ISMAILOV M.I.

THE INVERSE SCATTERING PROBLEM FOR THE FIRST ORDER SYMMETRIC HYPERBOLIC EQUATIONS SYSTEM ON THE WHOLE AXIS

Abstract

The scattering operators for the first order symmetric hyperbolic equations system on the whole axis are determined, and unique determination of coefficients of this system by the scattering operator is studied. Minimal information for solving the inverse scattering problem is given.

Consider a non-stationary system of the n-th order equations of the form

$$\frac{\partial}{\partial t}\psi(x,t)-\sigma\frac{\partial}{\partial x}\psi(x,t)=Q(x,t)\psi(x,t)\ (-\infty<x,\ t<+\infty), \tag{1}$$

where

$$\sigma=\begin{pmatrix}\xi_1 I_{n_1} & 0\\ 0 & \xi_2 I_{n_2}\end{pmatrix},\ Q(x,t)=\begin{pmatrix}0 & q_{12}(x,t)\\ q_{21}(x,t) & 0\end{pmatrix},$$

I_{n_i} $(i=1,2)$ are $n_i\times n_i$ unit matrices, $n_1+n_2=n$, $\xi_1>0>\xi_2, q_{ij}(x,t)\ (i\neq j=1,2)$ are $n_i\times n_j$ matrix functions with measurable complex valued elements whose Euclidean norms satisfy the inequalities

$$\|q_{ij}(x,t)\|\le c(1+|t|)^{-1-\varepsilon}(1+|x|)^{-1-\varepsilon},\ i\neq j=1,2, \tag{2}$$

where ε and C are positive numbers.

For the case $n_1=n_2=1$ the inverse scattering problem for a non-stationary system of Dirac equations was studied in papers [2], [3], in the case when $n_1=n_2>1$ - in [5].

Scattering problem. Let in the system (1) $Q(x,t)\equiv 0$ then one can easily verify that the vector-function

$$\psi(x,t)=\{\varphi_1(t+\xi_1 x),\dots,\varphi_{n_1}(t+\xi_1 x),\varphi_{n_1+1}(t+\xi_2 x),\dots,\varphi_n(t+\xi_2 x)\}=F_t\varphi(x),$$

where $\varphi_i(x)\ (i=\overline{1,n})$ are arbitrary locally integrable functions, in a generalized sense satisfies the non-perturbed system (1).

Under the solution of the system (1) we shall understand any function $\psi(x,t)\in L_\infty(R^2,C^n)$ satisfying the system (1) in a generalized sense.

Theorem 1. *For the equations system (1) satisfying the condition (1) the following statements are valid:*

1) *For any functions $a(x)\in L_\infty(R,C^n)$ there exists such a unique solution $\psi(x,t)\in L_\infty(R^2,C^n)$ of the system (1) that*

$$\vartheta\operatorname{raimax}_x|\psi(x,t)-F_t a(x)|\to 0\ for\ t\to-\infty$$

(the modulus sign here means the norm in C^n).

2) *For any solutions $\psi(x,t)\in L_\infty(R^2,C^n)$ of the system (1) there exists such a unique solution $F_t a(x)$ and $F_t b(x)\in L_\infty(R^2,C^n)$ of the non-perturbed system that*

$$\vartheta\operatorname{raimax}_x|\psi(x,t)-F_t a(x)|\to 0\ for\ t\to-\infty,$$

$$\vartheta \text{ raimax}_x \left|\psi(x,t) - F_t b(x)\right| \to 0 \text{ for } t \to +\infty .$$

The proof of the theorem is similar to the method of papers [2], [3].

By virtue of theorem 1 the Cauchy problem with data of infinity is identically solvable in a class of essentially bounded functions. This admits to determine the scattering as the operator transferring $F_t a(x)$ to $F_t b(x)$. More precisely, identifying the solutions $F_t a(x)$ and $F_t b(x)$ with initial data $a(x)$ and $b(x)$ for $t = 0$, we determine the scattering operator S (in correspondence with general definition) as the operator

$$S : a(x) \to b(x). \tag{3}$$

The operator S is a matrix and linear bounded operator in the space $L_\infty(R, C^n)$. Further, the operator S will be studied in the space $L_2(R, C^n)$, i.e. under S we'll understand the closure of the operator S, contracted in $L_2(R, C^n) \cap L_\infty(R, C^n)$.

The deeper properties of the scattering operator S will be studied below.

Transformation operators. Under the solution of the inverse scattering problem, i.e. the problem on restoration of the coefficients of the equation known by the scattering operator the integral representation of Volterra type solutions plays a great role. The following lemma is valid for the system (1).

Lemma 1. *Any bounded solution $\psi(x,t)$ of the system (1) with condition (2) admits the representation*

$$\psi(x,t) = F_t f^+(x) + \int_{-\infty}^{x} A^+(x,t,s) F_t f^+(s) ds , \tag{4}$$

$$\psi(x,t) = F_t f^-(x) + \int_{x}^{+\infty} A^-(x,t,s) F_t f^-(s) ds , \tag{5}$$

where

$$F_t f^+(x) = \left(f_1^\pm(x + \xi_1 t), f_2^\pm(x + \xi_2 t)\right), f_1^\pm(x) \in L_\infty\left(R, C^{n_1}\right), f_2^\pm(x) \in L_\infty\left(R, C^{n_2}\right);$$

$$A^\pm(x,t,s) = \left(A_{ij}^\pm(x,t,s)\right)_{i,j=1}^2 ,$$

$A_{ij}^\pm(x,t,s)$ $(i,j = 1,2)$ are $n_i \times n_j$ matrix functions. The kernels $A_{ij}^\pm(x,t,s)$ $(i,j = 1,2)$ are identically defined by the coefficients of the system (1), and for the fixed x summed with a square on t and s, i.e. they are Hilbert-Schmidt kernels.

The coefficients $q_{ij}(x,t)$ $(i \neq j = 1,2)$ of the system (1) are expressed by the kernel of transformation operators according to the formula

$$\begin{aligned} q_{12}(x,t) &= \pm(\xi_2 - \xi_1) A_{12}^\pm(x,t,x), \\ q_{21}(x,t) &= \pm(\xi_1 - \xi_2) A_{21}^\pm(x,t,x). \end{aligned} \tag{6}$$

Proof. Prove the representation (4). The bounded solution $\psi(x,t) = (\psi_1(x,t), \psi_2(x,t))$ of the system (1) with given asymptotics $F_t f^+(t)$ for $x \to -\infty$ satisfy the integral equations system

$$\begin{aligned} \psi_1(x,t) &= f_1^+(x + \xi_1 t) - \frac{1}{\xi_1} \int_{-\infty}^{x} (q_{12}\psi_2)\left(s, t + \frac{1}{\xi_1}(x - s)\right) ds, \\ \psi_2(x,t) &= f_2^+(x + \xi_2 t) - \frac{1}{\xi_2} \int_{-\infty}^{x} (q_{21}\psi_2)\left(s, t + \frac{1}{\xi_2}(x - s)\right) ds . \end{aligned} \tag{7}$$

If the solution of equations (7) is representable in the form of (4) for any $f_1^+(x)\in L_\infty(R,C^{n_1})$, $f_2^+(x)\in L_\infty(R,C^{n_2})$ then substituting (4) in (7) we get the equation system for the kernels

$$A_{11}^+(x,t,s)=-\int_t^{+\infty} q_{12}(x+\xi_1(t-\tau),\tau)A_{21}^+(x+\xi_1(t-\tau),\tau,s+\xi_1(t-\tau))d\tau\,,$$

$$A_{12}^+(x,t,s)=\frac{1}{\xi_2-\xi_1}q_{12}\left(x+\frac{\xi_1}{\xi_2-\xi_1}(x-s),t+\frac{1}{\xi_2-\xi_1}(s-x)\right)-$$

$$-\int_t^{t+\frac{x-s}{\xi_1-\xi_2}} q_{12}(x+\xi_1(t-\tau),\tau)A_{22}^+(x+\xi_1(t-\tau),\tau,s+\xi_2(t-\tau))d\tau,$$

$$A_{21}^-(x,t,s)=\frac{1}{\xi_1-\xi_2}q_{21}\left(x+\frac{\xi_2}{\xi_1-\xi_2}(x-s),t+\frac{1}{\xi_1-\xi_2}(s-x)\right)+$$

$$-\int_{t+\frac{s-x}{\xi_1-\xi_2}}^{t} q_{21}(x+\xi_2(t-\tau),\tau)\cdot A_{11}^+(x+\xi_2(t-\tau),\tau,s+\xi_1(t-\tau))d\tau,$$

$$A_{22}^-(x,t,s)=\int_{-\infty}^{t} q_{21}(x+\xi_2(t-\tau),\tau)A_{12}^+(x+\xi_2(t-\tau),\tau,s+\xi_2(t-\tau))d\tau\,,\ s\le x. \tag{8}$$

On the contrary, if the kernels $A_{ij}^+(x,t,s)$ satisfy the system (1), then the representation (4) gives the bounded solution for any $f_1^+(x)\in L_\infty(R,C^{n_1})$, $f_2^+(x)\in L_\infty(R,C^{n_2})$.

Thus, for proving the representation (4) it is sufficient to prove that the equations system (8) has a solution. This follows from the Volterra property of these systems on the basis of Theorem 4.1.1 [1].

The equality (5) directly follows from the system (8) for $s=x$. The proof of representations (5) is completely similar to the proof of the representation (4).

We can write the formulas (4) and (5) in the operator form

$$\psi(x,t)=(I+A^+(t))F_t f^+(x), \tag{4'}$$

$$\psi(x,t)=(I+A^-(t))F_t f^-(x), \tag{5'}$$

where indices "+" and "–" mean the polarity of volterrian integral operator.

It follows from lemma 1 that between the preimages (between $f^+(x)$ and $f^-(x)$ of the same solution of the equations system (1) there exists one-to one correspondence).

Thus, there exists a jump operator connecting the preimages of the same solution

$$\widetilde{S}: f^+(x)\to f^-(x). \tag{9}$$

This operator is a matrix operator in $L_2(R,C^n)$ and from (4') and (5') we see that the operator $F_t\widetilde{S}F_t^{-1}$ admits the factorization on the Hilbert-Schmidt's matrix volterrian integral operators, i.e.

$$F_t\widetilde{S}F_t^{-1}=(I+A^-(t))^{-1}(I+A^+(t)). \tag{10}$$

The structure of the operators S and $\widetilde{S}$ is studied in the following

Lemma 2. *For operators S and $\widetilde{S}$ the following representations are valid*

$$S=\begin{pmatrix} I_{n_1} & 0 \\ R_1 & I_{n_2}+R_{1+}\end{pmatrix}\begin{pmatrix} I_{n_1}+R_{2+} & R_2 \\ 0 & I_{n_2}\end{pmatrix}^{-1}=$$

$$=\begin{pmatrix} I_{n_1}+R_{2-} & R_3 \\ 0 & I_{n_2}\end{pmatrix}\begin{pmatrix} I_{n_1} & 0 \\ R_4 & I_{n_2}+R_{1-}\end{pmatrix}^{-1}, \qquad (11)$$

$$\widetilde{S}=\begin{pmatrix} I_{n_1} & 0 \\ R_4 & I_{n_2}+R_{1-}\end{pmatrix}^{-1}\begin{pmatrix} I_{n_1}+R_{2+} & R_2 \\ 0 & I_{n_2}\end{pmatrix}=$$

$$=\begin{pmatrix} I_{n_1}+R_{2-} & R_3 \\ 0 & I_{n_2}\end{pmatrix}^{-1}\begin{pmatrix} I_{n_1} & 0 \\ R_1 & I_{n_2}+R_{1+}\end{pmatrix}, \qquad (12)$$

where R_1, R_4 and R_2, R_3 are $n_2\times n_1$ and $n_1\times n_2$ Hilbert-Schmidt's matrix integral operators respectively, but R_{1+}, R_{1-} and R_{2+}, R_{2-} are respective $n_2\times n_2$ and $n_1\times n_1$ Hilbert-Schmidt's matrix volterrian operators of corresponding polarity.

Proof. Substituting the integral representation (4) into the system of integral equations of the scattering problem satisfying the asymptotics $F_t a(t)=(a_1(x+\xi_1 t), a_2(t+\xi_2 t))$, i.e. in the system

$$\psi_1(x,t)=a_1(x+\xi_1 t)+\int_{-\infty}^{t}(q_{12}\psi_2)(s+\xi_1(t-s),s)ds,$$

$$\psi_2(x,t)=a_2(t+\xi_2 t)+\int_{-\infty}^{t}(q_{21}\psi_1)(x+\xi_2(t-s),s)ds$$

subject to the equations for the kernels $A_{ij}^{+}(x,t,s)$ we obtain

$$\begin{pmatrix} a_1(x+\xi_1 t) \\ a_2(x+\xi_2 t)\end{pmatrix}=\begin{pmatrix} I_{n_1}+R_{2+}(t) & R_2(t) \\ 0 & I_{n_2}\end{pmatrix}\begin{pmatrix} f_1^{+}(x+\xi_1 t) \\ f_2^{+}(x+\xi_2 t)\end{pmatrix}, \qquad (13)$$

where $R_{2+}(t)$ and $R_2(t)$ are respective $n_1\times n_1$ and $n_1\times n_2$ matrix integral operators of the form

$$R_{2+}(t)f(x)=\int_{-\infty}^{x}R_{2+}(x,t,s)f(s)ds\,,,$$

where

$$R_{2+}(x,t,s)=-\int_{-\infty}^{+\infty}q_{12}(x+\xi_1(t-\tau),\tau)A_{21}^{+}(x+\xi_1(t-\tau),\tau,s+\xi_1(t-\tau))d\tau;$$

$$R_2(t)f(x)=\int_{-\infty}^{+\infty}R_2(x,t,s)f(s)ds,$$

where

$$R_2(x,t,s)=\frac{1}{\xi_2-\xi_1}q_{12}\left(x+\frac{\xi_1}{\xi_2-\xi_1}(x-s),t+\frac{1}{\xi_2-\xi_1}(s-x)\right)-$$

$$-\int_{-\infty}^{t+\frac{s-x}{\xi_2-\xi_1}}q_{12}(x+\xi_1(t-\tau),\tau)\cdot A_{22}^{+}(x+\xi_1(t-\tau),\tau,s-\xi_2(t-\tau))ds\,.$$

The following equalities are obtained analogously

$$F_t b(x)=\begin{pmatrix} I_{n_1} & 0 \\ R_1(t) & I_{n_2}+R_{1+}(t)\end{pmatrix}\cdot F_t f^{+}(x), \qquad (14)$$

$$F_t a(x)=\begin{pmatrix} I_{n_1} & 0 \\ R_4(t) & I_{n_2}+R_{1-}(t)\end{pmatrix}\cdot F_t f^-(x), \qquad (15)$$

$$F_t b(x)=\begin{pmatrix} I_{n_1}+R_{2-}(t) & R_3(t) \\ 0 & I_{n_2}\end{pmatrix}\cdot F_t f^-(x). \qquad (16)$$

The validity of formulas (11) and (12) yields from the equalities (13)-(16).

The next corollary follows from lemma 2.

Corollary. *The operators* S *and* $\widetilde{S}$ *are identically connected between themselves.*

The inverse scattering problem. The inverse scattering problem for the equation system (1) is in the finding of the matrix potential $Q(x,t)$ by the known scattering operator S.

Theorem 2. *Let* S *be a scattering operator for the differential equations system (1) with the potential* $Q(x,t)$ *satisfying the estimate (2). Then the potential* $Q(x,t)$ *is identically determined by the known scattering operator* S.

Proof. Let the potential $Q_1(x,t)$ correspond to the scattering operator S_1, and the potential $Q_2(x,t)$ - to the operator S_2. Show that if $S_1=S_2$, then $Q_1(x,t)=Q_2(x,t)$. By lemma 2 the scattering operator S_1 and S_2 admit the structure (11), i.e.

$$S_k=\begin{pmatrix} I_{n_1} & 0 \\ R_1^k & I_{n_2}+R_{1+}^k\end{pmatrix}\begin{pmatrix} I_{n_1}+R_{2+}^k & R_2^k \\ 0 & I\end{pmatrix}^{-1}=$$

$$=\begin{pmatrix} I_{n_1}+R_{2-}^k & R_3^k \\ 0 & I_{n_2}\end{pmatrix}\begin{pmatrix} I_{n_1} & 0 \\ R_4^k & I+R_{1-}^k\end{pmatrix}^{-1}, \; k=1,2 .$$

Since these representations are unique, and $S_1=S_2$, then $R_i^1=R_i^2\ \left(i=\overline{1,4}\right)$, $R_{i+}^1=R_{i+}^2, R_{i-}^1=R_{i-}^2\ (i=1,2)$. Therefore according to (12) the operators $\widetilde{S}^k$, $k=1,2$ also coincide.

By virtue of the uniqueness of factorization on volterrian multipliers, factorizational multipliers coincide in factorizations

$$F_t\widetilde{S}^k F_t^{-1}=\left(I+A_k^-(t)\right)^{-1}\left(I+A_k^+(t)\right),\ k=1,2 .$$

Hence $Q_k(x,t)$, $k=1,2$ obtained by these factorizational multipliers according to formulas (6) also coincide.

Thus, by the matrix kernel of the operator $F=S-I$, containing n^2 of given functions $F_{ij}(x,y)\left(i,j=\overline{1,n}\right)$ one can find the potential $Q(x,t)$ in the system (1) that contains $2n_1\cdot n_2$ desired functions, where $n_1+n_2=n$.

Under the scattering data we'll understand that minimal information, by which one can solve the inverse scattering problem. For a hyperbolic system of $n\geq 3$ equations ($\xi_1>\xi_2>...>\xi_n$) such minimal information is introduced in [4].

Definition. *The scattering data for the differential equations system (1) we'll call any of pairs from matrix functions* $\{R_1(x,y),R_3(x,y)\},\{R_2(x,y),R_4(x,y)\}$ *that are the kernels of the matrix integral operators* $\{R_1,R_3\},\{R_2,R_4\}$ *in representations (11) and (12).*

We can show that by these scattering data, for instance by $\{R_1, R_2\}$, the operators S and $\widetilde{S}$ are identically found.

The following equalities are obtained from the matrix equality (12)

$$(I_{n_1} + R_{2-})(I_{n_1} + R_{2+}) = I_{n_1} - R_3 \cdot R_1 , \tag{17}$$

$$(I_{n_2} + R_{1-})^{-1}(I_{n_2} + R_{1+})^{-1} = I_{n_2} + R_1(I_{n_1} - R_3 R_1)^{-1} R_3 , \tag{18}$$

$$R_2 = -(I_{n_1} + R_{2-})^{-1} R_3 (I_{n_2} + R_{1+}), \tag{19}$$

$$R_4 = -(I_{n_2} + R_{1-}) R_1 (I_{n_1} + R_{2+})^{-1}. \tag{20}$$

Really, if the integral operators $\{R_1, R_3\}$ are known, from the factorizational equalities (17) and (18) we can find factorizational multiplies, i.e. matrix operators $R_{2\pm}$ and $R_{1\pm}$. The problem on finding these multipliers has been studied well and it is reduced to the solution of Gelfand-Levitan-Marchenko type integral equations system (see [1]).

The equalities (19) and (20) make possible to determine the operators R_2 and R_4.

Thus, by pair of operators $\{R_1, R_3\}$ all other operators are determined in representations (11) and (12) for the operators S and $\widetilde{S}$.

The solution of the inverse scattering problem for the differential equations system (1) with a potential satisfying the estimate (2) is unique. The solution algorithm of the problem is that the operator $\widetilde{S}$ structured on the scattering data, according to (17)-(20) admits the factorization (10), on whose factorizational multipliers the potential is found by the formula (6).

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Ismailov M.I.
Institute of Mathematics and Mechanics of AS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

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