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STABILIZATION OF SOLUTIONS FOR A SEMI-LINEAR WAVE EQUATION

Abstract

Sufficient condition for the stabilization of solutions for a semi-linear wave equation is obtained. An example is cited also for the case when by not fulfilling this sufficient condition, the solutions of this considered norm are not stabilized.

Consider the following initial-boundary value problem:

$$\left.\begin{array}{l} \vartheta_{tt} + f(\vartheta)\vartheta_t - \vartheta_{xx} + g(\vartheta) = h(x),\ (t,x)\in(0,+\infty)\times(0,1) \\ \vartheta(t,0)=\vartheta(t,1)=0\ ,\ t\in(0,+\infty) \\ \vartheta(0,x)=\vartheta_0(x),\ \vartheta_t(0,x)=\vartheta_1(x),\ x\in(0,1) \end{array}\right\} \qquad (1)$$

where $f(\cdot)\in C^1(R^1)$ is positive function a.e. on R^1, $g(\cdot)\in C^1(R^1)$, $g(0)=0$, $g'(\cdot)\geq 0$, $h(\cdot)\in L_2(0,1)$, $\vartheta_0(\cdot)\in \overset{\circ}{W}{}_2^1(0,1)$, $\vartheta_1(\cdot)\in L_2(0,1)$.

In the case $f(\vartheta)=|\vartheta|^p$ $)p>1$, $g(\cdot)\equiv 0$, $h(\cdot)\equiv 0$ problem (1) has been considered in paper [1] and the decrease of solutions in the norm L_{p+2} has been proved.

Stabilization of solutions of problem (1) for $t\to+\infty$ in the norm W_2^1 is studied in the paper.

Using the transformation $\theta(t)=\begin{pmatrix}\vartheta(t)\\ \vartheta_t(t)\end{pmatrix}$ problem (1) is led to the Cauchy problem for the first order operator equation in the space $\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$ which according to [2] generates a strong- continuous group $V(t)$, $t\in R_+^1$ in $\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$.

Now for $\forall\theta_0=\begin{pmatrix}\vartheta_0\\ \vartheta_1\end{pmatrix}\in\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$ we construct a set

$$\tilde{\omega}(\theta_0)=\bigcap_{t\geq 0}\left[\bigcup_{s\geq t}V(s)\theta_0\right]_W ,$$

where $[\ \ ]_W$ is a weak closure in $\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$. Since for $\forall s\geq 0$ $\theta(s)=V(s)\theta_0$ is bounded in $\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$, the set $\left[\bigcup_{s\geq t}Vs\theta_0\right]_W$ for $\forall t\geq 0$ is weakly compact, and consequently the set $\tilde{\omega}(\theta_0)$ is not empty and weakly compact in $\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$.

We can easily show that

1) $\varphi\in\tilde{\omega}(\theta_0)\Leftrightarrow$ exists $\{t_n\}_{n=1}^{\infty}$, $t_n\to+\infty$, such that $V(t_n)\theta_0\to\varphi$ weakly in $\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$;
2) $V(t_n)\tilde{\omega}(\theta_0)=\tilde{\omega}(\theta_0)$, $\forall t\geq 0$.

Denote by $\varphi^0(x)$ the solution of the following problem

$$\left.\begin{array}{l} -\vartheta_{xx}+g(\vartheta)=h\ ,\ \text{ a.e. in } (0,1) \\ \vartheta(0)=\vartheta(1)=0 \end{array}\right\} \tag{2}$$

Theorem 1. $\widetilde{\omega}(\theta_0)=\begin{pmatrix}\varphi^0\\ 0\end{pmatrix}$.

Proof. Let $\varphi=\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}\in\widetilde{\omega}(\theta_0)$. Then $\begin{pmatrix}\vartheta(t_n)\\ \vartheta_t(t_n)\end{pmatrix}\equiv V(t_n)\theta_0\to\varphi$ weakly in $\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$, where $t_n\to+\infty$. In other words, there exists such $\{t_n\}_{n=1}^{\infty}$, $t_n\to+\infty$ that $\vartheta(t_n)\to\varphi^1$ weakly in $\overset{\circ}{W}{}_2^1(0,1)$, $\vartheta_t(t_n)\to\varphi^2$ weakly in $L_2(0,1)$. Hence it follows that

$$f(\vartheta(t_n))\cdot\vartheta_t(t_n)\to f(\varphi^1)\varphi^2 \text{ weakly in } L_2(0,1). \tag{3}$$

On the other hand, multiplying $(1)_1$ by ϑ_t and integrating on $(0,+\infty)\times(0,1)$ we get

$$\int_0^{\infty}\int_0^1 f(\vartheta)\vartheta_t^2\,dxdt<\infty$$

and since $f(\vartheta(\cdot))\in L_\infty\left(0,+\infty;W_2^1(0,1)\right)$, consequently

$$\int_0^{\infty}\|f(\vartheta)\vartheta_t\|^2_{W_2^{-1}(0,1)}dt\le\int_0^{\infty}\|f(\vartheta)\vartheta_t\|^2_{L_2(0,1)}dt<\infty .$$

Now to the function $y(t)=\|f(\vartheta)\vartheta_t\|^2_{W_2^{-1}(0,1)}$ we can apply the lemma from [3]. Then

$$f(\vartheta)\cdot\vartheta_t\xrightarrow[t\to+\infty]{}0 \text{ strongly in } W_2^{-1}(0,1). \tag{4}$$

From (3)-(4) it follows that for $\forall\varphi=\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}\in\widetilde{\omega}(\theta_0)$

$$f(\varphi^1)\varphi^2=0 \quad \text{a.e. in } (0,1).$$

Since for $\forall t\ge 0$ $\begin{pmatrix}\widetilde{\vartheta}(t)\\ \widetilde{\vartheta}_t(t)\end{pmatrix}\equiv V(t)\varphi\in\widetilde{\omega}(\theta_0)$, then

$$f(\widetilde{\vartheta}(t,\cdot))\widetilde{\vartheta}_t(t,\cdot)=0 \text{ a.e. in } (0,1),$$

and consequently

$$\frac{d}{dt}\Phi(\widetilde{\vartheta}(t,x))=0 \quad \text{a.e. in } (0,1),$$

where $\Phi(s)=\int_0^s f(t)dt$.

Hence for $\forall t\ge 0$ and $\forall x\in[0,1]$ we have:

$$\Phi(\widetilde{\vartheta}(t,x))=\Phi(\widetilde{\vartheta}(0,x)). \tag{5}$$

By virtue of that $\Phi:R^1\to R^1$ is a monotonically increasing function, from (5) we obtain

$$\widetilde{\vartheta}(t,x)=\widetilde{\vartheta}(0,x),\quad \forall t\ge 0,\ \forall x\in[0,1].$$

Since $\widetilde{\vartheta}(t,x)$ satisfies $(1)_1$ with initial conditions $\widetilde{\vartheta}(0,x)=\varphi^1(x)$, $\vartheta_t(0,x)=\varphi^2(x)$, then from the latter equality it follows

$$\varphi^2(x)=0\,,\ \text{a.e. in}\ \ (0,1)$$
$$\varphi^1(x)=\varphi^0(x)\,,\ \text{a.e. in}\ \ (0,1)$$

and this completes the proof of Theorem 1.

Theorem 2. *If* $h(\cdot)\not\equiv 0$ *, then for* $\forall\,\theta_0=\begin{pmatrix}\vartheta_0\\ \vartheta_1\end{pmatrix}\in \overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$

$$\begin{pmatrix}\vartheta(t)\\ \vartheta_t(t)\end{pmatrix}\equiv V(t)\theta_0\xrightarrow[t\to+\infty]{}\begin{pmatrix}\varphi^0\\ 0\end{pmatrix}\ \textit{strongly in}\ \overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1).$$

Proof. If $h(\cdot)\not\equiv 0$ then $\varphi^0(\cdot)\not\equiv 0$. Then there exist such a $[a,b]\subset(0,1)$ and such a $\delta>0$, that for $\forall x\in[a,b]$

$$f\left(\varphi^0(x)\right)\geq\delta\,.$$

By virtue of Theorem 1 we have

$$\left\|f(\vartheta(t,x))-f\left(\varphi^0(x)\right)\right\|_{C[0,1]}\xrightarrow[t\to+\infty]{}0$$

consequently for $\forall\varepsilon\in\left(0,\dfrac{\delta}{2}\right)$ there exists $T(\varepsilon)>0$, such that $\forall t\geq T(\varepsilon)$

$$f(\vartheta(t,x))>f\left(\varphi^0(x)\right)-\varepsilon\,,\quad \forall x\in[0,1].$$

Denoting $\alpha(x)=\max\left\{0,\,f\left(\varphi^0(x)\right)-\varepsilon\right\}$ we get that $\alpha(x)\not\equiv 0$ and

$$f(\vartheta(t,x))\geq\alpha(x)\geq 0\quad \forall t\geq T(\varepsilon),\quad \forall x\in[0,1]. \tag{6}$$

By definition $\vartheta(t,x)$ satisfies the problem (1). Represent it to the sum of the functions $u(t,x)$ and $w(t,x)$ that satisfies the following problems respectively:

$$\left.\begin{array}{l} u_{tt}+\alpha(x)u_t-u_{xx}+g(\vartheta)=h\\ u(t,0)=u(t,1)=0\\ u(0,x)=0\,,\ \ u_t(0,x)=0\end{array}\right\} \tag{7}$$

and

$$\left.\begin{array}{l} w_{tt}+\alpha(x)w_t-w_{xx}=\alpha(x)\vartheta_t-f(\vartheta)\vartheta_t\\ w(t,0)=w(t,1)=0\\ w(0,x)=\vartheta_0(x)\,,\ \ w_t(0,x)=\vartheta_1(x)\end{array}\right\} \tag{8}$$

For solutions of the problems (7) and (8) it is valid the following representations

$$\begin{pmatrix}u(t)\\ u_t(t)\end{pmatrix}=\int_0^t e^{A(t-s)}\begin{pmatrix}0\\ -g(\vartheta(s))+h\end{pmatrix}ds\,,$$

$$\begin{pmatrix}w(t)\\ w_t(t)\end{pmatrix}=e^{A(t-s)}\begin{pmatrix}w(s)\\ w_t(s)\end{pmatrix}+\int_s^t e^{A(t-\tau)}\begin{pmatrix}0\\ -\alpha\vartheta_t(\tau)-f(\vartheta)\vartheta_t(\tau)\end{pmatrix}d\tau\,,$$

where $A\equiv\begin{pmatrix}0 & I\\ \dfrac{\partial^2}{\partial x^2} & -\alpha I\end{pmatrix}$ is a linear operator on $\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$ with domain

$D(A)=W_2^2(0,1)\cap\overset{\circ}{W}{}_2^1(0,1)\times\overset{\circ}{W}{}_2^1(0,1)$ and $0\leq s\leq t$.

Now by using Theorem 1 from [3] estimate the solution of problems (7)-(8). Then

$$\left\|\begin{pmatrix} u(t) \\ u_t(t)\end{pmatrix}\right\|_{W_2^2(0,1)\cap \overset{\circ}{W}{}_2^1(0,1)\times \overset{\circ}{W}{}_2^1(0,1)} \le c_1\left(\|\theta_0\|_{\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)}\right) + c_2\|h\|_{L_2(0,1)}, \quad \forall t \ge 0, \tag{9}$$

$$\begin{aligned}\left\|\begin{pmatrix} w(t) \\ w_t(t)\end{pmatrix}\right\|_{\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)} &\le M\,e^{-\varepsilon(t-s)}\cdot\left\|\begin{pmatrix} w(s) \\ w_t(s)\end{pmatrix}\right\|_{\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)} + \\ &+ M\cdot\int_s^t e^{-\varepsilon(t-s)}\left\|\begin{pmatrix} 0 \\ -\alpha\vartheta_t(t) - f(\vartheta)\vartheta_t(\tau)\end{pmatrix}\right\|_{\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)} d\tau\ .\end{aligned} \tag{10}$$

Applying Hölder's inequality with regard to (6) we get from (10)

$$\left\|\begin{pmatrix} w(t) \\ w_t(t)\end{pmatrix}\right\|_{\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)} \le e^{-\varepsilon(t-s)}\left(\tilde{c}_1\left(\|\theta_0\|_{\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)}\right) + \tilde{c}_2\cdot\|h\|_{L_2(0,1)}\right) + \tilde{M}\int_s^t \|f(\vartheta)\vartheta_t\|^2_{L_2(0,1)}d\tau\ .$$

Now pass to the limit firstly on t, then on s.

Then

$$\overline{\lim_{t\to+\infty}}\left\|\begin{pmatrix} w(t) \\ w_t(t)\end{pmatrix}\right\|_{\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)} = 0. \tag{11}$$

Thus, it follows from (9) and (11) that for any $\{t_n\}_{n=1}^{\infty}$, $t_n \to +\infty$, the set $\{V(t_n)\theta_0\}_{n=1}^{\infty}$ is precompact in $\overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$.

By virtue of Theorem 1 for $\forall\theta_0 \in \overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$

$$V(t)\theta_0 \xrightarrow[t\to+\infty]{} \begin{pmatrix} \varphi^0 \\ 0 \end{pmatrix} \text{ weakly } \overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1).$$

Considering the abovementioned precompactness of the set $\{V(t_n)\theta_0\}_{n=1}^{\infty}$ from the latter limited relation the statement of Theorem 2 follows.

Note that if $h(\cdot)\equiv 0$, then Theorem 2 in generally speaking, is not true. As an example consider the case $f(\vartheta) = |\vartheta|^p$, $p > 2$, $h \equiv 0$, $g \equiv 0$. Assume, the statement of Theorem 2 is true. Then for $\forall\theta_0 \in \overset{\circ}{W}{}_2^1(0,1)\times L_2(0,1)$, $\theta_0 \ne 0$,

$$\vartheta(t) \xrightarrow[t\to+\infty]{} 0 \text{ strongly in } \overset{\circ}{W}{}_2^1(0,1) \text{ and } \vartheta_t(t) \xrightarrow[t\to+\infty]{} 0 \text{ strongly in } L_2(0,1),$$

consequently, multiplying $(1)_1$ by ϑ_t and integrating on $(s,+\infty)\times(0,1)$ for $\forall s > 0$ we have

$$2\int_s^{+\infty}\int_0^1 |\vartheta|^p\vartheta_t^2\,dxdt = \|\vartheta(s)\|^2_{\overset{\circ}{W}{}_2^1(0,1)} + \|\vartheta_t(s)\|^2_{L_2(0,1)} \ge \|\vartheta_t(s)\|^2_{L_2(0,1)}. \tag{12}$$

Now take $\{s_n\}_{n=1}^{\infty}$, $s_n \to +\infty$ such that $\|\vartheta_t(s_n)\|_{L_2(0,1)} = \sup_{t\ge s_n}\|\vartheta_t(t)\|_{L_2(0,1)}$. We shall assume that

$$\|\vartheta_t(s_n)\|_{L_2(0,1)} > 0, \quad \forall n \in N.$$

On the contrary there should exist such $n_0 \in N$ that

$$\|\vartheta_t(s_{n_0})\|_{L_2(0,1)} = \sup_{t\ge s_{n_0}}\|\vartheta_t(t)\|_{L_2(0,1)} = 0.$$

Hence, using equation $(1)_1$ we get

$$\vartheta(t,x)=0\,,\quad \forall t\ge s_{n_0}\,,\quad \forall x\in(0,1),$$

consequently

$$\vartheta_0(x)=0\,,\quad \vartheta_1(x)=0\,,\quad \forall x\in(0,1).$$

This contradicts to that $\theta_0\neq 0$.

It follows from (12) that

$$1\le 2\int_{s_n}^{+\infty}\int_0^1|\vartheta|^p\frac{\vartheta_t^2}{\|\vartheta_t(s_n)\|^2_{L_2(0,1)}}dxdt\le 2\int_{s_n}^{+\infty}\|\vartheta(t)\|^p_{C[0,1]}dt\,. \qquad (13)$$

Since $f(\vartheta)=|\vartheta|^p$, $g\equiv 0$, $h\equiv 0$, the from equation $(1)_1$ we can easily get the following estimates:

$$\int_0^{+\infty}\int_0^1|\vartheta|^p\vartheta_t^2dxdt<+\infty\,,\qquad \int_0^{+\infty}\int_0^1|\vartheta|^p\vartheta_x^2dxdt<+\infty$$

or

$$|\vartheta|^{\frac{p}{2}+1}\in W_2^1((0,+\infty)\times(0,1)).$$

From imbeddings

$$W_2^1((0,+\infty)\times(0,1))\subset W_q^1((0,+\infty)\times(0,1))\subset L_q\left(0,+\infty;W_q^1(0,1)\right),\ \ 1<q<2$$

we have

$$|\vartheta|^{\frac{p}{2}+1}\in L_q\left(0,+\infty;W_q^1(0,1)\right)\subset L_q\left(0,+\infty;C[0,1]\right),\quad q=\frac{2p}{p+2},$$

i.e.

$$\int_0^{+\infty}\|\vartheta(t)\|^p_{C[0,1]}dt<+\infty \qquad (14)$$

(13) and (14) contradict each other.

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Received February 22, 2000; Revised April 28, 2000.
Translated by Aliyeva E.T.