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ON THE BEHAVIOR OF THE SOLUTION OF A NONLINEAR NON-UNIFORMLY ELLIPTIC EQUATION IN NON-SMOOTH DOMAIN

Abstract

In this paper the smothness property of solution of nonlinear nonuniformly elliptic equation in nonsmooth domain is studied.

The paper is devoted to the study of properties of solutions of a nonlinear nonuniformly elliptic equation of the form

$$\frac{d}{dx_i}a_i(x,u,u_x)+a(x,u,u_x)=0 \ . \tag{1}$$

One of the main conditions assumed to be fulfilled for the considered equation is the condition of non-uniform ellipticity

$$\gamma(1+|p|)^{m-2}|\xi|^2 \le \frac{\partial a_i}{\partial p_j}\xi_i\xi_j \le \mu(1+|p|)^{m-2+\alpha}|\xi|^2 . \tag{2}$$

The exponent of non-uniform ellipticity determining besides the exponent $m>1$ the behavior of functions $a_i(x,u,u_x)$ under great p is the exponent α. $m>1$ and $0\le\alpha<1$ are assumed to be in conditions that must satisfy the functions $a_i(x,u,p)$, $a(x,u,p)$.

In case of smooth domains these problems have been studied in papers by A.P. Oskolkov [1], A.V. Ivanov [2], V.N. Tyshlek [3], and Gilbarg and Trudinger [4] and others.

1. In domain Ω with boundary $\partial\Omega$ consider the Dirichlet problem for a nonlinear equation

$$\frac{d}{dx_i}a_i(x,u,u_x)+a(x,u,u_x)=0 \, , \tag{3}$$

$$u|_{\partial\Omega}=0 \tag{4}$$

and assume that the functions $a_i(x,u,p)$ and $a(x,u,p)$ are defined for $x\in\overline{\Omega}$ and arbitrary u,p measurable and subjected to conditions

$$a_i(x,u,p)p_i \ge \gamma(|u|)|p|^{m+\alpha_1}-\mu(|u|)\left(1+|u|^{\alpha_2}\right), \tag{5}$$

$$|u\cdot a(x,u,p)| \le \mu(|u|)\left[\left(1+|u|^{\alpha_3}\right)+\left(1+|u|^{\alpha_4}\right)\cdot|p|^{m+\beta-1}\right]. \tag{6}$$

Here $\gamma(t)$ and $\mu(t)$, $0\le t<\infty$ are non-increasing and non-decreasing strictly positive functions respectively, $m>1$, and exponents $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta$ satisfy the following conditions

$$\alpha_1\ge 0,\ \beta\le\alpha_1+1 \, , \tag{7}$$

$$0\le\alpha_2<\frac{n(m+\alpha_1)}{n-(m+\alpha_1)}, \tag{8}$$

$$0\le\alpha_3<\frac{n(m+\alpha_1)}{n-(m+\alpha_1)}-1 \, , \tag{9}$$

$$0 \le \alpha_4 < \frac{n(\alpha_1 - \beta + 1)}{n - (m + \alpha_1)} - 1 . \qquad (10)$$

The generalized solution of the problem (3)-(4) we shall call the function $u(x)$ from $\overset{\circ}{W}{}^1_{m+\alpha_1}(\Omega)$ satisfying the integral identity

$$\int_\Omega [a_i(x,u,u_x)\eta_{x_i} - a(x,u,u_x)\eta]dx = 0 \qquad (11)$$

for any function $\eta(x) \in \overset{\circ}{W}{}^1_{m+\alpha_1}(\Omega)$.

The domain Ω has a non-smoth boundary, and with regard to the domain the fulfilment of conditions providing the validity of imbedding theorems is required.

Under these conditions it is proved the following

Theorem 1. *Let $u(x) \in \overset{\circ}{W}{}^1_{m+\alpha_1}(\Omega)$ be a generalized solution of the problem (3), (4), and the conditions with respect to the domain and coefficients (5)-(10) be fulfilled. Then the solution of the problem (3)-(4) is bounded and $\max_{\overline{\Omega}}|u|$ is estimated by the constant that depends on data of the problem.*

For the proof, in identity (11) substitute $\eta(x) = u^k(x)$, $k \ge 0$. As a result by applying conditions (5)-(6) we get the following

$$\nu k \int_\Omega |u|^{k-1}|\nabla u|^{m+\alpha_1} dx \le k \int_\Omega |u|^{k-1}\left(1 + |u|^{\alpha_2}\right)dx + \mu \int_\Omega |u|^k \left(1 + |u|^{\alpha_3}\right)dx +$$
$$+ \mu \int_\Omega |u|^k \left(1 + |u|^{\alpha_4}\right)|\nabla u|^{m+\beta-1} dx . \qquad (12)$$

We apply the Young's inequality

$$ab \le \frac{(\delta a)^p}{p} + \frac{b^{p'}}{\delta^{p'} p'} , \quad p' = \frac{p}{p-1} > 1 ,$$

where δ is any positive number, with exponents $p = \dfrac{m + \alpha_1}{m + \beta - 1} > 1$ for $\beta < \alpha_1 + 1$, and $p' = \dfrac{m + \alpha_1}{\alpha_1 - \beta + 1} > 1$, have

$$\mu |u|^{k-1}\left(|u| + |u|^{\alpha_4+1}\right)|\nabla u|^{m+\beta-1} \le \frac{\mu \delta^p}{p}|\nabla u|^{m+\alpha_1}|u|^{k-1} +$$
$$+ \frac{\mu}{p'\delta^{p'}}\left(|u| + |u|^{\alpha_4+1}\right)^{p'}|u|^{k-1} ,$$

δ is chosen such that $\dfrac{\mu\delta}{p} = \dfrac{\gamma k}{2}$.

Then we get from (12)

$$\nu(|u|)\int_\Omega |\nabla u|^{m+\alpha_1}|u|^{k-1} dx \le C_1 \cdot \mu \int_\Omega \left(1 + |u|^{\alpha_2}\right)|u|^{k-1} dx + \frac{\mu}{k}\int_\Omega \left(1 + |u|^{\alpha_3}\right)|u|^k dx +$$
$$+ \frac{C_2}{k}\int_\Omega \left(1 + |u|^{\alpha_4}\right)^{p'}|u|^{k+p'-1} dx .$$

Denoting by $\gamma = \max\{1, \alpha_2, \alpha_3 + 1, p'(\alpha_4 + 1)\}$ and using some inequalities we obtain the estimate

$$\nu \int_\Omega |\nabla u|^{m+\alpha_1} |u|^{k-1} dx \le C_3 \left(\int_\Omega |u|^{k+\gamma-1} dx + 1 \right), \tag{13}$$

where C_3 is some constant.

Consider the integral

$$\int_\Omega |u|^{p_k} dx ,$$

where $p_k = \dfrac{n(m+\alpha_1) + n(k-1)}{n-(m+\alpha_1)}$. For this integral we get following estimate

$$\int_\Omega |u|^{p_k} dx \le C_4 p_k^N \left\{ \int_\Omega |\nabla u|^{m+\alpha_1} |u|^{p_k \frac{n-(m+\alpha_1)}{n} - (m+\alpha_1)} dx \right\}^{\frac{n}{n-(m+\alpha_1)}}. \tag{14}$$

Later, using the well- known method we get the boundedness of the solution.

2. Now assume that with respect to the coefficients it is fulfilled the condition

$$\begin{aligned} &a_i(x,u,p)p_i \ge \gamma(|u|)|p|^{m+\alpha} - C_5 , \\ &\sum_{i=1}^n |a_i(x,u,p)| \le \mu(|u|)|p|^{m+\alpha-1} + C_6 , \\ &|\alpha(x,u,p)| \le \mu(|u|)|p|^{m+\beta-1} + C_7 , \end{aligned} \tag{15}$$

where $m>1$, C_5, C_6, C_7 are positive constants, and exponents α, β satisfy the conditions

$$\begin{aligned} &0 \le \alpha < 1 , \\ &\alpha \le \beta \le \alpha + 1 . \end{aligned} \tag{16}$$

Theorem 2. *Let* $u(x) \in \overset{\circ}{W}{}^1_{m+\alpha}(\Omega)$ *be a generalized solution of the problem (3), (4), and the conditions with respect to the domain and coefficients (15)-(16) be fulfilled. Then the solution of the problem (3)-(4) is Hölderian.*

For the proof, some auxiliary function $\vartheta(x) = \psi(u(x))$ is estimated. For this function it is valid the inequality

$$\int_{\Omega_{2R}} |\nabla \vartheta|^{m+\alpha} \xi^{m+\alpha} dx \le C R^{n-(m+\alpha)} . \tag{17}$$

Later, by the well-known scheme of the proof by the Mozer's method we get the boundedness of $\vartheta(x)$ in Ω_{2R} and the Hölder property of the function $u(x)$.

3. Proving the boundedness of derivatives with respect to coefficients it is required the fulfillment of conditions for $x \in \overline{\Omega}$ and arbitrary u, p of the function $a_i(x,u,p), a(x,u,p)$ are measurable, the functions $a_i(x,u,p)$ are differentiable with respect to x,u,p and all of them are subjected to inequalities

$$\begin{aligned} &\nu(1+|p|)^{m-2}|\xi|^2 \le \frac{\partial a_i}{\partial p_j}\xi_i\xi_j \le \mu(1+|p|)^{m-2+\alpha}|\xi|^2 , \\ &\sum_{i=1}^n \left|\frac{\partial a_i}{\partial u}\right|(1+|p|)^2 + \sum_{i,j=1}^n \left|\frac{\partial a_i}{\partial x_j}\right|(1+|p|) \le \mu(1+|\mu|)^{m+\gamma} , \\ &|a(x,u,p)| \le \mu(1+|\mu|)^m \end{aligned} \tag{18}$$

with $m>1$ and $0<\alpha, \gamma<1$.

The domain Ω is assumed to be n-dimensional parallelepiped.

Under these conditions it is proved the following

Theorem 3. *Let* $u(x) \in \overset{\circ}{W}{}^{1}_{m+\alpha}(\Omega)$ *be a generalized solution of the problem (3), (4), and the condition with respect to the domain and coefficients (18) be fulfilled. Then* $\max\limits_{\Omega_R} |\nabla u|$ *will be bounded.*

Remark. Conditions with respect to the domain could be done less inflexible, more exactly could be considered and n-dimensional conical domains of angle less than π.

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