le C_5(n,p,\alpha,\gamma) \int_{E} |\mathcal{L}^{x'} u|^p dx$$

is valid.

Make a remark. It follows from the condition (5) that there exists such a non-negative, continuous and non-decreasing function $\omega(t)$ on $[0, \mathrm{diam} D]$ that $\omega(0) = 0$ and

$$|b_{ij}(x) - b_{ij}(y)| \le \omega(|x-y|); \; x, y \in \overline{D}; \; i, j = 1,...,n. \tag{5'}$$

Besides, it follows from the condition (30) that the matrix $\|b_{ij}(x)\|$ is uniformly positive determined in $\overline{D}$, i.e. there exists such a positive constant b_0 that

$$|b_{ij}(x)| \le b_0;\ x \in D;\ i,j = 1,...,n.$$

Lemma 6. *If the conditions (3) and (5) are fulfilled with respect to the coefficients of the operator* $\mathcal{L}$, *then for any function* $u(x) \in C_0^\infty(E)$ *for* $0 < r \le r_0(n,p,\alpha,\gamma,\omega,b_0)$ *and any* $p \in \left(1, \frac{n}{2}\right]$ *the estimate*

$$\int_E \sum_{i,j=1}^n (\lambda_i(x)\lambda_j(x))^{\frac{p}{2}} |u_{ij}|^p dx \le C_6(n,p,\alpha,\gamma,\omega,b_0) \int_E |\mathcal{L}u|^p dx \tag{14}$$

is valid.

Proof. We have for $i,j = 1,...,n$

$$\sqrt{\lambda_i(x')\lambda_j(x')} = \sqrt{\lambda_i(x')\lambda_j(x')} - \sqrt{\lambda_i(x)\lambda_j(x)} + \sqrt{\lambda_i(x)\lambda_j(x)} = K_1 + \sqrt{\lambda_i(x)\lambda_j(x)}. \tag{15}$$

On the other hand, subject to (7)

$$|K_1| \le \sqrt{\lambda_j(x')}|\lambda_i(x') - \lambda_i(x)| + \sqrt{\lambda_i(x)}\left|\sqrt{\lambda_j(x')} - \sqrt{\lambda_j(x)}\right| \le$$

$$\le \sqrt{\frac{C_2}{C_1}}\sqrt{\lambda_i(x)\lambda_j(x)}\left|\sqrt{\frac{\lambda_i(x')}{\lambda_i(x)}} - 1\right| + \sqrt{\lambda_i(x)\lambda_j(x)}\left|\sqrt{\frac{\lambda_j(x')}{\lambda_j(x)}} - 1\right| =$$

$$= \sqrt{\lambda_i(x)\lambda_j(x)}\left(\sqrt{\frac{C_2}{C_1}}\left|\sqrt{\frac{\lambda_i(x')}{\lambda_i(x)}} - 1\right| + \left|\sqrt{\frac{\lambda_j(x')}{\lambda_j(x)}} - 1\right|\right). \tag{16}$$

Then we get for $i = 1,...,n$

$$J_i = \left|\sqrt{\frac{\lambda_i(x')}{\lambda_i(x)}} - 1\right| = \frac{\left|\frac{\lambda_i(x')}{\lambda_i(x)} - 1\right|}{\sqrt{\frac{\lambda_i(x')}{\lambda_i(x)}} + 1} \le \frac{|\lambda_i(x') - \lambda_i(x)|}{\lambda_i(x)} \le$$

$$\le \frac{1}{C_1}\frac{\left|(|x'|_\alpha)^{\alpha_i} - (|x|_\alpha)^{\alpha_i}\right|}{R^{\alpha_i}} = \frac{\alpha_i}{C_1}\frac{\left||x'|_\alpha - |x|_\alpha\right|\theta^{\alpha_i - 1}}{R^{\alpha_i}},$$

where θ belongs to the interval that connects the points $|x'|_\alpha$ and $|x|_\alpha$. Thus

$$J_i \le C_7(n,\alpha)\frac{\left||x'|_\alpha - |x|_\alpha\right|}{R} \le C_7 \frac{\sum_{i=1}^n \left||x_i'|^{\frac{2}{2+\alpha_i}} - |x_i|^{\frac{2}{2+\alpha_i}}\right|}{R} \le$$

$$\le C_7 \frac{\sum_{i=1}^n |x_i' - x_i|^{\frac{2}{2+\alpha_i}}}{R} \le C_7 \frac{\sum_{i=1}^n \left(\frac{r}{2}R^{1+\frac{\alpha_i}{2}}\right)^{\frac{2}{2+\alpha_i}}}{R} \le C_8(n,\alpha) r^{\frac{2}{2+\alpha^+}}.$$

Hence, subject to (16) we deduce

$$|K_1| \le C_9(n,\alpha)\sqrt{\lambda_i(x)\lambda_j(x)}\, r^{\frac{2}{2+\alpha^+}};\ i,j = 1,...,n.$$

Then it follows from (15)

$$\left|\mathcal{L}^{x'}u\right| \le C_{10}(n,\alpha,b_0)r^{\frac{2}{2+\alpha^+}}\sum_{i,j=1}^{n}\sqrt{\lambda_i(x)\lambda_j(x)}\left|u_{ij}\right| + \left|\sum_{i,j=1}^{n}\sqrt{\lambda_i(x)\lambda_j(x)}b_{ij}(x')u_{ij}\right| \le$$

$$\le\left(C_{10}r^{\frac{2}{2+\alpha^+}} + \omega(|x'-x|)\right)\sum_{i,j=1}^{n}\sqrt{\lambda_i(x)\lambda_j(x)}\left|u_{ij}\right| + \left|\sum_{i,j=1}^{n}\sqrt{\lambda_i(x)\lambda_j(x)}b_{ij}(x)u_{ij}\right|. \qquad (17)$$

Note that $|x'-x| \le \frac{\sqrt{n}}{2}r$ and $\sqrt{\lambda_i(x)\lambda_j(x)}b_{ij}(x) = a_{ij}(x)$, $i,j=1,\ldots,n$. There we get from (17)

$$\left|\mathcal{L}^{x'}u\right| \le \left(C_{10}r^{\frac{2}{2+\alpha^+}} + \omega\left(\sqrt{\frac{n}{2}}r\right)\right)\sum_{i,j=1}^{n}\sqrt{\lambda_i(x)\lambda_j(x)}\left|u_{ij}\right| + \left|\mathcal{L}u\right|, \qquad (18)$$

i.e.

$$\left|\mathcal{L}^{x'}u\right| \le 2^{p-1}\left(C_{10}r^{\frac{2}{2+\alpha^+}} + \omega\left(\sqrt{\frac{n}{2}}r\right)\right)^p\left(\sum_{i,j=1}^{n}\sqrt{\lambda_i(x)\lambda_j(x)}\left|u_{ij}\right|\right)^p + \left|\mathcal{L}u\right|^p \le$$

$$\le\left(2n^2\right)^{p-1}\left(C_{10}r^{\frac{2}{2+\alpha^+}} + \omega\left(\sqrt{\frac{n}{2}}r\right)\right)^p\sum_{i,j=1}^{n}\left(\lambda_i(x)\lambda_j(x)\right)^{\frac{p}{2}}\left|u_{ij}\right|^p + \left|\mathcal{L}u\right|^p.$$

Let $r_1 > 0$ be such that

$$\left(2n^2\right)^{p-1}\left(C_{10}r_1^{\frac{2}{2+\alpha^+}} + \omega\left(\sqrt{\frac{n}{2}}r_1\right)\right)^p \le \frac{1}{2C_5}.$$

Fix $r_0 = \min\left\{r_1, \frac{1}{n-1}\right\}$. Then the required estimate (14) follows from the corollary to lemma 5 and inequality (18). The lemma is proved.

In the sequel, not stipulating this particularly, we'll assume $r = r_0$, and also consider that $\overline{\mathcal{E}_R^0(1+r)} \subset D$.

Lemma 7. *If the conditions (3) and (5) are fulfilled with respect to the coefficients of the operator* $\mathcal{L}$, *then for any function* $u(x) \in C_0^\infty(E)$ *for any* $p \in \left(1, \frac{n}{2}\right]$ *the estimate*

$$\|u\|_{W^p_{2,\alpha}(E)} \le C_{11}(n,p,\alpha,\gamma,\omega,b_0)\|\mathcal{L}u\|_{Lp(E)} \qquad (19)$$

is valid.

Proof. Consider a parallelepiped $E^0 = \left\{x : |x_i - x_i'| < \frac{r}{2}R^{1+\frac{\alpha_i}{2}}; i=1,\ldots,n\right\}$. For $x \in E^0$ $\sqrt{\sum_{i=1}^{n}\frac{(x_i - x_i')^2}{R^{\alpha_i}}} < \frac{r\sqrt{n}}{2}R$. Therefore

$$\sqrt{\sum_{i=1}^{n}\frac{x_i^2}{R^{\alpha_i}}} \ge \sqrt{\sum_{i=1}^{n}\frac{(x_i')}{R^{\alpha_i}}} - \sqrt{\sum_{i=1}^{n}\frac{(x_i - x_i')^2}{R^{\alpha_i}}} > \left(1+\frac{r}{2}\right)R - \frac{r\sqrt{n}}{2}R = R\left(1 - \frac{r(\sqrt{n}-1)}{2}\right).$$

Taking into account that $r\left(\sqrt{n}-1\right)<r(n-1)\le 1$, we deduce $E^0\subset \mathbf{E}_n\setminus \mathcal{E}_R^0\left(\frac{1}{2}\right)$. The latter by analogy with (9) gives

$$|x|_\alpha \ge C_{12}(n,\alpha)R\,;\; x\in E^0 . \tag{20}$$

Continue the function $u(x)$ by the zero in $E^0\setminus E$ and denote the continued function again by $u(x)$. It is clear that $u(x)\in C_0^\infty\left(E^0\right)$. Let $\widetilde{x}=(x_2,\dots,x_n)$. We have for $x_1\in\left(x_1'-\frac{r}{2}R^{1+\frac{\alpha_1}{2}},x_1'+\frac{r}{2}R^{1+\frac{\alpha_1}{2}}\right)$

$$u(x_1,\widetilde{x})=\int\limits_{x_1'-\frac{r}{2}R^{1+\frac{\alpha_1}{2}}}^{x_1}u_1(t,\widetilde{x})dt=\int\limits_{x_1'-\frac{r}{2}R^{1+\frac{\alpha_1}{2}}}^{x_1}(\lambda_1(t,\widetilde{x}))^{-\frac{1}{2}}(\lambda_1(t,\widetilde{x}))^{\frac{1}{2}}u_1(t,\widetilde{x})dt .$$

Hence, subject to (20) it follows

$$|u(x_1,\widetilde{x})|^p\le C_{13}(n,p,\alpha)R^{p-1-\frac{\alpha_1}{2}}\int\limits_{x_1'-\frac{r}{2}R^{1+\frac{\alpha_1}{2}}}^{x_1'+\frac{r}{2}R^{1+\frac{\alpha_1}{2}}}(\lambda_1(t,\widetilde{x}))^{-\frac{p}{2}}|u_1(t,\widetilde{x})|^p dt$$

and further

$$\int\limits_{E^0}|u|^p dx\le C_{13}R^p\int\limits_{E^0}(\lambda_1(x))^{\frac{p}{2}}|u_1|^p dx .$$

Thus

$$\int\limits_{E}|u|^p dx\le C_{13}R^p\int\limits_{E}\sum_{i=1}^{n}(\lambda_i(x))^{\frac{p}{2}}|u_i|^p dx . \tag{21}$$

Similarly we have

$$u_1(x_1,\widetilde{x})=\int\limits_{x_1'-\frac{r}{2}R^{1+\frac{\alpha_1}{2}}}^{x_1}(\lambda_1(t,\widetilde{x}))^{-1}(\lambda_1(t,\widetilde{x}))u_{11}(t,\widetilde{x})\,dt ,$$

and further

$$|u_1(x_1,\widetilde{x})|^p\le C_{14}(n,p,\alpha)R^{\left(1+\frac{\alpha_1}{2}\right)(p-1)-\alpha_1 p}\int\limits_{x_1'-\frac{r}{2}R^{1-\frac{\alpha_1}{2}}}^{x_1'+\frac{r}{2}R^{1+\frac{\alpha_1}{2}}}(\lambda_1(t,\widetilde{x}))^{-p}|u_{11}(t,\widetilde{x})|^p dt .$$

On the other hand, if $x\in E^0$, then $\sqrt{\sum_{i=1}^{n}\frac{(x_i-x_i')^2}{R^{\alpha_i}}}<\frac{r\sqrt{n}}{2}R$, therefore $E^0\subset$ $\subset\mathcal{E}_R^0\left(1+\frac{r}{2}\left(1+\sqrt{n}\right)\right)\subset\mathcal{E}_R^0(2)$. Thus

$$|x|_\alpha\le C_{15}(n,\alpha)R\,;\, x\in E^0\,;$$

and we get

$$(\lambda_1(x_1,\widetilde{x}))^{\frac{p}{2}}|u_1(x,\widetilde{x})|^p \le C_{16}(n,p,\alpha)R^{\left(1+\frac{\alpha_1}{2}\right)(p-1)-\frac{\alpha_1 p}{2}}\int_{x_1'-\frac{r}{2}R^{1+\frac{\alpha_1}{2}}}^{x_1'+\frac{r}{2}R^{1+\frac{\alpha_1}{2}}}(\lambda_1(t,\widetilde{x}))^p|u_{11}(t,\widetilde{x})|^p\,dt.$$

Thus

$$\int_{E^0}(\lambda_1(x))^{\frac{p}{2}}|u_1|^p\,dx \le C_{16}R^p\int_{E^0}(\lambda_1(x))^p|u_{11}|^p\,dx.$$

The estimate

$$\int_{E^0}(\lambda_i(x))^{\frac{p}{2}}|u_i|^p\,dx \le C_{16}R^p\int_{E^0}(\lambda_i(x))^p|u_{ii}|^p\,dx,\ i=2,\dots,n$$

is deduced analogously.

In fact proved the inequality

$$\int_E\sum_{i=1}^n(\lambda_i(x))^{\frac{p}{2}}|u_i|^p\,dx \le C_{16}R^p\int_E\sum_{i,j=1}^n(\lambda_i(x)\lambda_j(x))^{\frac{p}{2}}|u_{ij}|^p\,dx. \qquad (22)$$

Now, the required estimate (19) follows from lemma 6 and inequalities (21) and (22). The lemma is proved.

Let $E_1=\mathcal{E}_R^{x'}\left(\frac{r}{4}\right)$.

Lemma 8. *If the conditions (3) and (5) are fulfilled with respect to the coefficients of the operator* $\mathcal{L}$, *then for any function* $u(x)\in C^\infty(\overline{E})$ *for any* $p\in\left(1,\frac{n}{2}\right]$ *and* $\varepsilon>0$, *the estimate*

$$\|u\|^p_{W^p_{2,\alpha}(E_1)} \le C_{17}(n,p,\alpha,\gamma,\omega,b_0)\int_E|\mathcal{L}u|^p\,dx+\varepsilon\|u\|^p_{W^p_{2,\alpha}(E)}+\frac{C_{18}(n,p,\alpha,\gamma,\omega,b_0)}{\varepsilon R^{2p}}\int_E|u|^p\,dx \quad (23)$$

is valid.

Proof. Consider such a function $\eta(x)\in C_0^\infty(E)$ that $\eta(x)=1$ for $x\in E_1$, $\eta(x)=0$ for $x\notin\mathcal{E}_R^{x'}\left(\frac{3r}{8}\right)$, $0\le\eta(x)\le1$, and for $i,j=1,\dots,n$

$$|\eta_i|\le\frac{C_{19}(n,\alpha,r)}{R^{1+\frac{\alpha_i}{2}}},\ \ |\eta_{ij}|\le\frac{C_{19}}{R^{2+\frac{\alpha_i+\alpha_j}{2}}}. \qquad (24)$$

Put $\vartheta(x)=u(x)\eta(x)$. Then $\vartheta(x)\in C_0^\infty(E)$, and from lemma 7 we get

$$\|u\|_{W^p_{2,\alpha}(E_1)}\le C_{11}\|\mathcal{L}\vartheta\|_{L_p(E)}. \qquad (25)$$

On the other hand

$$\mathcal{L}\vartheta=\eta\mathcal{L}u+u\mathcal{L}\eta+2\sum_{i,j=1}^n a_{ij}(x)u_i\eta_j.$$

According to (24) and lemma 4 we have

$$|L\eta|=\left|\sum_{i,j=1}^n b_{ij}(x)\sqrt{\lambda_i(x)\lambda_j(x)}\eta_{ij}\right|\le\frac{C_{20}(n,\alpha,\gamma,\omega,b_0)}{R^2},$$

$$2\left|\sum_{i,j=1}^n a_{ij}(x)u_i\eta_j\right|\le 2\sqrt{\sum_{i,j=1}^n a_{ij}(x)u_iu_j\sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j}\le$$

$$\leq 2\gamma^{-1}\sqrt{\sum_{i=1}^{n}\lambda_i(x)u_i^2\sum_{i=1}^{n}\lambda_i(x)\eta_i^2}\leq\frac{C_{21}(n,\alpha,\gamma,\omega,b_0)}{R}\sqrt{\sum_{i=1}^{n}\lambda_i(x)u_i^2}\,.$$

Therefore

$$|\mathcal{L}\vartheta|^p\leq C_{22}(n,p)|\mathcal{L}u|^p+\frac{C_{23}(n,p,\alpha,\gamma,\omega,b_0)}{R^{2p}}|u|^p+$$

$$+\frac{C_{24}(n,p,\alpha,\gamma,\omega,b_0)}{R^p}\left(\sum_{i=1}^{n}\lambda_i(x)u_i^2\right)^{\frac{p}{2}}.$$

Taking into account that

$$\left(\sum_{i=1}^{n}\lambda_i(x)u_i^2\right)^{\frac{p}{2}}\leq C_{25}(n,p)\sum_{i=1}^{n}(\lambda_i(x))^{\frac{p}{2}}|u_i|^p,$$

we finally get

$$\int_E|\mathcal{L}\vartheta|^p dx\leq C_{22}\int_E|\mathcal{L}u|^p dx+\frac{C_{23}}{R^{2p}}\int_E|u|^p dx+\frac{C_{26}(n,p,\alpha,\gamma,\omega,b_0)}{R^p}\int_E\sum_{i=1}^{n}(\lambda_i(x))^{\frac{p}{2}}|u_i|^p dx.\quad(26)$$

Make a change of variables $y_i=R^{-\frac{\alpha_i}{2}}x_i$, $i=1,...,n$ and let $\widetilde{E}$ and $\widetilde{u}(y)$ be the images of the ellipsoid E and the function $u(x)$, respectively. By the interpolational inequality [11] for any $\varepsilon_1>0$ there exists such a constant $C_{27}(n,p)$ that

$$\int_{\widetilde{E}}\sum_{i=1}^{n}\left|\frac{\partial\widetilde{u}}{\partial y_i}\right|^p dy\leq\varepsilon_1\int_{\widetilde{E}}\sum_{i,j=1}^{n}\left|\frac{\partial^2\widetilde{u}}{\partial y_i\partial y_j}\right|^p dy+\frac{C_{27}}{\varepsilon_1}\int_{\widetilde{E}}|\widetilde{u}|^p dy.$$

Returning to the variables x and taking into account lemma 4, we get

$$\int_E\sum_{i=1}^{n}(\lambda_i(x))^{\frac{p}{2}}|u_i|^p dx\leq C_{28}(n,p,\alpha)\varepsilon_1\int_E\sum_{i,j=1}^{n}(\lambda_i(x)\lambda_j(x))^{\frac{p}{2}}|u_{ij}|^p dx+\frac{C_{29}(n,p,\alpha)}{\varepsilon_1}\int_E|u|^p dx.(27)$$

Now fix an arbitrary $\varepsilon>0$. Not losing generality, we'll assume that $\varepsilon\leq 1$. Put $\varepsilon_1=\frac{\varepsilon R^p}{C_{26}(C_{11})^p}$. Then we obtain from (25)-(27)

$$\|u\|^p_{W^p_{2,\alpha}(E_1)}\leq(C_{11})^p C_{22}\int_E|\mathcal{L}u|^p dx+\varepsilon\|u\|^p_{W^p_{2,\alpha}(E)}+\frac{(C_{11})^pC_{23}+\frac{(C_{11})^{2p}C_{26}C_{29}}{\varepsilon}}{R^{2p}}\int_E|u|^p dx,$$

whence the required estimate (23) follows. The lemma is proved.

Lemma 9. *If the conditions (3) and (5) are fulfilled with respect to the coefficients of the operator* $\mathcal{L}$, *then for any function* $u(x)\in W^p_{2,\alpha}(\mathcal{E}^0_R(1+r))$ *for any* $p\in\left(1,\frac{n}{2}\right]$ *and* $\varepsilon>0$ *the estimate*

$$\|u\|^p_{W^p_{2,\alpha}\left(\mathcal{E}^0_R\left(1+\frac{r}{2}+\frac{r^2}{64}\right)\right)}\leq C_{30}(n,p,\alpha,\gamma,\omega,b_0)\int_{\mathcal{E}^0_R(1+r)}|\mathcal{L}u|^p dx+\varepsilon\|u\|^p_{W^p_{2,\alpha}(\mathcal{E}^0_R(1+r))}+$$

$$+\frac{C_{31}(n,p,\alpha,\gamma,\omega,b_0)}{\varepsilon}\left(\operatorname*{ess\,sup}_{\mathcal{E}^0_R(1+r)}|u|\right)^p.$$

is valid.

Proof. It is sufficient to consider the case $u(x)\in C^{\infty}\left(\overline{\mathcal{E}_R^0(1+r)}\right)$. In the sequel, everywhere for the brevity of records we'll denote $\operatorname{ess\,sup}$ by sup. It is easy to see that

$$R^{-2p}\int_E |u|^p dx \le C_{31}(n,\alpha)R^{n-2p+\frac{|\alpha|}{2}}\left(\sup_{\mathcal{E}_R^0(1+r)}|u|\right)^p \le C_{31}R^{\frac{|\alpha|}{2}}\left(\sup_{\mathcal{E}_R^0(1+r)}|u|\right)^p,$$

and according to our assumption $|\alpha|>0$. Put for $m=0,1,2,\ldots$, $R_m=Ra^m$, where the number a has the same sense as in lemma 3. Then by lemma 8 for each m

$$\|u\|^p_{W^p_{2,\alpha}(E_1(m))} \le C_{17}\int_{E_m}|\mathcal{L}u|^p dx + \varepsilon\|u\|^p_{W^p_{2,\alpha}(E(m))} + \frac{C_{18}C_{31}}{\varepsilon}R_m^{\frac{|\alpha|}{2}}\left(\sup_{\mathcal{E}_R^0(1+r)}|u|\right)^p, \qquad (28)$$

where $E_1(m)=\mathcal{E}^0_{R_m}\left(\frac{r}{4}\right)$, $E(m)=\mathcal{E}^0_{R_m}\left(\frac{r}{2}\right)$. Now to complete the proof it is sufficient to sum the inequality (28) with respect to m, consider lemma 3, its remark, and the convergence of series $\sum_{m=0}^{\infty} R_m^{\frac{|\alpha|}{2}}$.

Remark. Since the operator $\mathcal{L}$ is degenerated only at the point 0, then the affirmation of the given lemma holds and in ellipsoids $\mathcal{E}_R^{x^*}\left(1+\frac{r}{2}+\frac{r^2}{64}\right)$ and $\mathcal{E}_R^{x^*}(1+r)$, if only $\overline{\mathcal{E}_R^{x^*}(1+r)}\subset D$ and $\mathcal{E}_R^{x^*}(1+r)\cap\mathcal{E}_R^0(r)=\varnothing$.

3^0. The main coercive inequality.

Let for $\rho>0$ $D_\rho=\{x: x\in D, dist(x,\partial D)>\rho\}$.

Lemma 10. *If the conditions (3) and (5) are fulfilled with respect to the coefficients of the operator $\mathcal{L}$, then for any function $u(x)\in W^p_{2,\alpha}(D)$ for any $p\in\left(1,\frac{n}{2}\right]$, $\varepsilon>0$ and $\rho>0$ the estimation*

$$\|u\|^p_{W^p_{2,\alpha}(D_\rho)} \le C_{32}(n,p,\alpha,\gamma,\omega,b_0,\rho,D)\left(\int_D|\mathcal{L}u|^p dx + \varepsilon\|u\|^p_{W^p_{2,\alpha}(D)} + \frac{1}{\varepsilon}\left(\sup_D|u|\right)^p\right). \qquad (29)$$

is valid.

Proof. To cover a closed domain $\overline{D}$ with a finite number $N_2(\rho,n,\alpha,D)$ of ellipsoids $\mathcal{E}_R^{x^\nu}\left(1+\frac{r}{2}+\frac{r^2}{64}\right)$ so that $\mathcal{E}_R^{x^\nu}(1+r)\subset D$; $\nu=1,\ldots,N_2$ (here, obviously, $R=R(\rho,n,\alpha)$), and then at each ellipsoid to apply lemma 9 is subject for the proof.

Lemma 11. *If the conditions (3) and (5) are fulfilled with respect to the coefficients of the operator $\mathcal{L}$, then for any function $u(x)\in \dot{W}^p_{2,\alpha}(D)$ for any $p\in\left(1,\frac{n}{2}\right]$, $\varepsilon>0$ and $\rho>0$ the estimate*

$$\|u\|^p_{W_2^p(D\setminus D_\rho)} \le C_{34}(n,p,\alpha,\gamma,\omega,b_0,\rho,D)\left(\int_D |\mathcal{L}u|^p dx + \left(\sup_D |u|\right)^p\right). \tag{30}$$

is valid.

Proof. Since $\partial D \in C^2$, then by [1] for sufficiently small ρ, that satisfy the condition $(D \setminus D_\rho) \cap \mathcal{E}_\rho^0(1) = \varnothing$, the inequality

$$\|u\|^p_{W_2^p(D\setminus D_\rho)} \le C_{34}(n,p,\alpha,\gamma,\omega,b_0,\rho,D)\left(\int_D |\mathcal{L}u|^p dx + \int_D |u|^p dx\right).$$

holds.

Now to complete the proof it is sufficient to consider that $\|u\|^p_{W^p_{2,\alpha}(D\setminus D_\rho)} \le$ $\le C_{35}(n,p,\alpha)\|u\|_{W_2^p(D\setminus D_\rho)}$ and the estimate $\int_D |u|^p dx \le mesD\left(\sup_D |u|\right)^p$.

Lemma 12. *If the conditions (3) and (5) are fulfilled with respect to the coefficients of the operator $\mathcal{L}$, then for any function $u(x) \in \dot{W}^p_{2,\alpha}(D)$ the estimate*

$$\|u\|_{W^p_{2,\alpha}(D)} \le C_{36}(n,p,\alpha,\gamma,\omega,b_0,D)\left(\|\mathcal{L}u\|_{Lp(D)} + \sup_D |u|\right)$$

is valid.

Proof. Fix a sufficiently small $\rho > 0$ and sum the inequalities (29) and (30). We get

$$\|u\|^p_{W^p_{2,\alpha}(D)} \le (C_{32} + C_{33})\int_D |\mathcal{L}u|^p dx + C_{32}\varepsilon\|u\|^p_{W^p_{2,\alpha}(D)} + \left(\frac{C_{32}}{\varepsilon} + C_{33}\right)\left(\sup_D |u|\right)^p.$$

To complete the proof it is sufficient to choose and fix $\varepsilon = \frac{1}{2C_{32}}$.

Theorem 1. *If the conditions (3)-(5) are fulfilled with respect to the coefficients of the operator $\mathcal{L}$, then for any function $u(x) \in \dot{W}^p_{2,\alpha}(D)$ for any $q \in \left(\frac{n(2+\alpha^+)}{2-\alpha^+}, \infty\right)$ the estimate*

$$\|u\|_{W^p_{2,\alpha}(D)} \le C_{37}(n,p,q,\alpha,\gamma,\omega,b_0,D)\|\mathcal{L}u\|_{Lq(D)} \tag{31}$$

is valid.

Proof. By A.D.Aleksandrov's estimate [12]

$$\sup_D |u| \le C_{38}(n,\alpha,D)\left(\int_D \frac{|\mathcal{L}u|^n}{\det\|a_{ij}(x)\|} dx\right)^{\frac{1}{n}}. \tag{32}$$

But on the other hand

$$\det\|a_{ij}(x)\| \ge C_{39}(n,\alpha,\gamma)\prod_{i=1}^n \lambda_i(x) = C_{39}\prod_{i=1}^n (|x|_\alpha)^{\alpha_i} \ge C_{39}\prod_{i=1}^n |x_i|^{\frac{2\alpha_i}{2+\alpha_i}}.$$

Therefore we deduce from (32)

$$\sup_D |u| \le C_{40}(n,\alpha,\gamma,D)\|\mathcal{L}u\|_{Lq(D)}\left(\int_D \prod_{i=1}^n |x_i|^{-\frac{2\alpha_i}{2+\alpha_i}\frac{q}{q-n}} dx\right)^{\frac{q-n}{nq}}.$$

The integral at the right hand side of the last inequality is finite, if for $i = 1,...,n$

$$\frac{2\alpha_i}{2+\alpha_i}\frac{q}{q-n} < 1, \quad \text{i.e.} \quad \frac{2\alpha^+}{2+\alpha^+} < \frac{q-n}{q}.$$

But this condition is satisfied by virtue of that $q > \frac{n(2+\alpha^+)}{2-\alpha^+}$. Thus

$$\sup_D |u| \le C_{41}(n,\alpha,\gamma,q,D)\|\mathcal{L}u\|_{Lq(D)}. \tag{33}$$

Now it is sufficiently to consider

$$\|\mathcal{L}u\|_{Lp(D)} \le (mesD)^{\frac{q-p}{qp}}\|\mathcal{L}u\|_{Lq(D)},$$

and the required estimate (31) follows from lemma 12 and the inequality (33).

4^0. Strong solvability of the first boundary value problem.

Theorem 2. *Let in the bounded domain* $D \subset E_n, n \ge 3, 0 \in D, \partial D \in C^2$ *the coefficients of the operator* $\mathcal{L}$ *satisfying the conditions (3)-(5) be determined. Then the first boundary value problem (1)-(2) is uniquely strongly solvable in* $\dot{W}^p_{2,\alpha}(D)$ *for* $p \in \left(1, \frac{n}{2}\right]$ *for any* $f(x) \in L_q(D)$, $q \in \left(\frac{n(2+\alpha^+)}{2-\alpha^-}, \infty\right)$. *Moreover for the solution* $u(x)$ *of the problem (1)-(2) the estimate*

$$\|u\|_{W^p_{2,\alpha}(D)} \le C_{37}\|f\|_{Lq(D)} \tag{34}$$

is valid.

Proof. Put for $\delta > 0$ $\quad T_\delta = \{x : |x|_\alpha < \delta\}$. Not losing generality, we'll assume that $\overline{T_1} \subset D$. Let for natural m

$$\lambda_i^m(x) = \begin{cases} \lambda_i(x), \text{ if } x \in D \setminus T_{\frac{1}{m}}; \\ m^{-\alpha_i}, \text{ if } x \in T_{\frac{1}{m}}; \end{cases}$$

$a_{ij}^m(x) = \sqrt{\lambda_i^m(x)\lambda_j^m(x)}\, b_{ij}(x)$; $i, j = 1,...,n$. From the condition (13) it follows that for $x \in D, \xi \in E_n$

$$\gamma\sum_{i=1}^n \lambda_i^m(x)\xi_i^2 \le \sum_{i,j=1}^n a_{ij}^m(x)\xi_i\xi_j \le \gamma^{-1}\sum_{i=1}^n \lambda_i^m(x)\xi_i^2 .$$

Now introduce in spaces $W^p_{2,\alpha^m}(D)$ and $\dot{W}^p_{2,\alpha^m}(D)$ by analogy with spaces $W^p_{2,\alpha}(D)$ and $\dot{W}^p_{2,\alpha}(D)$ by a change of functions $\lambda_i(x)$ on $\lambda_i^m(x)$. By means of arguments fully analogously to that of applied in the proof of lemma 1, we can show that for any function $u(x) \in \dot{W}^p_{2,\alpha^m}(D)$, the inequality

$$\|u\|_{W^p_{2,\alpha^m}(D)} \le C_{37}\|\mathcal{L}^m u\|_{Lq(D)}, \tag{35}$$

where $p \in \left(1, \frac{n}{2}\right]$, $q \in \left(\frac{n(2+\alpha^+)}{2-\alpha^+}, \infty\right)$, $\mathcal{L}^m = \sum_{i,j=1}^m a_{ij}^m(x)\frac{\partial^2}{\partial x_i \partial x_j}$; $m = 1,2,...$ is fulfilled.

For natural m consider a family of Dirichlet problems

$$\mathcal{L}^m u^m = f(x),\ \ x \in D\,;\ u^m\big|_{\partial D} = 0\,. \tag{36}$$

Since for each natural m an operator $\mathcal{L}^m$ is not degenerated, then for each $f(x) \in L_q(D)$ the boundary value problem (36) by [11] has a unique strong solution $u_m(x) \in \dot{W}_2^p(D)$. But $\dot{W}_2^p(D) \subset \dot{W}^p_{2,\alpha^m}(D) \subset \dot{W}^p_{2,\alpha}(D)$, therefore we get from (35)

$$\left\|u^m\right\|_{W^p_{2,\alpha}(D)} \le \left\|u^m\right\|_{W^p_{2,\alpha^m}(D)} \le C_{37}\left\|\mathcal{L}^m u^m\right\|_{Lq(D)} = C_{37}\|f\|_{Lq(D)}.$$

Thus, the family $\{u^m(x)\}$ is strongly bounded in $\dot{W}^p_{2,\alpha}(D)$, and according to [13] it is weakly compact in this space. Hence, in particular, it follows that there exist such a function $u(x) \in \dot{W}^p_{2,\alpha}(D)$ and subsequence of natural numbers $m_k \to \infty$ for $k \to \infty$, that for any function $\varphi(x) \in C_0^\infty(D)$

$$\lim_{k\to\infty}\left(\mathcal{L}u^{m_k},\varphi\right) = (\mathcal{L}u,\varphi), \tag{37}$$

where $(\psi,\varphi) = \int\limits_D \psi(x)\varphi(x)dx$.

But on the other hand

$$\left(\mathcal{L}u^{m_k},\varphi\right) = \left(\left(\mathcal{L} - \mathcal{L}^{m_k}\right)u^{m_k},\varphi\right) + \left(\mathcal{L}^{m_k}u^{m_k},\varphi\right) = \left(\left(\mathcal{L} - \mathcal{L}^{m_k}\right)u^{m_k},\varphi\right) + (f,\varphi) \tag{38}$$

and further

$$\left|\left(\mathcal{L} - \mathcal{L}^{m_k}\right)u^{m_k},\varphi\right| \le C_{42}(n,\gamma,b_0)\int\limits_{T_{\frac{1}{m_k}}}\sum_{i,j=1}^{n}\sqrt{\lambda_i^{m_k}(x)\lambda_j^{m_k}(x)}\left|u_{ij}^{m_k}\right||\varphi|dx \le$$

$$\le C_{42}\left\|u^{m_k}\right\|_{W^p_{2,\alpha^{m_k}}(D)}\|\varphi\|_{L_{\frac{p}{p-1}}\left(T_{\frac{1}{m_k}}\right)} \le C_{42}\|f\|_{Lq(D)}\|\varphi\|_{L_{\frac{p}{p-1}}\left(T_{m_k}\right)} \to 0\ (k\to\infty). \tag{39}$$

We deduce from (37)-(39) that $(\mathcal{L}u,\varphi) = (f,\varphi)$ for any function $\varphi(x) \in C_0^\infty(D)$. Thus $\mathcal{L}u = f(x)$ a.e. in D, i.e. the function $u(x)$ is a strong solution of the boundary value problem (1)-(2). The estimate (34) and the uniqueness of the solution follow from the inequality (31). The theorem is proved.

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