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ON THE MAXIMUM PRINCIPLE FOR ELLIPTIC EQUATIONS OF THE SECOND ORDER WITH NON-NEGATIVE CHARACTERISTIC FORM

Abstract

For the generalized solution of the non-uniform degeneralized elliptic equation of the second order in the divergent form the estimation in the norm of the space CD is proved trough the weight Lebeg norm of the right-hand side of the equation.

In the paper the maximum principle of one class of linear elliptic equations of the second order with the nonnegative characteristic form. Earlier this maximum principle was obtained in [1], [2], [3] for the linear elliptic uniformly equations.

Let us consider in the bounded domain $D \subset R^n$ $(n \geq 3)$ the following equation of the form:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = F(x). \tag{1}$$

We will suppose with respect to the coefficient of this equation that the following condition is fulfilled:

$$\varphi(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \psi(x)|\xi|^2, \tag{2}$$

for $\forall \xi = (\xi_1, \xi_2, \dots, \xi_n) \in R^n$, $x = (x_1, x_2, \dots, x_n)$, where $\varphi(x)$ and $\psi(x)$ are the nonnegative measurable functions determined in D and subjected to determination: for any subdomain $E \subset D$ with the smooth bound such that $\partial E \cap \partial D = \varnothing$ there exist the positive constants C and α such that it is fulfilled the condition

$$\int_{\partial E} \varphi(y) d_y s \geq C \left(\int_E \psi(x) dx \right)^{\alpha}. \tag{3}$$

Moreover $\frac{1}{2} < \alpha < 1$. We will not give the condition of smoothness for function $\varphi(x)$, supposing that its trace on the smooth surface ∂E exists and it is summable. It is clear, that for $\varphi(x) = \psi(x) = const$ the condition (3) is fulfilled automatically. Indeed, in this case condition (3) turns over the isoperimetric inequality, which is valid for any set E. And $\alpha = \frac{n-1}{n}$.

We will suppose with respect to $F(x)$ that it is fulfilled the condition

$$\varphi^{-\frac{p-1}{p}}(x) F(x) \in L_p(D),\ p > 1. \tag{4}$$

Denote by $\overset{\circ}{W}{}^{1,2}_{\varphi,\psi}(D)$ - the space obtained by adding the set of functions $g(x) \in C^{0,1}(D)$ by the

$$\|g\| = \left(\int_D \varphi(x) g^2(x) + \psi(x) |\nabla g(x)|^2 dx \right)^{\frac{1}{2}}.$$

We will understand under the solution of equation (1) the function $u(x) \in \overset{\circ}{W}{}^{1,2}_{\varphi,\psi}(D)$ satisfying the integral identity

$$-\int_D \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial g}{\partial x_j} dx = \int_D g(x) F(x) dx, \tag{5}$$

where $g(x)$ is any function from $\overset{\circ}{W}{}^{1,2}_{\varphi,\psi}(D)$.

Theorem (maximum principle). *Let the equation (1) be given in the bounded domain $D \subset R^n$ $(n \geq 3)$. Let in D the conditions (2), (3) and (4) be fulfilled. Then for any solution of equation (1) $u(x) \in \overset{\circ}{W}{}^{1,2}_{\varphi,\psi}(D)$ the following inequality has place*

$$\|u\|_{C(D)} \leq A \left\| \varphi^{-\frac{p-1}{p}} F \right\|_{L_p(D)} \quad for \quad P > \frac{1}{2(1-\alpha)},$$

where α is the constant of inequality (3), $A > 0$ is the constant depending only on n, $mes_n D$ and the coefficients of equation (1).

We will need the following lemmas for proofing the theorem.

Lemma 1. *Let $u(x) \in \overset{\circ}{W}{}^{1,2}_{\varphi,\psi}(D)$ and in D the condition (2) be fulfilled. Let D_t be the set of points $x \in D$ for which $u(x) > t > 0$ (t is a number) and $E_t = \{x \in D,\ u(x) = t\}$. Let*

$$W(t) = \int_{D_t} \sum_{i,j=1}^{n} a_{ij} \frac{u_{x_i}}{|\nabla u|} \frac{u_{x_j}}{|\nabla u|} dx \quad and \quad V(t) = \int_{D_t} \varphi(x) dx.$$

Then almost for all t the inequality takes place:

$$W'(t) \geq V'(t).$$

Proof. It is known that if $\Phi(x)$ is the function measured in D and $u(x)$ has partial derivatives of the first order by all x_i then the following formula is valid:

$$\int_D \Phi(x) |\nabla u(x)| dx = \int_{\inf u}^{\sup u} \left(\int_{E_\tau} \Phi ds \right) d\tau. \tag{6}$$

Let $D = D_t$, $E_t = \{x \in D,\ u(x) = t\}$ then by (6) we have

$$W(t) = \int_{D_t} \sum_{i,j=1}^{n} a_{ij} \frac{u_{x_i}}{|\nabla u|} \frac{u_{x_j}}{|\nabla u|} dx = \int_t^{\sup u} \left(\int_{E_\tau} |\nabla u|^{-1} \sum_{i,j=1}^{n} \right), \tag{7}$$

$$V(t) = \int_{D_t} \varphi(x) dx = \int_{D_t} \frac{\psi(x)}{|\nabla u|} |\nabla u| dx = \int_t^{\sup u} \left(\int_{E_\tau} \frac{\varphi(y) d_y s}{|\nabla u|} \right) d\tau. \tag{8}$$

The formulas (7) and (8) are valid for almost all t. So it follows from (7) and (8)

$$-W'(t) = \int_{E_t} \frac{\sum_{i,j=1}^{n} a_{ij} \cos\alpha_i \cos\alpha_j}{|\nabla u|} ds \tag{9}$$

and

$$-V'(t)=\int_{E_t}\frac{\varphi(y)}{|\nabla u|}d_y s\,. \qquad (10)$$

By virtue of the condition (2) from the equalities (9) and (10) we have

$$-W'(t)\le\int_{E_t}\frac{\varphi(y)d_y s}{|\nabla u|}=-V'(t)$$

i.e. $W'(t)\ge V'(t)$. Lemma has been proved.

Lemma 2. *Let* $u(x)\in \overset{\circ}{W}{}^{1,2}_{\varphi,\psi}(D)$ *and* D_t, E_t *have the same sense which was in Lemma 1. The inequality is valid*

$$\int_{E_t}|\nabla u|\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j\ge\frac{C_1\left(\int_{D_t}\psi(x)dx\right)^{2\alpha}}{-V'(t)},$$

where $C_1=C^2$ *is the constant,* $\cos\alpha_i$ *are the direction cosines of the gradient of function* $u(x)$.

Proof. It is clear, that

$$\int_{E_t}\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds=\int_{E_t}\frac{\left(\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j\right)^{\frac{1}{2}}}{|\nabla u|^{1/2}}\frac{|\nabla u|^{1/2}\left(\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j\right)^{\frac{1}{2}}}{1}ds\,.$$

Using Hölder's inequality we obtain

$$\int_{E_t}\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds\le\left(\int_{E_t}\frac{\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j}{|\nabla u|}ds\right)^{1/2}\left(\int_{E_t}|\nabla u|\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds\right)^{1/2}.$$

Hence we have

$$\int_{E_t}|\nabla u|\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds\ge\frac{\left(\int_{E_t}\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds\right)^2}{\int_{E_t}\frac{\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds}{|\nabla u|}}.$$

From (6) it is obtained that

$$\int_{D_t}\sum_{i,j=1}^{n}a_{ij}(x)\frac{u_{x_i}}{|\nabla u|}\frac{u_{x_j}}{|\nabla u|}dx=\int_{D_t}\frac{|\nabla u|}{|\nabla u|}\sum_{i,j=1}^{n}a_{ij}(x)\frac{u_{x_i}}{|\nabla u|}\frac{u_{x_j}}{|\nabla u|}dx=\int_t^{\sup u}\left(\int_{E_\tau}|\nabla u|\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds\right)d\tau$$

i.e.

$$W(t)=\int_t^{\sup u}\left(\int_{E_\tau}\frac{1}{|\nabla u|}\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds\right)d\tau\,.$$

Hence we find that

$$-W'(t)=\int_{E_\tau}\frac{1}{|\nabla u|}\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds.$$

By virtue of the last formula from (10) we obtain, that

$$\int_{E_\tau}|\nabla u|\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds\geq\frac{\left(\int_{E_t}\varphi(y)d_y s\right)^2}{-W'(t)}$$

hence by virtue of Lemma 1 and condition (3) we have

$$\int_{E_\tau}|\nabla u|\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds\geq\frac{C^2\left(\int_{D_t}\psi(x)dx\right)^2}{-V'(t)}=\frac{C^2V^{2\alpha}(t)}{-V'(t)}. \qquad (11)$$

Lemma 2 has been proved.

Now let's prove the theorem. Let's take as the test function $g(x)$

$$g(x)=\begin{cases}u(x)-t, & npu\quad u(x)>t\\ 0, & npu\quad u(x)\leq t.\end{cases}$$

Then from (5) we have

$$\int_{D_t}\sum_{i,j=1}^{n}a_{ij}(x)u_{x_i}u_{x_j}=\int_{D_t}(u(x)-t)F(x)dx. \qquad (12)$$

By virtue of (6) from (12) we have

$$\int_t^{\sup u}\left(\int_{E_\tau}\frac{\sum_{i,j=1}^{n}a_{ij}(x)u_{x_i}u_{x_j}}{|\nabla u|}ds\right)d\tau=\int_t^{\sup u}\left(\int_{E_\tau}\frac{(\tau-t)F}{|\nabla u|}ds\right)d\tau=\int_t(\tau-t)\left(\int_{E_\tau}F\frac{ds}{\nabla u}\right)d\tau.$$

Or

$$\int_t^{\sup u}\left(\int_{E_\tau}|\nabla u|\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds\right)d\tau=-\int_t^{\sup u}\left(\int_{E_\tau}F\frac{ds}{\nabla u}\right)d\tau.$$

Differentiating both sides of this equality we have

$$\int_{E_t}|\nabla u|\sum_{i,j=1}^{n}a_{ij}\cos\alpha_i\cos\alpha_j ds=-\int_{D_t}F(x)dx. \qquad (13)$$

On the base of Lemma 2 from equation (13) we obtain that

$$-\int_{D_t}F(x)dx\geq\frac{C^2V^{2\alpha}(t)}{-V'(t)}$$

or

$$\frac{-V'(t)}{C^2V^{2\alpha}(t)}\int_{D_t}|F(x)|dx\geq 1. \qquad (14)$$

Using Hölder's inequality we have

$$\int_{D_t}|F(x)|dx=\int_{D_t}\frac{|F(x)|}{\varphi^{\frac{p-1}{p}}(x)}\cdot\varphi^{\frac{p-1}{p}}(x)dx\le\left(\int_{Dt}\frac{|F(x)|^p}{\varphi^{p-1}(x)}\right)^{\frac{1}{p}}\left(\int_{Dt}\varphi(x)dx\right)^{\frac{1}{q}}\le$$

$$\le\left(\int_D\varphi^{-(p-1)}(x)|F(x)|^p dx\right)^{\frac{1}{p}}\cdot V^{\frac{1}{q}}(t),\tag{15}$$

$$\frac{1}{p}+\frac{1}{q}=1.$$

Further from (14) and (15) we have

$$-V'(t)V(t)^{\frac{1}{q}2\alpha}\cdot C^{-2}\left(\int_D\varphi^{-(p-1)}(x)|F(x)|^p dx\right)^{\frac{1}{p}}\ge 1.$$

Integrating the last inequality from zero up to $\sup u$ by t we obtain

$$\operatorname{Sup}u\le\frac{1}{C^2\left(\frac{1}{q}-2\alpha+1\right)}V^{\frac{1}{q}2\alpha+1}(o)\left(\int_D\varphi^{-(p-1)}|F|^p dx\right)^{\frac{1}{p}},$$

where $V(0)=\int_D\psi(x)dx$ and $q<\frac{1}{2\alpha-1}$ and $p>\frac{1}{2(1-\alpha)}\left(\frac{1}{2}<\alpha<1\right)$. It is proved by analogue that

$$\sup(-u)\le\frac{1}{C^2\left(\frac{1}{q}-2\alpha+1\right)}V^{\frac{1}{q}-2\alpha+1}(o)\left\|\varphi^{-\frac{(p-1)}{p}}F\right\|_{L_p(D)}$$

for

$$p>\frac{1}{2(1-\alpha)}.$$

So the theorem has been proved completely.

Let us note that for $\varphi(x)\cong\lambda_1>0$, $\psi(x)\cong\lambda_2>0$,, where $\lambda_1>0$, $\lambda_2>0$ $(\lambda_1<\lambda_2)$ the numbers from our theorem the earlier known results are obtained [1], [2], [3]. The analogous result is valid also for the complete equation of the form:

$$\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i}+C(x)u=F(x).$$

It's desired for the coefficients $b_i(x)$ and $C(x)$ some additional conditions of the form as their belongity to some weight functional spaces.

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