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ON THE EXISTENCE AND ASYMPTOTICS OF THE SOLUTIONS OF NEUMANN'S PROBLEM FOR POISSON EQUATION IN THE LAYER TYPE MULTIVARIATE DOMAINS

Abstract

Let $\Pi = R^n \times \Omega$, where $\Omega \in R^m$ is a bounded domain with Lipshits boundary. In the layer Π we consider the equation

$$\sum_{i=1}^{n+m} \frac{\partial^2 u}{\partial z_i^2} = \sum_{i=1}^{n+m} \frac{\partial f_i}{\partial z_i}.$$

We establish the theorems on existence, uniqueness and asymptotics of the generalised solution of the given equation in Π satisfying the boundary condition

$$\left. \frac{\partial u}{\partial \vec{u}} \right|_{\partial\Omega} = 0,$$

where $\vec{n}$ is the outward normal to $\partial\Omega$ in R^m.

Let's consider the layer $\Pi = \{z : z = (z_1, z_2, ..., z_n, z_{n+1}, ..., z_{n+m})\}$, $x = (x_1, x_2, ..., x_n) = (z_1, z_2, ..., z_n) \in R^n$, $y = (y_1, y_2, ..., y_m) = (z_{n+1}, ..., z_{n+m}) \in \Omega$, where Ω is the bounded Lipshits domain of the space R^m with the bound $\partial\Omega$.

1. let's consider in the domain $\Pi = R^n \times \Omega$ the equation

$$\Delta u = \sum_{i=1}^{n+m} \frac{\partial f_i}{\partial z_i}, \tag{1}$$

where $\Delta u = \sum_{i=1}^{n+m} \frac{\partial^2 u_i}{\partial z_i^2}$ is Laplacian operator. The generalised solution of equation (1) is considered which satisfies the boundary-valued condition

$$\left. \frac{\partial u}{\partial \vec{n}} \right|_{\partial\Omega} = 0. \tag{2}$$

It is supposed, that $f_i(z) \in L_{2,loc}(\Pi)$,

$$\sum_{i=n+1}^{n+m} \int_{\partial\Omega \times R^n} f_i(z) \cos(\vec{n}, z_i) dx d_s \Omega = 0. \tag{3}$$

By the generalised solution of the problem (1), (2), where $f_i(z)$ satisfy the condition (3) such $u(z) \in W_{2,loc}^1(\Pi)$ is named that

$$\int_\Pi \sum_{i+1}^{n+m} \frac{\partial u}{\partial z_i} \frac{\partial \psi}{\partial z_i} dz = \sum_{i=1}^{n+m} \int_\Pi f_i(z) \frac{\partial \psi}{\partial z_i} dz, \tag{4}$$

if only $\psi(z) \in W_2^1(\Pi)$ and vanishes for $|x| > \rho$.

Theorem 1. *If*

$$J_a^*(f) = \int_\Pi \left(1 + |x|^a\right) \left[\sum_{i=1}^{n+m} f_i^2(z)\right] dz < \infty$$

$a<n-2$, $2s+a-2+n\neq 0$ for no entire nonnegative s, then there exists the only solution of the problem (1), (2) such that

$$J_a(u)=\int_\Pi (1+|x|)^a \sum_{i=1}^{n+m}\left(\frac{\partial u}{\partial z_i}\right)^2 dz+\int_\Pi (1+|x|)^{a-2}u^2(z)dz+$$
$$+\int_\Pi |x|^{a-2}\left[\int_\Omega u(z)dy\right]^2 dx<\infty$$

and for that

$$J_a(u)\le cJ_a^*(f), \tag{5}$$

where $C=const$ doesn't depend on u,f.

Proof. Let's prove first the existence of the necessary solution of the problem (1), (2).

Expand each $f_i(z)$ into the series

$$f_i(z)=\sum_{k=0}^{\infty} f_{ik}(x)\Phi_k(y),\ i=1,2,...,n+m \tag{6}$$

by the ortnomalized eigen-functions of the boundary-valued problem:

$$\Delta_y\Phi+\lambda\Phi=0,$$
$$\left.\frac{\partial\Phi}{\partial n}\right|_{\partial\Omega}=0. \tag{7}$$

Nearly for every x the series (6) converges in $L_2(\Omega)$. We will seek the solution of the problem (1)-(2) in the form of the series

$$u(z)=\sum_{k=0}^{\infty} u_k(x)\Phi_k(y), \tag{8}$$

where the functions $u_k(x)$ are unknown. Let's take as $u_0(x)$ the solution of the equation

$$\Delta u_0(x)=\sum_{i=1}^{n}\frac{\partial f_{i0}(x)}{\partial x_i}+f_{00}(x),\ x\in R^n \tag{9}$$

satisfying the inequality

$$\int_{R^n}(1+|x|)^a|\nabla_x u_0(x)|^2 dx\le\sum_{i=1}^{n}\int_{R^n}(1+|x|)^a|f_{i0}(x)|^2 dx. \tag{10}$$

Such solution is the function

$$u_0(x)=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}[\Gamma(x)*f_{i0}(x)], \tag{11}$$

where $\Gamma(x)$ is the fundamental solution of Laplacian equation.

The inequality (10) for $u_0(x)$ is fulfilled by virtue of the theorem on restriction of the singular integral in the weight space (7).

From (10) and from Hardy's inequality (for the domains of the form $\Omega\times R^n$ it was proved in [8]) it follows:

$$u(x)=c+u_0(x), \tag{12}$$

where

$$\int_{R^n}(1+|x|)^{a-2}u_0^2(z)dz \le c\int_{\Pi}|x|^a\sum_{i=1}^{n+m}\left(\frac{\partial u}{\partial z_i}\right)^2 dz \qquad (13)$$

for $a+n\neq 2$.

Therefore, the constructed by (12) the function R^n is the solution of equation (9) in R^n and satisfies the inequality

$$\int_{R^n}\left[(1+|x|)^{a-2}u_0^2(x)+(1+|x|)^a\sum_{i=1}^{n}\left(\frac{\partial u_0}{\partial x_i}\right)^2\right]dx \le$$
$$\le c\sum_{i=1}^{n}\int_{R^n}(1+|x|)^a|f_{i0}(x)|^2 dx\ . \qquad (14)$$

We will seek the coefficients of the expansion (8) $u_k(x)$ for $k>1$ from the equation:

$$\Delta u_k(x)-\lambda_k u_k(x)=\left[\sum_{i=1}^{n}\frac{\partial}{\partial x_i}f_{ik}(x)\right]-\sum_{i=n+1}^{n+m}\int_{\Omega}f_i(z)\frac{\partial\Phi_k}{\partial y_i}dy, \qquad (15)$$

$x\in R^n$.

For investigation of (15) let's make the substitution $u_k(x)=v(x)\left[1+|x|^2\right]$. As the result we obtain the equation

$$\Delta v-\lambda_k v+\sum_{i=1}^{n}\frac{\partial v}{\partial x_i}a_i(x)+a_0(x)v=$$
$$=\left(1+|x|^2\right)^{\frac{a}{2}}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}f_{ik}(x)-\sum_{i=n+1}^{n+m}\left(1+|x|^2\right)^{\frac{a}{2}}\int_{\Omega}f_i(z)\frac{\partial\Phi_k}{\partial y_i}dy= \qquad (16)$$
$$=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}f_{ik}^{(1)}(x)+\sum_{i=n+1}^{n+m}f_{ik}^{(2)}(x),$$

where $a_0(x)\in C^\infty(R^n)$, $\quad a_i(x)\in C^\infty(R^n)$

$$\lim_{|x|\to\infty}a_i(x)=0,\qquad i=0,1,\ldots,n,$$

$$\sum_{i=1}^{n+m}\int_{R^n}\left|f_{ik}^{(1)}(x)\right|^2dx\le c\sum_{i=1}^{n+m}\int_{R^n}(1+|x|)^a f_{ik}^{(2)}(x)dx\,. \qquad (17)$$

The equation (16) determines Fredholm's operator

$$W_2^1(R^n)\to\underbrace{L_2(R^n)\times\ldots\times L_2(R^n)}_{n+m\ \text{times}}.$$

It's solvable uniquely if the corresponding homogeneous equation has only a trivial solution. Let's show that it has that. Let $v_0(x)$ be the solution of (16) for $f_{ik}^{(1)}(x)=0$, $v_0(x)\in W_2^1(R^n)$. In this case, the function $\widetilde{u}(x)=v(x)\left(1+|x|^2\right)^{-\frac{a}{2}}$ satisfies the equation

$$\Delta u-\lambda_k u=0 \qquad (18)$$

and

$$\int_{R^n}\left(1+|x|^2\right)^a u^2(x)dx<\infty\,. \qquad (19)$$

It is known that from

$$\int_{R^n} u^2(x) e^{\varepsilon|x|} dx < \infty \tag{20}$$

it follows that $u(x) \equiv 0$. As for as the condition (19) is weaker than (20), then (19) means, that $u(x) \equiv 0$. Thus, the equation (16) is solvable uniquely and gives a possibility to find such $u_k(x)$ - the solution of equation (15), that

$$\sum_{i=1}^{n} \int_{R^n} \left(1+|x|^a\right)\left(\frac{\partial u_k}{\partial x_i}\right)^2 dx + (1+\lambda_k) \int_{R^n} \left(1+|x|^a\right) u_k^2(x) dx \le$$
$$\le c_1 \int_{R^n} \sum_{i=1}^{n} \left(1+|x|^a\right) f_{ik}^{(2)}(x) dx + \sum_{i=n+1}^{n+m} c_k \int_{R^n} \left(1+|x|^a\right) \left| f_i(z) \frac{\partial \Phi_k}{\partial y_i} dy \right|^2 dx. \tag{21}$$

The structure of (1) is such that, we can consider

$$\int_\Omega f_i(z) = 0, \quad i = n+1, \ldots, n+m.$$

Indeed, if $f_i(z)$ an arbitrary function, then if we denote by $f_i^*(z) = f_i(z) - \frac{1}{mes\Omega} \int_\Omega f_i(z) dy$, it is obvious, that

$$\int_\Omega f_i^*(z) dz = 0$$

$\frac{\partial f_i}{\partial z_i} = \frac{\partial f^*}{\partial z_i}$ for $i > n$. So the equation (1) won't be changed, if we substitute $f_i(z)$ by $f_i^*(z)$.

The condition (4) and also the equality

$$\sum_{i=n+1}^{n+m} \int_\Omega f_i(z) \cos(n, z_i) dx d_s \Omega = 0$$

are saved, too.

Using the above said, let us estimate the last sum in the right-hand side of the inequality (21). Let's take for that $\Delta_y w_i = f_i(x,y), \left.\frac{\partial w_i}{\partial n}\right|_{\partial\Pi} = 0$.

Then

$$\int_\Omega f_i \frac{\partial \Phi_k}{\partial y_i} dy = -\lambda_k \int_\Omega w_i \Phi_k dy.$$

Therefore, from (15) it follows:

$$\sum_{i=1}^{n} \int_{R^n} \left[\left(1+|x|^a\right)\left(\frac{\partial u_k}{\partial x_i}\right)^2 + \left(1+|x|^a\right) u_k^2 \right] dx \le$$
$$\le c_k \int_{R^n} \sum_{i=1}^{n} \left(1+|x|^a\right) f_{ik}^2(x) dx + c_k \int_{R^n} \lambda_k \left(1+|x|^a\right) \left[\int_\Omega w_i \Phi_k dy \right]^2 dx. \tag{22}$$

Let's show that the constant in the right-hand of the inequality (21) doesn't depend on k, if k is sufficient great. From the equation (16) after multiplication by v and integration by R^n it follows

$$\int_{R^n}\left[\sum_{i=1}^{n}\left(\frac{\partial v}{\partial x_i}\right)^2+\lambda_k v^2\right]dx \le c\int_{R^n} v^2dx+\frac{1}{2}\int_{R^n}\left(\frac{\partial v}{\partial x_i}\right)^2dx+$$
$$+c\left[\int_{R^n}\left(\sum_{i=n+1}^{n+m} f_i^2\right)dx\right].$$

As far as $\lambda_k \to \infty$ for $k \to \infty$, then for $\lambda_k > 4c$ we obtain (21), with the constant c_k which is single for all $k = 1,2,\ldots$.

Let's sum the inequalities (22) by all k. As result we obtain:

$$\sum_{k=1}^{\infty}\left\{\sum_{i=1}^{n}\int_{R^n}\left(1+|x|^a\right)\left(\frac{\partial u_k}{\partial x_i}\right)^2dx+\left(1+\lambda_k\right)\int_{R^n}\left(1+|x|^a\right)u_k^2dx\right\}\le$$
$$\le c\int_{\Pi}\sum_{i=1}^{n}\left(1+|x|^a\right)f_i^2dx+\int_{\Pi}\left(1+|x|^a\right)\sum_{k=1}^{\infty}\lambda_k\left[\int_{\Omega}w_i\Phi_k dy\right]^2dx. \tag{23}$$

From the theory of eigen-functions of Laplacian operator it follows that

$$\sum_{k=1}^{\infty}\lambda_k\left[w_i\Phi_k dy\right]^2\le\int_{\Omega}\sum_{j=1}^{n}\left(\frac{\partial w_i}{\partial y_j}\right)^2dy.$$

Hence from the definition of w_i it follows that

$$\sum_{i=n+1}^{n+m}\sum_{k=1}^{\infty}l_k\left(\int_{\Omega}w_i\Phi_k dy\right)^2\le\int_{\Omega}\sum_{i=n+1}^{n+m}f_i^2dy.$$

Therefore

$$\sum_{i=1}^{n}\int_{R^n}\left(1+|x|^a\right)\left(\frac{\partial u}{\partial x_i}\right)^2dx+\int_{R^n}\left(1+|x|\right)^{a-2}u^2dx\le$$
$$\le c\int_{\Pi}\sum_{i=1}^{n}\left(1+|x|\right)^a f_i^2(z)dx, \tag{24}$$

where $u(z)$ is determined by (8). The series (8) converges in the mean, the series obtained from (8) converges in the mean, too by the differentiation term by term by x.

Let's estimate now $\dfrac{\partial u}{\partial x_i}$ for $i > n$.

It's easy to see, that for $j > n$

$$\int_{R^n}dx\int_{\Omega}\left|u(x)\frac{\partial\Phi_k}{\partial y_j}\right|^2dy=\lambda_k=\lambda_k\int_{R^n}u_k^2(x)dx.$$

Hence and from (23) it follows, that

$$\sum_{k=1}^{\infty}u_k(x)\frac{\partial\Phi_k}{\partial y_j}\left(1+|x|^a\right)^{1/2},\ \ j>n$$

converges in the mean and consequently

$$\sum_{i=1}^{n+m}\int_{R^n}\left(1+|x|^a\right)\left(\frac{\partial u}{\partial x_i}\right)^2dx+\int_{R^n}\left(1+|x|^a\right)u^2dx\le J_a(f).$$

Now let's prove the uniqueness of the solution of the problem (1), (2) in the statement of Theorem 1.

Let $f_i \equiv 0,\ u(z)$ is the solution of the problem (1), (2), $J_a(u) < \infty$. Let's represent $u(z)$ by the formula (8). Each of $u_k(x)$ is the solution of the equation

$$\Delta u_k - \lambda_k u_k = 0,\ \ x \in R^n$$

and

$$\sum_{i=1}^{n} \int_{R^n} \left[\left(1 + |x|^a\right) \left(\frac{\partial u_k}{\partial x_i} \right)^2 + \left(1 + |x|^a\right) u_k^2 \right] dx < \infty .$$

Also $u_k(x)$ satisfies the inequality (20) and it means that $u_k(x) \equiv 0$ for $k > 1$.

If $k = 0$, then $\lambda_k = 0$ and $u_0(x)$ is the polinom by virtue of Livill's theorem. As far as $J_a(u) < \infty$, then $J_a(u_0) < \infty$ and $u_0 \equiv 0$.

2. Let's consider the problem on the asymptotics of the solution of the equation

$$\Delta u = \sum_{i=1}^{n+m} \frac{\partial f}{\partial z_i}$$

in the domain Π for fulfilment of the boundary-valued condition (2).

Suppose, that

$$\int_{\Pi} \left(1 + |x|^{a_1}\right) u^2(z) dz + \int_{\Pi} \left(1 + |x|^{a_2}\right) \sum_{i=1}^{n+m} \left(\frac{\partial u}{\partial z_i} \right)^2 dz < \infty \qquad (25)$$

$f_i(z) \equiv 0$ for $|x| > \rho,\ x \in \Pi$

Represent $u(z)$ in the form of the servies (8). Let's consider $u_k(x), k > 1$. This function is the solution of the equation (15).

Let's fix arbitrary $a_2 > a_1$. The inequality (21) has been proved for any $a = const$, particularly, it is valid also for $a = a_2$. Summing all the inequalities (21) for $k = 2,3,\ldots$ we obtain

$$\sum_{k=2}^{\infty} \sum_{i=1}^{n} \int_{R^n} R^n \left(1 + |x|^{a_2}\right) \left(\frac{\partial u_k}{\partial x_i} \right)^2 dx + \sum_{k=2}^{\infty} \int_{R^n} \left(1 + |x|^{a_2}\right) u_k^2(x) dx \le c$$

since $f_i(z) \equiv 0$ is out of some compact.

Now let's consider $u_1(x)$. This function is the solution of Laplacian equation

$$\Delta u_1 = 0,\ x \in R^n,\ |x| \ge \pi_1 = const$$

and

$$\sum_{i=1}^{n} \int_{R^n} \left(1 + |x|^a\right) \left(\frac{\partial u_1}{\partial x_i} \right)^2 dx + \int_{R^n} \left(1 + |x|^a\right) u_1^2(x) dx < \infty .$$

From the formulas for the asymptotic expansion in the neighbour of the infinity (for example, (1)) it follows, that for any N

$$u_1(x) = P(x) + \sum_{|\alpha| < 2-n+N} a_\alpha D^\alpha \Gamma(x) + u_1^*(x),$$

where $P(x)$ is the harmonic polinom of power less than $1 - \dfrac{a+n}{2}$,

$$a_\alpha = const,\ \left|u_1^*(x)\right| = O\left(|x|^{-N}\right),\ \left|\nabla u_1^*\right| = O\left(|x|^{-N-1}\right).$$

From (8) it follows

$$u(z)=u_1(x)\Phi_1(y)+\sum_{k=2}^{\infty}u_k(x)\Phi_k(y)=\left[P(x)+\sum_{|\alpha|<2-n+N}a_\alpha D^\alpha\Gamma(x)+u_1^k(x)\right]\Phi_1(y)+$$

$$+\sum_{k=2}^{\infty}u_k(x)\Phi_k(y)=P(x)\Phi_1(y)+\sum_{|\alpha|<2-n+N}b_\alpha(y)D^\alpha\Gamma(x)+u_1\Phi_1(y)+\sum_{k=2}^{\infty}u_k(x)\Phi_k(y)=$$

$$=F(x,y)+\sum_{|\alpha|<2-n+N}b_\alpha(y)D^\alpha\Gamma(x)+V(z). \qquad (26)$$

The function $V(z)$ is the solution of Laplacian equation in Π for $|x|>\rho$, it satisfies Neyman's boundary-valued condition and

$$\int_\Pi |V(z)|^2\left(1+|x|^{a_2}\right)dx<\infty .$$

Hence it follows, that

$$V^2(z)\le c|x|^{-a_2-n},\quad |\nabla V(z)|^2\le c|x|^{-a_2-n-2}. \qquad (27)$$

From (26), (27) it follows

Theorem 2. *If $f_i(z)$ have a compact carrier, $u(z)$ is the solution of the problem (1), (2) such that (25) to be fulfilment, then*

$$u(z)=\sum_{|\alpha|<2-n+N}b_\alpha(y)D^\alpha\Gamma(x)+F(x,y)+V(z),$$

where $F(x,y)$ is the harmonic polinom whose power is less than $2-n+N$;

$\Gamma(x)$ is the fundamental solution of Laplacian operator in R^n ;

$$|V(z)|\le c|x|^{-N},\quad |\nabla V(z)|\le c|x|^{-N-1}$$

for $|x|>\rho,\ x\in\Pi$.

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