KALANTAROV V.K.

GLOBAL SOLUTION OF COUPLED KURAMOTO-SIVASHINSKY AND GINZBURG-LANDAU EQUATIONS

Abstract

Global existence and uniqueness of solution of periodic initial boundary value problem for coupled Kuramoto-Sivashinsky and Ginzburg-Landau equations is proven.

Introduction.

We consider the following periodic initial boundary value problem:

$$A_t = A + A_{xx} - |A|^2 A + Ah, \quad x \in R, \ t > 0, \tag{A}$$

$$h_t = -h_{xx} - h_{xxxx} + \alpha |A|^2_{xx}, \quad x \in R, \ t > 0, \tag{H}$$

$$A(x,0) = A_0(x), \quad h(x,0) = h_0(x), \quad x \in R,$$

$$A(x,t) = A(x+l,t), \quad h(x,t) = h(x+l,t), \quad x \in R, \ t > 0,$$

where α, l are given positive numbers, $A(x,t) = (A_1(x,t),...,A_N(x,t))$ is the unknown vector- function, $h(x,t)$ is the unknown scalar function, $A_0(x)$ and $h_0(x)$ are given functions.

When A is complex valued function, that is when $N = 2$, the system (A), (H) has been derived by A.A. Golovin *et al.* [2] as a simplified model of the surface tension driven Marangoni convection. D.Kazhdan *et al.* [4] has done numerical simulation of the system (A), (H) under periodic boundary conditions. J.Duan *et al.* [1] proved a theorem on global existence and uniqueness of regular solution of periodic initial boundary value problem for this system under the condition $0 < \alpha < 2$. Here we prove the global unique solvability of the initial boundary value problem for (A), (H) without the above-mentioned restriction, that is for each positive α.

In what follows we'll use the following notations:

for $f(x) = (f_1(x),...,f_N(x))$ and $g(x) = (g_1(x),...,g_N(x))$

$$\|f\| = \left(\int_0^l |f(x)|^2 dx\right)^{1/2}, \quad (f,g) = \int_0^l \langle f(x), g(x)\rangle dx,$$

where $\langle f,g\rangle = f_1 g_1 + f_2 g_2 + \cdots + f_N g_N$ and $|f|^2 = \langle f,f\rangle$. For real valued functions $f(x)$ and $g(x)$, $\|f\| = \left(\int_0^l f^2(x)dx\right)^{1/2}$ and $(f,g) = \int_0^l f(x)g(x)dx$. For notational easy we will use the bracket $(.,.)$ also for the duality pairing between the elements of $\dot{H}^1_{per}(0,l)$ and $\dot{H}^{-1}_{per}(0,l)$. $\|\cdot\|_l$ will denote the norm in $\dot{H}^1_{per}(0,l)$.

We will use also the Agmon's inequality, which is valid for each function from $H^1(0,l)$:

$$\|f\|^2_{L_\infty(0,l)} \le d_0 \|f\| \|f\|_{H^1(0,l)}, \quad d_0 > 0. \tag{I$_1$}$$

We refer to [5] for the definition of the spaces $H^s_{per}(0,l)$ and $\dot{H}^s_{per}(0,l)$.

1. Apriory Estimates.

Integrating the second equation of the system over $(0,l)$ with respect to x we obtain that

$$\frac{d}{dt}\int_0^l h(x,t)dx = 0,$$

thus we have:

$$\int_0^l h(x,t)dx = \int_0^l h_0(x)dx.$$

Let us consider the function $u(x,t) = h(x,t) - \frac{1}{l}\int_0^l h_0(x)dx$. Then $\int_0^l u(x,t)dx = 0$ and the paire $A(x,t)$, $u(x,t)$ is a solution of the following problem:

$$A_t = \lambda A + A_{xx} - |A|^2 A + Au, \quad x \in R, \ t > 0, \tag{1}$$

$$P^2 u_t = u + u_{xx} + \alpha |A|^2, \quad x \in R, \ t > 0, \tag{2}$$

$$A(x,0) = A_0(x), \quad u(x,0) = u_0(x), \quad x \in R, \tag{3}$$

$$A(x,t) = A(x+l,t), \quad u(x,t) = u(x+l,t), \quad x \in R, \ t > 0, \tag{4}$$

where

$$h_l = \frac{1}{l}\int_0^l h_0(x)dx, \quad \lambda = 1 + h_l, \quad u_0 = h_0 + h_l$$

and P^2 is the inverse of the operator $B = -\frac{d^2}{dx^2}$ with the domain $D(B) = \dot{H}^2_{per}(0,l)$.

Multiplying the equation (1) by A and (2) by u we obtain the following equalities:

$$\frac{1}{2}\frac{d}{dt}\|A(.,t)\|^2 = \lambda\|A(.,t)\|^2 - \|A_x(.,t)\|^2 - \int_0^l |A(x,t)|^4 dx + \int_0^l |A(x,t)|^2 u(x,t)dx, \tag{5_1}$$

$$\frac{1}{2}\frac{d}{dt}\|Pu(.,t)\|^2 = \|u(.,t)\|^2 - \|u_x(.,t)\|^2 - \alpha\int_0^l |A(x,t)|^2 u(x,t)dx. \tag{5_2}$$

Multiplying (5_1) by α and adding to the (5_2) we obtain:

$$\frac{1}{2}\frac{d}{dt}\left[\alpha\|A(.,t)\|^2 + \|Pu(.,t)\|^2\right] + \alpha\|A_x(.,t)\|^2 + \alpha\int_0^l |A(x,t)|^4 dx + \|u_x(.,t)\|^2 = $$
$$= \alpha\lambda\|A(.,t)\|^2 + \|u(.,t)\|^2. \tag{6}$$

Since α is a positive number we have

$$\frac{1}{2}\frac{d}{dt}\left[\alpha\|A(.,t)\|^2 + \|Pu(.,t)\|^2\right] + \|u_x(.,t)\|^2 \le \alpha|\lambda|\|A(.,t)\|^2 + \|u(.,t)\|^2. \tag{7}$$

Due to the Schwarz inequality we have:

$$\|u\|^2 = \left(Pu, P^{-1}u\right) \le \|Pu\|\left\|P^{-1}u\right\| \le e\|u_x\|^2 + C_1(\varepsilon)\|Pu\|^2. \tag{I_2}$$

Using the last inequality in (7) we obtain:

$$\frac{1}{2}\frac{d}{dt}\left[\alpha\|A(.,t)\|^2 + \|Pu(.,t)\|^2\right] + (1-\varepsilon)\|u_x(.,t)\|^2 \le \left(|\lambda| + C_1(\varepsilon)\right)\left[\alpha\|A(.,t)\|^2 + \|Pu(.,t)\|^2\right]. \tag{8}$$

It follows from (8) that

$$\alpha\|A(.,t)\|^2 + \|Pu(.,t)\|^2 \le \left(\alpha\|A_0\|^2 + \|Pu_0\|^2\right)\exp\{t(|\lambda| + C_1(\varepsilon))\}, \quad \forall t \in R^+ \tag{9}$$

and

$$\int_0^t \|A_x(.,s)\|^2 ds\,,\quad \int_0^t \|u_x(.,s)\|^2 ds,\quad \int_0^l\int_0^t |A(x,s)|^4 dxds \le C(t),\quad \forall t \in R^+\,, \qquad (10)$$

where $C(t)$ is some positive continuous function on $[0,\infty)$.

The estimates (9) and (10) allow us by using the standard Faedo-Galiorkin method to prove the existence of a global weak solution $[A,u]$ of the problem (1)-(4) with the following properties:

$A \in L_\infty(0,T;L_2(0,l)) \cap L_2(0,T;H^1_{per}(0,l))$ and $u \in L_\infty(0,T;\dot{H}^{-1}_{per}(0,l)) \cap L_2(0,T;\dot{H}(0,l))$.

2. Uniqueness.

Assume that $[A,u]$ is weak solution of the problem (1)-(4) corresponding to the initial data $[A_0,u_0]$ and $[\widetilde{A},\widetilde{u}]$ is the weak solution of this problem corresponding to $[\widetilde{A}_0,\widetilde{u}_0]$. Then $[a,U]=[A-\widetilde{A},u-\widetilde{u}]$ is a solution of the following problem:

$$a_t - \lambda a - a_{xx} + |A|^2 A - |\widetilde{A}|^2 \widetilde{A} = Au - \widetilde{A}\widetilde{u}\,,\quad x \in R\,,\ t>0\,, \qquad (11)$$

$$P^2 U_t - U - U_{xx} = -\alpha|A|^2 + \alpha|\widetilde{A}|^2\,,\quad x\in R\,,\ t>0\,, \qquad (12)$$

$$a(x,0) = A_0(x) - \widetilde{A}_0(x),\ \ U(x,0) = u_0(x) - \widetilde{u}_0(x),\ \ x \in R\,,$$
$$a(x,t) = a(x+l,t),\quad U(x,t) = U(x+l,t),\quad x\in R\,,\ t>0\,.$$

Since

$$Au - \widetilde{A}\widetilde{u} = Au - \widetilde{A}u + \widetilde{A}u - \widetilde{A}\widetilde{u} = au + \widetilde{A}U$$

and

$$-\alpha|A|^2 + \alpha|\widetilde{A}|^2 = -\alpha\left(\langle A,A\rangle - \langle \widetilde{A},A\rangle + \langle \widetilde{A},A\rangle - \langle \widetilde{A},\widetilde{A}\rangle\right) = -\alpha\langle a,A\rangle - \alpha\langle \widetilde{A},a\rangle$$

it follows from (11) and (12) that $[a,U]$ satisfies the following system:

$$a_t - \lambda a - a_{xx} + |A|^2 A - |\widetilde{A}|^2\widetilde{A} = au + \widetilde{A}U\,, \qquad (13)$$

$$P^2 U_t - U - U_{xx} = -\alpha\langle \widetilde{A},a\rangle - \alpha\langle a,A\rangle\,. \qquad (14)$$

Multiplying (13) by a we obtain:

$$\frac{1}{2}\frac{d}{dt}\|a(\cdot,t)\|^2 - \lambda\|a(\cdot,t)\|^2 + \|a_x(\cdot,t)\|^2 + \left(|A(\cdot,t)|^2 A(\cdot,t) - |\widetilde{A}(\cdot,t)|^2\widetilde{A}(\cdot,t), A(\cdot,t) - \widetilde{A}(\cdot,t)\right) =$$
$$= \left(|a(x,t)|^2, u(x,t)\right) + \left(\langle A(\cdot,t), a(\cdot,t)\rangle U(\cdot,t)\right). \qquad (15)$$

We can use the inequality (I_1) and estimate the first integral in the right hand side of (15) as follows:

$$\left|\left(|a(\cdot,t)|^2, u(\cdot,t)\right)\right| \le \max_{x\in[0,l]}|a(x,t)|^2 \int_0^l |u(x,t)|dx \le \sqrt{l}\max_{x\in[0,l]}|a(x,t)|^2\|u(\cdot,t)\| \le$$
$$\le \sqrt{l}d_0\|a(\cdot,t)\|\|a(\cdot,t)\|_{H^1(0,l)}\|u(\cdot,t)\| \le \sqrt{l}d_0\|a(\cdot,t)\|^2\|u(\cdot,t)\| + \sqrt{l}d_0\|a(\cdot,t)\|\|a_x(\cdot,t)\|\|u(\cdot,t)\| \le$$
$$\le \varepsilon_1\|a_x(\cdot,t)\|^2 + \left(\sqrt{l}d_0\|u(\cdot,t)\| + \frac{ld_0^2}{4\varepsilon_1}\|u(\cdot,t)\|^2\right)\|a(\cdot,t)\|^2\,. \qquad (16)$$

By using (I_1) and (I_2) we can estimate the second term in the right hand side of (15) as follows

$$\left|\left(\left(A(\cdot,t),a(\cdot,t)\right),U(\cdot,t)\right)\right|\le\int_0^l\left|\widetilde{A}(x,t)\right|\left|a(x,t)\right|\left|U(x,t)\right|dx\le\max_{x\in[0,l]}\left|\widetilde{A}(x,t)\right|\left\|a(\cdot,t)\right\|\left\|U(\cdot,t)\right\|\le$$
$$\le\max_{x\in[0,l]}\left|\widetilde{A}(x,t)\right|^2\left\|a(\cdot,t)\right\|^2+\frac{1}{4}\left\|U(\cdot,t)\right\|^2\le d_0\left\|\widetilde{A}(\cdot,t)\right\|\left\|\widetilde{A}(\cdot,t)\right\|_{H^1(0,l)}\left\|a(\cdot,t)\right\|^2+$$
$$+\varepsilon_2\left\|U_x(\cdot,t)\right\|^2+C_2(\varepsilon_2)\left\|PU(\cdot,t)\right\|^2. \tag{17}$$

Since the expression $\left(\left|A(\cdot,t)\right|^2A(\cdot,t)-\left|\widetilde{A}(\cdot,t)\right|^2\widetilde{A}(\cdot,t),A(\cdot,t)-\widetilde{A}(\cdot,t)\right)$ is non-negative, by using (16) and (17) we can obtain from (15) the inequality:

$$\frac{1}{2}\frac{d}{dt}\left\|a(\cdot,t)\right\|^2+(1-\varepsilon_1)\left\|a_x(\cdot,t)\right\|^2\le|\lambda|\left\|a(\cdot,t)\right\|^2+\frac{ld_0^2}{4\varepsilon_1}\left\|u(\cdot,t)\right\|^2\left\|a(\cdot,t)\right\|^2+$$
$$+d_0\left\|\widetilde{A}(\cdot,t)\right\|\left\|\widetilde{A}(\cdot,t)\right\|_{H^1(0,l)}\left\|a(\cdot,t)\right\|^2+\varepsilon_2\left\|U_x(\cdot,t)\right\|^2+C_2(\varepsilon_2)\left\|PU(\cdot,t)\right\|^2. \tag{18}$$

Multiplying (14) by U we get:

$$\frac{1}{2}\frac{d}{dt}\left\|PU(\cdot,t)\right\|^2-\left\|U(\cdot,t)\right\|^2+\left\|U_x(\cdot,t)\right\|^2=$$
$$=-\alpha\left(\left(\widetilde{A}(\cdot,t),a(\cdot,t)\right),U(\cdot,t)\right)-\alpha\left(\left(a(\cdot,t),A(\cdot,t)\right),U(\cdot,t)\right). \tag{19}$$

It follows from (17) that the right hand side of (19) has the estimate:

$$\left|\alpha\left(\left(\widetilde{A}(\cdot,t),a(\cdot,t)\right),U(\cdot,t)\right)+\left(\left(a(\cdot,t),A(\cdot,t)\right),U(\cdot,t)\right)\right|\le$$
$$\le\alpha d_0\left(\left\|\widetilde{A}(\cdot,t)\right\|\left\|\widetilde{A}_x(\cdot,t)\right\|_{H^1(0,l)}+\left\|A(\cdot,t)\right\|\left\|A(\cdot,t)\right\|_{H^1(0,l)}\right)\left\|a(\cdot,t)\right\|^2+$$
$$+2e\varepsilon_2\left\|U_x(\cdot,t)\right\|^2+2C_2(\varepsilon_2)\left\|PU(\cdot,t)\right\|^2.$$

By using the last inequality and (I_2) we get from (19) the following inequality:

$$\frac{1}{2}\frac{d}{dt}\left\|PU(\cdot,t)\right\|^2+(1-2\varepsilon_2-\varepsilon_3)\left\|U_x(\cdot,t)\right\|^2\le$$
$$\le\alpha d_0\left(\left\|\widetilde{A}(\cdot,t)\right\|\left\|\widetilde{A}_x(\cdot,t)\right\|+\left\|A(\cdot,t)\right\|\left\|A_x(\cdot,t)\right\|\right)\left\|a(\cdot,t)\right\|^2+\left(2C_2(\varepsilon_2)+C_3(\varepsilon_3)\right)\left\|PU(\cdot,t)\right\|^2. \tag{20}$$

Adding (18) and (20) and choosing $\varepsilon_1=\frac{1}{2}$, $3\varepsilon_2+\varepsilon_3=\frac{1}{2}$ we can get:

$$\frac{d}{dt}\left[\left\|a(\cdot,t)\right\|^2+\left\|PU(\cdot,t)\right\|^2\right]\le 2H(t)\left[\left\|a(\cdot,t)\right\|^2+\left\|PU(\cdot,t)\right\|^2\right], \tag{21}$$

where

$$H(t)=|\lambda|+\frac{ld_0^2}{2}\left\|u(\cdot,t)\right\|^2+(\alpha+1)d_0\left\|\widetilde{A}(\cdot,t)\right\|\left\|\widetilde{A}(\cdot,t)\right\|_{H^1(0,l)}+$$
$$+d_0\alpha\left\|A(\cdot,t)\right\|\left\|A(\cdot,t)\right\|_{H^1(0,l)}+3C_2(\varepsilon_2)+C_3'(\varepsilon_3).$$

Due to (9) and (10) $H\in L_1(0,T)$ for each $T>0$. Thus it follows from (21) that

$$\left\|A(\cdot,t)-\widetilde{A}(\cdot,t)\right\|^2+\left\|P\left(u(\cdot,t)-\widetilde{u}(\cdot,t)\right)\right\|^2\le$$
$$\le\exp\left(\int_0^t 2H(s)ds\right)\left(\left\|A_0-\widetilde{A}_0\right\|^2+\left\|P\left(u_0-\widetilde{u}_0\right)\right\|^2\right).$$

So we have proved the following theorem:

Theorem. *If*

So we have proved the following theorem:

Theorem. *If*

$$A_0 \in L_2(0,l), \quad u_0 \in \dot{H}^{-1}_{per}(0,l)$$

then the problem (1)-(4) has a unique weak solution

$$[A,u] \in L_\infty(0,T;L_2(0,l)) \cap L_2\left(0,T;H^1_{per}(0,l)\right) \times L_\infty\left(0,T;\dot{H}^{-1}_{per}(0,l)\right) \cap L_2\left(0,T;\dot{H}^1_{per}(0,l)\right),$$

which continuously depends on initial data.

References

[1]. Duan J., Bu C., Gao H., Taboada M. *On coupled Kuramoto-Sivashinsky and Ginzburg-Landau type model for the Marangony convection.* J.Math.Phys., 38(5), 1997.

[2]. Golovin A.A., Nepomnyashichy A.A., Pismen L.M. *Interaction between short-scale Maragoni convection and long-scale deformational instability.* Phys.Fluids, vol.6, 1994, p.34-48.

[3]. Ladyzhenskaya O.A., Solonnikov V.A., Uraltseva N.N. *Linear and quasilinear equations of parabolic types.* A.M.S., Providence, Rhode Island, 1968.

[4]. Kazhdan D., Shtilman L., Golovin A.A., Pismen L.M. *Nonlinear waves and turbulence in Maragoni convection.* Phys.Fluids, vol. 7, 1995, p. 2679-2685.

[5]. Temam R. *Infinite Dimensional Dynamical Systems in Mechanics and Physics.* Springer, New-York, 1988.

Kalantarov V.K.
Institute of Mathematics and Mechanics of AS Azerbaijan,
9, F.Agayev str., 370141, Baku, Azerbaijan.
Hacettepe University, Department of Mathematics, Faculty of Sciences,
06 532 Beytepe, Ankara, Turkey.
E-mail: varga@hacettepe.edu.tr

Received March 27, 2000; Revised May 24, 2000.
Translated by author.