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ON THE EXISTING OF THE PROBLEM CLASSES WITH NON-SUMMABLE EXPANSIONS BY THE EIGEN FUNCTIONS

Abstract

In this article the irregular boundary-valued problem with the non-summable Fourier series by the eigen functions for the even order differential equation has been constructed.

The series of works is dedicated to the investigation of non-regular boundary value problems for differential expressions of non-keldeshevian type

$$l(y,\lambda) \equiv \sum_{i+j=n} p_i \lambda^i y^{(j)}, \tag{1}$$

(where $p_i = const$, $p_0, p_n \neq 0$, $\sum_{i=1}^{n-1} |p_i| > 0$, λ is complex parameter) with separated boundary conditions

$$\begin{cases} u_j(y) \equiv y^{(\varkappa_j)}(0) = 0, & j = \overline{1,l} \\ u_j(y) \equiv y^{(\varkappa_j)}(1) = 0, & j = l + \overline{1,n} \end{cases} \tag{2}$$

in the space $L_2[0,1]$. In particular, in [1]-[7] was established conditions of k-multiple $(1 \leq k \leq n)$ completeness of system of eigen and adjusting functions (s.e.a.f.) of such problems. The presence of complete system of functions raise the question on convergence of expansions to the Fourier series by this system of functions from domain of determination of operator L corresponding to the problem (1)-(2):

$$\sum_{i=1}^{\infty} c_i Y_i(x,\lambda) = \sum_{i=1}^{\infty} (f, z_i) Y_i(x,\lambda). \tag{3}$$

However, because of exponential growth of Green function $G(x,\xi,\lambda)$ of non-regular boundary value problems the summability of series (3) could be considered only as summability by some method. We will consider Abel's method.

Definition. *We will say, that s.e.f.* $\{Y_i(x,\lambda)\}$ $i = \overline{1,\infty}$ *of the problem have the property of* n*-multiple summability by the Abel's method of the order* $\gamma = (\gamma_1, \ldots, \gamma_p)$ *of the Fourier series for* $F(x) = (f_0(x), \ldots, f_{n-1}(x)) \in L_2^n[0,1] \cap D(L)$, $f_{n-1}(0) = f_{n-1}(1) = 0$, *if in the sense of norm* $L_2[0,1]$ *there exists*

$$\lim_{t \to +0} u_\nu(x,t) = f_\nu(x) \qquad \left(\nu = \overline{0, n-1}\right),$$

where

$$u_\nu(x,t) = \sum_{s=1}^{\infty} \sum_{k=1}^{p} \left(l_{s,k}^{\nu} f_\nu, z_s\right) Y_s \exp\left(-\lambda_{s,k}\right)^{\gamma_k} t. \tag{4}$$

So, in [8] it was shown that for the differential expressions (1) of the order $n = 2m$ the following theorem takes place.

Theorem 1. *Let* $l = m$. *Then, if s.e.f.* $\{Y_i(x,\lambda)\}_{i=\overline{1,\infty}}$ *of the problem (1)-(2) is* n*-multiple complete in* $L_2[0,1]$, *then Fourier series by this system is* n*-multiple summable* *by the Abel's method of the order* $\gamma=(\gamma_1,...,\gamma_p)$ *to the function* $F(x)=(f_0(x),...,f_{n-1}(x))\in L_2^n[0,1]\cap D(L)$, $f_{n-1}(0)=f_{n-1}(1)=0$, *moreover, summability is uniform by* $x\in[0;1]$.

In [9] the analogous result was obtained in case of so-called "right" arrangement of roots $\{k_i\}_{i=\overline{1,n}}$ of characteristic equation

$$p_0k^n+p_1k^{n-1}+\cdots+p_n=0. \qquad (5)$$

In present paper we will show, that arrangement of roots of equation (5) plays essential role in investigated question. So, in case, when arrangement $\{k_i\}_{i=\overline{1,n}}$ is not "right" already in case $k=|2l-n|=2$, the classes with non-summable Abel's method by expansion (3) arises.

In $L_2[0;1]$ for $n=6$ we consider differential expression (1) corresponding to the following arrangement at roots of characteristic equation:

$$k_1=\alpha_1>0,$$
$$k_2=\alpha_1+i\beta_2,$$
$$k_3=\alpha_1+i\beta_3,$$
$$k_4=\alpha_4<\alpha_1,$$
$$k_5=\alpha_1-i\beta_5,$$
$$k_6=\alpha_1-i\beta_6$$

with boundary conditions

$$\begin{cases} u_j(y)\equiv y^{(j)}(0)=0, & j=\overline{0,3} \\ u_j(y)\equiv y^{(j)}(1)=0, & j=\overline{0,1} \end{cases} \qquad (2')$$

As it was shown in [8], the convergence of (4) follows from convergence of integral

$$J=\frac{1}{2\pi i}\oint_{\Gamma} l^{-\lambda^{\gamma} t}\int_0^1 G(x,\xi,\lambda)f(\xi)d\xi d\lambda,$$

more exactly, takes place

Lemma 1. *For Fourier series by s.e.a.f. of operator* L *to be summable by Abel method of the order* $\gamma=(\gamma_1,...,\gamma_p)$ *it is necessary and sufficient the existence in the sense of* L_2 *of the limit*

$$\lim_{t\to+0}J(x,t)=J(x,0),$$

for this the series is uniformly summable if this limit exists uniformly by x.

Therefore we constract and study function $G(x,\xi,\lambda)$ [10]:

$$G(x,\xi,\lambda)=\frac{H(x,\xi,\lambda)}{2\Delta(\lambda)}.$$

It is well-known, that $\Delta(x)\equiv\det\{u_i(y_j)\}_{i,j=\overline{1,6}}$. Denote by J_Δ indicator diagram of function $\Delta(x)$. As it follows from [11], J_Δ is convex quadrangular with vertexes at the points B_i, $B_1=k_2+k_3$; $B_2=k_3+k_4$, $B_3=k_5+k_4$, $B_4=k_5+k_6$.

Let $\{\lambda_i\}_{i=\overline{1,\infty}}$ be an eigen values of problem (1)-(2'). They makes discrete set asymptotically situated at the angles, which contains mean perpendiculars to the sides of indicator diagram J_Δ.

We denote $\widetilde{C}=C\setminus\bigcup_{i=1}^{\infty}U_{\varepsilon}(\lambda_i)$ and all further estimations we will do in $\widetilde{C}$. Let $\lambda=R\exp i\varphi$.

Introduce into consideration the indicator of increasing function $\Delta(\lambda)$:

$$h_{\Delta}(\varphi)=\lim_{R\to+\infty}\frac{\ln|\Delta(R\exp i\varphi)|}{R}.$$

According to Poliah theorem [12] $h_{\Delta}(\varphi)=\varkappa(-\varphi)$, where $\varkappa(-\varphi)$ is supporting function of conjugate diagram. By definition

$$\varkappa(\varphi)=\max_s R_s\cos(\varphi-\theta_s),$$

where

$$R_s=\left|\overline{OB_s}\right|,\quad \theta_s=\arg\overline{OB_s}\qquad \left(s=\overline{1,4}\right).$$

Then due to $\varphi\in S_1:\left\{\varphi\middle|0\le\varphi\le\frac{\pi}{2}+\theta_1\right\}$

$$\varkappa_{\Delta}(\varphi)=\max_s R_s\cos(\varphi-\theta_s)=R_1\cos(\varphi-\theta_1)=\varkappa_1$$

and

$$\varkappa_{\Delta}(\varphi)=\max_s R_s\cos(\varphi-\theta_s)=R_4\cos(\varphi-\theta_1)=\varkappa_2$$

in direction $\varphi\in S_2:\left\{\varphi\middle|-\frac{\pi}{2}-\theta_4\le\varphi\le 0\right\}$.

In remaining sector S_0 the spread in $\pi(\theta_1+\theta_4)$ the supporting function is equal to zero (fig. 1).

Thus, we have

$$\varkappa_1(-\varphi)=h_{\Delta}^1=k_5-k_6\quad\text{in}\quad S_1=\left\{\varphi\middle|0\le\varphi\le\frac{\pi}{2}+|\theta_4|\right\},$$

$$\varkappa_2(-\varphi)=h_{\Delta}^2=k_2+k_3\quad\text{in}\quad S_2=\left\{\varphi\middle|-\left(\frac{\pi}{2}+\theta_1\right)\le\varphi\le 0\right\}.$$

In each of S_1 and S_2 we rewrite $\Delta(\lambda)$ in the form:

$$S_1:\Delta(\lambda)=\lambda^7\cdot A_1\exp\lambda(k_5+k_6)\cdot\left[1+\sum_{i=1}^{14}A_iA_1^{-1}\exp\lambda\left[(k_i+k_j)-(k_5+k_6)\right]\right],$$

$$S_2:\Delta(\lambda)=\lambda^7\cdot A_2\exp\lambda(k_2+k_3)\cdot\left[1+\sum_{i=1}^{14}A_iA_2^{-1}\exp\lambda\left[(k_i+k_j)-(k_2+k_3)\right]\right].$$

By virtue of the fact, that in S_1

$$\operatorname{Re}\lambda\left[(k_i+k_j)-(k_5+k_6)\right]\le 0$$

for all members, and in S_2

$$\operatorname{Re}\lambda\left[(k_i+k_j)-(k_2+k_3)\right]\le 0$$

we obtain

Lemma 2. *The characteristic determinant $\Delta(\lambda)$ of problem (1)-(2') is integer function of the first order of increasing and satisfies to the estimation*

$$|\Delta(\lambda)|_{|\lambda|\to\infty}\ge|\lambda|^7 A_i\exp\lambda\varkappa_i$$

in sectors S_i $(i=1,2)$ *(* $A_i \neq 0$ *are determinants). In the rest part of plane* $\widetilde{C} \setminus S_1 \bigcup S_2$

$$|\Delta(\lambda)|_{|\lambda|\to\infty} \geq |\lambda|^7 A_3 \text{ (} A_3 \text{ is determinant } \neq 0\text{)}.$$

Calculating necessary expressions, for function $H(x,\xi,\lambda)$ we obtain:

$$H(x,\xi,\lambda)=\lambda^2\begin{cases}\sum\limits_{\substack{i,j,s=1\\ i\neq j\neq s}}^{6} M_{ijs}\exp\left[\lambda k_s(x-\xi)+\lambda(k_i+k_j)\right]+\sum\limits_{\substack{i\neq j\neq s\\ i,j,s=1}}^{6} D_{ijs}\times\\ \times\exp\left[\lambda k_s x+\lambda k_j(1-\xi)+\lambda k_i\right], \quad x\geq\xi,\\ \sum\limits_{\substack{i\neq j\neq s\\ i,j,s=1}}^{6} D_{ijs}\exp\left[\lambda k_s x+\lambda k_j(1-\xi)+\lambda k_i\right]+\sum\limits_{\substack{s,i=1\\ i\neq s}}^{6} N_{is}\times\\ \times\exp\left[\lambda k_s(1+x-\xi)+\lambda k_i\right], \quad x<\xi.\end{cases} \tag{6}$$

Takes place

Lemma 3. *Function* $H(x,\xi,\lambda)$ *is integer function of the first order of increasing and have four indicators of increasing.*

Proof. Using (6) we could construct indicator diagram J_H. According to [11] it is a convex hexagon with vertexes at points C_i $\left(i=\overline{1,6}\right)$:

$$C_1=B_1+k_1,$$
$$C_2=B_1;\, C_3=B_2;\, C_4=B_3;\, C_5=B_4,$$
$$C_6=B_4+k_1.$$

So as $J_H \cap J_\Delta \neq \varnothing$, we have (fig. 2):

$$\varkappa_H^1(-\varphi)=\max_s \widetilde{R}_s\left(\varphi-\widetilde{\theta}_s\right)=\widetilde{R}_1\cos\left(\varphi-\widetilde{\theta}_1\right),$$

where

$$\widetilde{R}_1=\left|\overline{OC_1}\right|, \quad \widetilde{\theta}_1=\arg\overline{OC_1}$$

and consequently,

$$\varkappa_H^1=k_1+k_5+k_6 \text{ in the sector } \widetilde{S}_1=\left[0;\frac{\pi}{2}\right],$$

$$\varkappa_H^2(-\varphi)=k_5+k_6 \text{ in the sector } \widetilde{S}_2=\left[\frac{\pi}{2};\frac{\pi}{2}+|\theta_4|\right].$$

Analogously,

$$\varkappa_H^3(-\varphi)=k_1+k_2+k_3 \text{ in the sector } \widetilde{S}_3=\left[\frac{3\pi}{2};2\pi\right],$$

$$\varkappa_H^4(-\varphi)=k_2+k_3 \text{ in the sector } \widetilde{S}_4=\left[-\left(\theta_1+\frac{\pi}{2}\right);\frac{3\pi}{2}\right]$$

and so on.

Using lemmas 1 and 2 it is easy to prove.

Theorem 2. *Green function* $G(x,\xi,\lambda)$ *of the problem (1)-(2') is integer function of the first order of the increasing and satisfies to the following asymptotic* estimations:
1) for $x<\xi$ *and large* λ

$$|G(x,\xi,\lambda)|\leq\frac{M}{|\lambda|^5}, \qquad M=const\neq 0, \tag{7}$$

2) for $x \ge \xi$

$$\left|G(x,\xi,\lambda)\right| \le \frac{M \exp \lambda k_1 (x-\xi)}{|\lambda|^5}, \qquad M = const \ne 0. \tag{8}$$

Proof. As it follows from [11] along any direction of $\varphi \in [0;2\pi]$

$$h_G(\varphi) = h_H(\varphi) - h_\Delta(\varphi).$$

Then in sector $S_1 \cap \widetilde{S}_1 = \left[0;\frac{\pi}{2}\right]$ we have:

$$h_G(\varphi) = k_1 + k_5 + k_6 - (k_5 + k_6) = k_1.$$

Similarly, in $S_1 \cap \widetilde{S}_2$ $\quad h_G(\varphi) = (k_5 + k_6) - (k_5 + k_6) = 0$.

Further, in $S_2 \cap \widetilde{S}_3 = \left[\frac{3\pi}{2};2\pi\right]$

$$h_G(\varphi) = k_1 + k_2 + k_3 - (k_2 + k_3) = k_1$$

and, finally, in $S_2 \cap \widetilde{S}_4$

$$h_G(\varphi) = k_2 + k_3 - (k_2 + k_3) = 0.$$

In rest part of the plane $h_G(\varphi) = 0$. The last means that

$$G(x,\xi,\lambda) = G_1(x,\xi,\lambda) + \frac{M \exp \lambda k_1 (x-\xi)}{\lambda^5},$$

where $\left|G_1(x,\xi,\lambda)\right| \le \frac{M_1}{|\lambda|^5}$ for large λ.

It is obvious, that for $x < \xi$

$$\left|G_1(x,\xi,\lambda)\right| \le \frac{M}{|\lambda|^5}$$

and for $x \ge \xi$, $\operatorname{Re} \lambda k_1 (x-\xi) \ge 0$ and consequently, (8) takes place. And so forth.

So, $G(x,\xi,\lambda)$ is increasing in half-plane $\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ and have one and the same indicator of increasing $h_G(\varphi) = k_1$. It means, that $J(x,t)$ and together with it the series (4) diverges.

The possibility of summation of (4) is giving by choosing number γ, *satisfying* the conditions:

1. $\gamma > 1$, [13]
2. $\operatorname{Re} \lambda^\gamma > 0$,
3. $\operatorname{Re} \lambda k_1 (x-\xi) - \lambda^\gamma t \le 0$.

We will show, that such γ doesn't existists. So as $-\frac{\pi}{2} < \arg \lambda = \varphi < \frac{\pi}{2}$, then

$$-\frac{\pi \cdot \gamma}{2} < \arg \lambda^\gamma = \gamma\varphi < \frac{\gamma\pi}{2},$$

because $\gamma > 1$, then $\operatorname{Re}\lambda^\gamma < 0$ on the interval $\left(-\frac{\pi\gamma}{2};-\frac{\pi}{2}\right)$ as well as on interval $\left(\frac{\pi}{2};\frac{\pi\gamma}{2}\right)$, and condition 2 is broken by this.

Thus, the following theorem takes place

Theorem 3. *Fourier series by s.e.f.* $\{Y_i(x,\lambda)\}_{i=\overline{1,\infty}}$ *of the boundary value problem (1)-(2') is not summable by the Abel's method of the whatever order.*

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