MATHEMATICS

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GLOBAL SOLVABILITY OF THE MIXED PROBLEM FOR QUASILINEAR HYPERBOLIC EQUATION OF THE FOURTH ORDER WITH INTEGRAL NON-LINEARITY

Abstract

In this article we study the mixed problem for the quasilinear hyperbolic equation of the fourth order. We prove that for the sufficient small initial data this problem has the global solution.

Let $\Omega \subset R^n$ be a bounded domain with the smooth boundary Γ. In the half infinite cylinder $Q = [0,\infty)\times\Omega$ let's consider the mixed problem for the quasilinear hyperbolic equation of the fourth order

$$u_{tt} + \alpha u_t + \Delta^2 u + \sum_{|\alpha|+|\beta|\le 2} (-1)^{|\alpha|} D^\alpha \left(a_{\alpha\beta}(t,x,[u]) D^\beta u\right) = f(t,x,[u]), \tag{1}$$

where $a_{\alpha\beta}(t,x,[u]) = a_{\alpha\beta}\left(t,x,\int_\Omega |u|^2 dx, \int_\Omega |Du|^2 dx, \int_\Omega |D^2 u|^2 dx\right), \alpha = (\alpha_1,\dots,\alpha_n),\ |\alpha| = \alpha_1 + \dots + \alpha_n$

$D^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n},\ D_{x_i} = \frac{\partial}{\partial x_i},\ i = 1,\dots,n,\ \Delta = \sum_{i=1}^n D_{x_i}^2$, with the boundary conditions

$$u(t,x) = 0,\ \Delta u(t,x) = 0,\ (t,x) \in [0,\infty)\times\Gamma \tag{2}$$

and the initial conditions

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x). \tag{3}$$

Before formulation of conditions at which we will investigate the problem (1)-(3), let's remark that J_k^δ will indicate the sphere with radius $\delta > 0$ and with centre at the origin in the space R^k.

Suppose, that the following conditions are satisfied:

1^0. The functions $a_{\alpha\beta}(t,x,\xi)$ are defined for all $(t,x,\xi) \in [0,\infty)\times\overline{\Omega} \in J_3^\delta$, continuously differentiable by t and ξ, twice continuously differentiable by x, and but the function $a'_{\alpha\beta\xi}(t,x,\xi)$ is continuously differentiable by t and twice continuously differentiable by x. For any $(t,x,\xi) \in [0,\infty)\times\overline{\Omega}\times J_3^\delta$ and for any $|\alpha|,|\beta| \le 2 \quad a_{\alpha\beta}(t,x,\xi) = a_{\beta\alpha}(t,x,\xi)$.

There exists the constants $p > 1$ and $c > 0$ such that for all $(t,x,\xi) \in [0,\infty)\times\overline{\Omega}\times J_3^\delta$ the inequalities

$$\left|Da_{\alpha\beta}(t,x,\xi)\right| \le c|\xi|^p$$

are satisfied, where $Dg(t,x,) = \left(g, g_t, Dg, g_\xi, D^2 g, Dg_t, D^2 g_\xi\right)$. For all $(t,x,\xi) \in [0,\infty)\times\Gamma\times J_\delta^3$, $|\alpha|,|\beta| \le 2 \quad a_{\alpha\beta}(t,x,\xi) = 0$.

are satisfied, where $Dg(t,x,)=\left(g,g_t,Dg,g_\xi,D^2g,Dg_t,D^2g_\xi\right)$. For all $(t,x,\xi)\in$ $[0,\infty)\times\Gamma\times J_\delta^3$, $|\alpha|,|\beta|\le 2$ $a_{\alpha\beta}(t,x,\xi)=0$.

2^0. The function $f(t,x,\xi)$ is defined for all $t,x,\xi\in[0,\infty)\times\overline{\Omega}\times J_3^\delta$, and continuously differentiable by t,x,ξ. The function $f'_\xi(t,x,\xi)$ is continuously differentiable by t and twice continuously differentiable by x.

There exist the constants $c_1>0$, $p_1>0$ such that for all $(t,x,\xi)\in[0,\infty)\times\overline{\Omega}\times J_3^\delta$

$$|Df(t,x,\xi)|\le c|\xi|^p.$$

Suppose that for all $(t,x,\xi)\in[0,\infty)\times\Gamma\times J_3^\delta$ $f(t,x,\xi)=0$.

Let $H^k(\Omega)$ is Sobolev space. We denote by H_0^k the subspace of functions from $H^k(\Omega)$ which are equal to zero on Γ. Let's consider the space $H=H_0^2\times L^2(\Omega)$ with the scalar product:

$$\left\langle w^1,w^2\right\rangle=\int_\Omega\Delta u_1\Delta u_2dx+\int_\Omega v_1(x)v_2(x)dx,$$

where $w^i=(u_i,v_i)\in H$, $i=1,2$.

We also determine the space H_1 by:

$$H_1=\left\{(u,v):(u,v)\in H_0^4\times H_0^2,\ \Delta u\in H_0^2\right\}.$$

Let $U_\delta\subset H_1$ be the sphere with radius δ with the centre in zero.

We introduce the denotations:

$$\varepsilon_1(w(t))=\|w(t)\|_H,\ \varepsilon_2(w(t))=\|w(t)\|_{H_1}+\|w'(t)\|_H,$$

where $w(t)\in L_\infty(0,\infty;H_1)$, $w'(t)\in L_\infty(0,\infty;H)$.

Theorem. *Let the condition $1^0,2^0$ are satisfied. Then there exists $\delta_1\in(0,\delta_0)$ such that for any $(u_0,u_1)\in U_{\delta_1}\subset H_1$ the problem (1)-(3) has a unique solution* $u(\cdot)\in C\left([0,\infty);H_0^4\right)\cap C^1\left([0,\infty);H_0^2\right)\cap C^2\left([0,\infty);L^2(\Omega)\right)$.

Before the proof we will give some auxiliary confirmations.

Let's consider in Hilbert space H the bilinear forms:

$$\left\langle z^1,z^2\right\rangle_{Ht,w}=\left\langle z^1,z^2\right\rangle+\sum_{\substack{|\alpha|\le 2\\|\beta|\le 2}}\int_\Omega a_{\alpha\beta}(t,x,[w])D^\alpha z_1^1D^\beta z_1^2dx,$$

where $z^i=\left(z_1^i,z_2^i\right)\in H$, $i=1,2$, $w=(w_1,w_2)\in U_\delta$, $0<\delta_1\le\delta_0$, $t\in[0,\infty)$.

Lemma 1. *For sufficient small $\delta>0$ the set of bilinear forms $\langle\cdot,\cdot\rangle_{H(t,w)}$, $w\in U_\delta, t\in[0,\infty)$ are determined uniformly equivalent scalar products in H and the mapping $(t,w)\to\langle h,h\rangle_{H(t,\omega)}$ satisfies the local Lipshitz condition, i.e. there exist $0<\delta_1<\delta_0$ such that for any $w,w_1,w_2\in U_\delta$, and $h\in H$ the following inequalities are valid:*

$$c_1\|h\|_H\le\|h\|_{H(t,w)}\le c_2\|h\|_H, \tag{4}$$

$$\left|\|h\|_{H(t_1,w_1)}^2-\|h\|_{H(t_2,w_2)}^2\right|\le c\left(\varepsilon_1(w_1(t_1))+\varepsilon_1(w_2(t_2))\right)\left[|t_1-t_2|+\|w_1-w_2\|_H\right]\|h\|_H^2,$$

where $c(\xi)=o\left(\xi^1\right)$, $\xi\to 0, p>1$, $c_1,c_2>0$ and depend only on $\delta_1>0$.

We determine the operator $A(t,w)$ in space H by the following expressions:

$$\begin{cases} D(A(t,w)) = H_1 \\ A(t,w) = \begin{pmatrix} 0 & I \\ -\Delta^2 - \sum\limits_{\substack{|\alpha|\le 2 \\ |\beta|\le 2}} (-1)^{|\alpha|} D^\alpha \left(a_{\alpha\beta}(t,x,[w])D^\beta\right) & \alpha \end{pmatrix} \end{cases}$$

Lemma 2. *For any* $t\in[0,\infty)$, $w\in U_\delta$ *the operator* $A(t,w)$ *generates the strong continuous contraction semi-group in Hilbert space* $H(t,w)$, *and for any* $h\in H_1$

$$\langle A(t,w)h,h\rangle_{H(t,w)} \le 0.$$

Lemma 3. *For any* $h\in H_1$ *the mapping* $(t,w)\to A(t,w)h$ *satisfied the local Lipshitz condition, i.e. for any* $t_1,t_2\in[0,\infty)$, $w_1,w_2\in U_\delta$

$$\left\|[A(t_1,w_2 - A(t_2,w_2))h]\right\|_H \le c(\varepsilon_1(w_1(t)) + \varepsilon_1(w_2(t)))\left[|t_1-t_2| + \|w_1-w_2\|_H\right]\|h\|_{H_1}.$$

Lemma 4. *The operator* A_0 *generates the strong continuous decreasing semi-group* $\exp(tA_0)$ *in space* H, *i.e. there exist the constants* $M\ge 1, \mu>0$ *such that for all* $t\ge 0$

$$\|\exp tA_0\| \le M\exp(-\mu t),$$

where $A_0 = A(0,0)$.

Let's determine the non-linear operator-function $F(t,w)$ as follows:

$$F(t,w) = \begin{pmatrix} 0 \\ f(t,x,[u]) \end{pmatrix}, \ w = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Lemma 5. *The operator-function* $(t,w)\to F(t,w)$ *act from* H_1 *to* H_1 *and satisfies Lipshitz condition in the sense:*

for any $t_1,t_2\in[0,\infty)$, $w_1,w_2\in H_1$ *the following inequality is satisfied:*

$$\|F(t_1,w_1) - F(t_2,w_2)\| \le c(\varepsilon_1(w_1(t)) + \varepsilon_1(w_2(t)))\left[|t_1-t_2| + \|w_1-w_2\|_H\right],$$

where $c(\xi) = O(\xi^p)$, $\xi\to 0$, $p>1$.

Now we can prove Theorem 1.

The problem (1)-(3) is reduced to the problem

$$\begin{cases} w' = A(t,w)w + F(t,w) & (5) \\ w(0) = w_0 & (6) \end{cases}$$

with substitution $u_1 = u$, $u_2 = u_t$, where $w = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $w_0 = \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix}$.

We will investigate the problem (5)-(6) in Hilbert space $H = H_0^2 \times L_2(\Omega)$.

Introduce the denotations

$$W = \{\omega : \omega\in L_\infty(0,\infty;H_1), \omega'\in L_\infty(0,\infty,H)\}.$$

Let $W_\delta \subset W$ be the sphere with the centre in the origin and with the radius δ.

Using Lemma 1 we can prove that for any $w\in W_\delta$, $h\in W$

$$\frac{d}{dt}\left(\|h(t)\|_{H(w(t),t)}\right) \le 2\langle h'(t),h(t)\rangle_{H(w(t),t)} + c(\xi)\|h(t)\|^2, \tag{7}$$

where $\xi = \varepsilon_2(w(t)),\ c(\xi) = O(\xi^p),\ \xi \to 0,\ p > 1$. From Lemma 3 it follows that for any $w \in W_\delta,\ h \in H_1$

$$\|[A_w(t) - A_w(s)]h\| \le c(\varepsilon_2(t))|t - s|\|h\|_{H_1}, \tag{8}$$

where $A_w(t) = A(t, [w(t)])$.

Hence, particularly, we obtain that

$$\left\|\frac{d}{dt}A_w(t)h\right\| \le c(\varepsilon_2(t))\|h\|_{H_1}. \tag{9}$$

Introduce the denotations

$$P_\lambda(t) = P_\lambda(t, w(t)) = A(t, [w])R_\lambda(t, w)$$

where $R_\lambda(t, w) = (I + \lambda A(t, w))^{-1}$ is the resolvent of operator $A(t, w),\ \lambda > 0$. (Note that the existence of $R_\lambda(t, w)$ follows from Lemma 2).

Lemma 6. *$P_\lambda(t)$ is the bounded operator acting in H for any $(t, w) \in [0, \infty) \times W_\delta$ and $\lambda > 0$ and the following estimations are satisfied:*

$$\|P_\lambda(t)\|_{H_1 \to H} \le c,\ \lambda > 0, \tag{10}$$

$$\left\|\frac{d}{dt}P_\lambda(t)\right\|_{H_1 \to H} \le c(\varepsilon_1(w(t))), \tag{11}$$

where $c > 0$ is the constant depending only on $\delta_0 > 0$ and the non-negative function $c(\cdot)$ satisfies the condition: $c(\xi) = 0(\xi^p),\ \xi \to 0,\ p > 1$.

From (7)-(11) follows that the fixed $w \in W_\delta$ the problem

$$\begin{cases} u'(t) = A_w(t)u(t) + F_w(t) & (12) \\ u(0) = u_0 & (13) \end{cases}$$

has a unique solution $u(\cdot) \in c([0, \infty), H_1) \cap c'([0, \infty), H_1)$, where $F_w(t) = F(t, w)$.

Using (7)-(11) we can prove that

$$\varepsilon_2^2(u(t)) \le \bar{c}\varepsilon_2^2(u_0) + c_1\int_0^t \varepsilon_1^{2p}(w(s))\varepsilon_2^2(w(s))[\varepsilon_2^2(u(s)) + \varepsilon_2^2(w(s))]ds. \tag{14}$$

The solution of the problem (12)-(13) satisfies the following integral equation

$$u(t) = e^{tA_0}u_0 + \int_0^t e^{(t-s)A_0}[\widetilde{A}_w(s)u(s) + F_w(s)]ds$$

where $\widetilde{A}_w(t) = A_w(t) - A_0$.

Hence we have

$$\|u(t)\|_H \le M_0 e^{-\alpha t}\|u_0\|_H + \int_0^t e^{-\alpha(t-s)}[\|A_w(s)u(s)\|_H + \|F_w(s)\|_H]ds.$$

Using Lemma 4 and (7)-(10) we obtain from last inequality the following:

$$e^{\alpha t}\varepsilon_1(u(t)) \le M_0\varepsilon_1(u_0) + \int_0^t Me^{\alpha s}\varepsilon_1^p(w(s))(\varepsilon_2(u(s)) + \varepsilon_2(w(s)))ds. \tag{15}$$

Now let's determine the set

$$D_\delta = \left\{ u : u \in W,\ \sup_{t\ge 0} \varepsilon_2(u(t)) \le \delta,\ \sup_{t\ge 0} e^{\alpha t} \varepsilon_1(u(t)) \le \delta \right\}.$$

Let $u_0 \in U_\varepsilon$ such that $\bar{c}\varepsilon_2^2(u_0) \le \varepsilon^2$, $M_0\varepsilon_2(u_0) \le \varepsilon$. Then from (14)-(15) we have

$$\varepsilon_2^2(u(t)) \le \varepsilon^2 + c\int_0^t \varepsilon_1^{\alpha p}(\omega(s))\varepsilon_2^2(\omega(s))\varepsilon_2^2(u(s))ds$$

$$e^{\alpha t}\varepsilon_1(u(t)) \le \varepsilon + M\int_0^t e^{\alpha s}\varepsilon_1^p(\omega(s))[\varepsilon_2(u(s)) + \varepsilon_2(w(s))]ds.$$

Using the last inequalities choose ε and δ such small that for any $w \in D_\delta$ the corresponding solution of the problem (12), (13) belong to D_δ.

Consequently, the sequence $\{u_n\} \subset D_\delta$ can be constructed so:

$$\begin{cases} u_0(t) = u_0, \\ u_n'(t) = A(t, u_{n-1}(t))u_n(t) + F(t, u_{n-1}(t)), \\ u_n(0) = u_0. \end{cases}$$

Further, using Lemmas 1-4 we can prove that the sequence $\{u_n\}$ is fundamental in $C([0,T];H)$.

So as $\{u_n\} \subset D_\delta$, so we can separate the subsequence $\{u_{nk}\}$ from the sequence $\{u_n\}$ from the sequence, that

$u_{nk} \to u$ strong in $C([0,\infty), H)$,

$u_{nk} \to u$ $*$-weak in $L_\infty(0,T;H)$,

$e^{-\alpha t}u_{nk} \to e^{-\alpha t}u$ $*$-weak in $L_\infty(0,T;H)$,

$u'_{nk} \to u'$ $*$-weak in $L_\infty(0,T;H)$,

where $u(\cdot) \in D_{\delta_0} \cap C([0,\infty);H)$.

It is proved that $u(\cdot)$ is the solution of the problem (5)-(6) and $u(\cdot) \in C([0,\infty);H_1) \cap ([0,\infty);H)$.

Using Lemmas 1-4 we prove that the problem (5)-(6) has a unique solution.

References

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