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ON THE FREDHOLM INDEX OF THE KOSZUL COMPLEX FOR MODULES OVER NILPOTENT LIE ALGEBRAS

The conditions for Koszul complexes for modules over finite-dimensional nilpotent Lie algebras to have zero Fredholm index are studied. The main result is the following: if a finite-dimensional nilpotent Lie algebra of bounded operators on a Banach space contains a nonzero compact operator and the related Koszul complex is Fredholm and the kernels of its boundary operators are complementable then its index is zero.

The Koszul complex for modules over Lie algebras and some its modifications were used in several papers [2, 3, 4, 5, 7, 8] to develop spectral theory for noncommutative operator families. In particular a notion of joint essential spectrum was studied in [5] and some results on Fredholm Koszul complexes and the joint essential spectrum were announced in [8].

In this paper we study the index of Fredholm Koszul complexes for modules over nilpotent Lie algebras. Remind that a complex is called to be Fredholm if its homology spaces are finite-dimensional and the index of a Fredholm complex is defined as alternating sum of dimensions of its homology spaces. The results of the paper give conditions for Fredholm Koszul complexes to have zero index.

At first we prove Fredholm versions of some results of [7]. Let E and F be complex finite-dimensional nilpotent Lie algebras and a complex vector space X be a E-module. See below for the exact definition of the Koszul complex $\mathrm{Kos}(E,X)$. It is a corollary of the projection property [7] that if F is a Lie subalgebra of E and $\mathrm{Kos}(F,X)$ is exact then $\mathrm{Kos}(E,X)$ is also exact. We prove that if $\mathrm{Kos}(F,X)$ is Freholm then $\mathrm{Kos}(E,X)$ is Fredholm and its index is zero. Further it is proved in [7] that if $h:F\to E$ is a Lie algebra epimorphism then the Koszul complexes $\mathrm{Kos}(E,X)$ and $\mathrm{Kos}(F,X)$ (the structure of F-module on X is induced by h) are simultaneously exact. We prove here that these complexes are simultaneously Fredholm and if h fails to be an isomorphism then the index of $\mathrm{Kos}(F,X)$ is zero.

Now let $L(X)$ be the algebra of all linear operators on X (operators are considered to be bounded if X is a Banach space) and let $E\subset L(X)$. We prove that if E contains a nonzero finite-dimensional operator then $\mathrm{Kos}(E,X)$ has zero index. The main result is that if X is a Banach space, E contains a nonzero compact operator and $\mathrm{Kos}(E,X)$ is λ-Fredholm (i.e. it has finite-dimensional homology spaces and the kernels of its boundary operators are complementable as subspaces of Banach spaces, in particular if X is a Hilbert space) then the index of $\mathrm{Kos}(E,X)$ is zero.(This is the answer to a question of Yu.V. Turovskii.)

The author wishes to thank Yuri Turovskii, Anar Dosiev and Daniel Beltita for useful discussions.

1. Let E be a finite-dimensional complex Lie algebra, X be a complex vector space, $L(X)$ be the algebra of all linear operators on X. Say that X is E-module if there is a Lie algebra morphism $E\to L(X)$. The action of E on X is denoted by ux with $u\in E$ and $x\in X$. Denote by ΛE the exterior algebra generated by E, by $\Lambda^p E$ the subspace of elements of ΛE of rank p. For definitions of chain complexes, homology spaces, Fredholmness and index of complexes see for example [1]. The chain Koszul complex Kos(E,X) is the complex of spaces

$$0 \longleftarrow X \xleftarrow{\alpha_0} X\otimes E \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_{p-1}} X\otimes \Lambda^p E \xleftarrow{\alpha_p} \dots$$

with $\alpha_{p-1}x\otimes \underline{u} = \sum_{i=1}^{p}(-1)^{i-1}u_i x\otimes \underline{u}^{\wedge i} + \sum_{i<j}(-1)^{i+j-1}x\otimes [u_i,u_j]\wedge \underline{u}^{\wedge i,j}$, $x\otimes\underline{u}\in X\otimes \Lambda^p E$,

$\underline{u}= u_1\wedge\ldots\wedge u_p\in \Lambda^p E$, ^ denotes the omission of the element with the respective index. Because of E is finite-dimensional Kos(E,X) has a finite length. Homology spaces of Kos(E,X) are denoted by $H_p(E,X)$. If Kos(E,X) is Fredholm we denote its index by ind(E,X).

We remind the construction of the cone of a morphism of a chain complex. Let

$$0 \longleftarrow X_0 \xleftarrow{\alpha_0} X_1 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_{p-1}} X_p \xleftarrow{\alpha_p} \dots$$

be a chain complex (X,α) and operators $\beta_p:X_p\to X_p$ define its morphism β i.e. $\alpha_p\beta_{p+1}=\beta_p\alpha_p$. The cone of β is the following complex Con($(X,\alpha),\beta$):

$$0 \longleftarrow X_0 \xleftarrow{\gamma_0} X_1\oplus X_0 \xleftarrow{\gamma_1} \dots \xleftarrow{\gamma_p} X_{p+1}\oplus X_p \xleftarrow{\gamma_{p+1}} \dots$$

with $\gamma_p(x,y)=(\alpha_p x+\beta_p y, -\alpha_{p-1}y)$, $(x,y)\in X_{p+1}\oplus X_p$.

We use below the following well known properties of Euler characteristic formulated here for the index of Fredholm complexes.

Lemma 1. [9] *Let*

$$0\leftarrow (X,\alpha)\leftarrow (Y,\beta)\leftarrow (Z,\gamma)\leftarrow 0$$

be an exact sequence of complexes. If any two of these complexes are Fredholm then the third one is also Fredholm and the following equality is valid:

$$\operatorname{ind}(Y,\beta)=\operatorname{ind}(X,\alpha)+\operatorname{ind}(Z,\gamma).$$

Lemma 2. [9] *If the spaces* $X_0,\ldots,X_n$ *of a complex* (X,α) *are finite-dimensional then* $\operatorname{ind}(X,\alpha)=\sum_{i=0}^{n}(-1)^i \dim X_i$.

Corollary 3. *If the space X is a finite-dimensional module over a finite-dimensional Lie algebra E then* ind(E,X) = 0.

Proof. According to Lemma 2 ind(E,X) = $\sum_{p=0}^{n}(-1)^p C_n^p \dim X = 0$ where $n=\dim E$, C_n^p are binomial coefficients.

The next two lemmas are essentially known.

Lemma 4. *If* (X,α) *is a Fredholm complex and* β *is its morphism then* Con($(X,\alpha),\beta$) *is also Fredholm and its index is zero.*

Proof. There is a short sequence of complexes

$$0\leftarrow (X,\alpha)\xleftarrow{\pi} \operatorname{Con}((X,\alpha),\beta)\xleftarrow{j} (X,\alpha)\leftarrow 0$$

with $j: X_p\to X_p\oplus X_{p-1}$, $jx=(x,0)$, $\pi: X_{p+1}\oplus X_p\to X_p$, $\pi(x,y)=(-1)^p y$. It is a corollary of Lemma 1 that Fredholmness of (X,α) implies Fredholmness of Con($(X,\alpha),\beta$) and its index is equal to the sum of indices of (X,α) with opposite signs i.e. is zero.

Let I be a Lie ideal of E. The space $X\otimes\Lambda I$ is a E-module with respect to the Lie algebra morphism $\theta: E\to L(X\otimes\Lambda I)$,

$$\theta(u)(x\otimes\underline{v}) = ux\otimes\underline{v} + \sum_{i=1}^{p}(-1)^{i-1}x\otimes [u,v_i]\wedge\underline{v}^{\wedge i} \text{ with } x\in X,\ u\in E,\ \underline{v}=v_1\wedge\ldots\wedge v_p\in \Lambda^p I.$$

It is clear that X is a I-module and Kos(I,X) is a subcomplex of Kos(E,X). It is easy to verify (cf. [7]) that the operator $\theta(u)$ is a morphism of Kos(I,X). Here and below the equality of complexes denotes their isomorphism.

Lemma 5. *Let I be an ideal of E of codimension one, $u \in E\backslash I$, X is a E-module. Then*

$$\mathrm{Kos}(E,X) = \mathrm{Con}(\mathrm{Kos}(I,X),\theta(u)) .$$

Proof. cf. [7, Lemma 1.5].

Remind that a Lie algebra E is called to be nilpotent if the decreasing sequence of its ideals $\{E_i\}$ with $E_0 = E$, $E_{i+1} = [E,E_i]$ terminates at some k-th step i.e. $E_k = \{0\}$. It is clear that a nilpotent Lie algebra of dimension n has such a basis $e_1, \dots ,e_n$ that $[e_i,e_{i+p}]$ belongs to the linear span of $e_{i+p+1}, \dots , e_n$.

Proposition 6. *Let E be a finite-dimensional nilpotent Lie algebra, F be its proper Lie subalgebra, X be a E-module. If* Kos(F,X) *is Fredholm then* Kos(E,X) *is also Fredholm and* ind(E,X) *is zero.*

Proof. Let $e_1,\dots,e_n$ be the basis of E with the mentioned property. Denote by Lie(S) the Lie subalgebra of E generated by a set S. Consider the following increasing sequence of Lie subalgebras of E: $L_0 = F$, $L_1 = \mathrm{Lie}(F,e_n)$, $L_2 = \mathrm{Lie}(F,e_{n-1},e_n)$, … , $L_n = \mathrm{Lie}(F,e_1,\dots,e_n) = E$. It is clear that every L_i is an ideal in L_{i+1} and if it is a proper one then its codimension is one. Now the proposition is a consequence of Lemmas 4 and 5.

In what follows up to the end of this section F is a finite-dimensional Lie algebra, $h{:}F \to E$ is a Lie algebra epimorphism which fails to be an isomorphism, G = Ker$h$, X is a E-module and the F-module structure in X is induced by h.

Proposition 7. *Let $[F,G] = 0$. Then the complexes* Kos(F,X) *and* Kos(E,X) *are simultaneously Fredholm and* ind(F,X) = 0.

Proof. We use the construction of [7,Proposition 2.5]. Denote by $\Lambda^q G \wedge \Lambda^p F$ the subspace of $\Lambda^{p+q} F$ generated by $g_1\wedge\dots\wedge g_q\wedge f_1\wedge\dots\wedge f_p$ with $g_i \in G, f_j \in F$. It is easy to prove that the following sequence is exact:

$$0 \leftarrow X \otimes \Lambda^q G \otimes \Lambda^p E \xleftarrow{\hat h} X \otimes \Lambda^q G \wedge \Lambda^p F \xleftarrow{i} X \otimes \Lambda^{q+1} G \wedge \Lambda^{p-1} F \leftarrow 0$$

with i is the inclusion and

$$\hat h\,(x\otimes g_1\wedge\dots\wedge g_q\wedge f_1\wedge\dots\wedge f_p) = x\otimes g_1\wedge\dots\wedge g_q\otimes h(f_1)\wedge\dots\wedge h(f_p) .$$

The operators $\hat h$ and i commute with the relative boundary operators due to the fact that $GX = 0$ and $[G,F] = 0$. So we have the exact sequence of complexes:

$$0 \leftarrow \Lambda^q G \otimes \mathrm{Kos}(E,X) \leftarrow \Lambda^q G \wedge \mathrm{Kos}(F,X) \leftarrow \Lambda^{q+1} G \wedge \mathrm{Kos}(F,X) \leftarrow 0.$$

Consider for every $q \geq 0$ the relative exact sequence of homology spaces (we denote for brevity $H_i(X,E) = H_i$, $H_i(\Lambda^q G \wedge \mathrm{Kos}(F,X)) = H_i^q$, in particular $H_i(X,F) = H_i^0$; it's clear that $H_i(\Lambda^q G \otimes \mathrm{Kos}(E,X)) = H_i \otimes \Lambda^q G$):

$$0 \leftarrow H_0 \otimes \Lambda^q G \leftarrow H_0^q \leftarrow 0 \leftarrow H_1 \otimes \Lambda^q G \leftarrow H_1^q \leftarrow H_0^{q+1} \leftarrow H_2 \otimes \Lambda^q G \leftarrow H_2^q \leftarrow$$
$$H_1^{q+1} \leftarrow \dots \quad .$$

Repeating the arguments of [7] we show that $H_i(X,E)$ are finite-dimensional for every i if and only if $H_i(X,F)$ are finite-dimensional for every i. It remains to show that ind(F,X) = 0.

Denote for brevity $\chi =$ ind(E,X), $\chi_q =$ ind($\Lambda^q G \wedge$Kos(F,X)) in particular $\chi_0 =$ ind(F,X), m = dim G. It is clear that ind($\Lambda^q G \otimes$ Kos(E,X)) = C_m^q. Applying Lemmas 1 and 2 to the exact sequence of homology spaces above we have the following equality for every q, $0\leq q\leq m$: $C_m^q\chi-\chi_q-\chi_{q+1}=0$ (with $\chi_{m+1}=0$). Multiplying each equality by $(-1)^q$ and summing we get:

$$0=\sum_{q=0}^{m}(-1)^q C_m^q\chi=(\chi_0+\chi_1)-(\chi_1+\chi_2)+\ldots+(-1)^{m-1}(\chi_{m-1}+\chi_m)+(-1)^m\chi_m=\chi_0 .$$

The proposition is proved.

Proposition 8. *If the Lie algebra F is nilpotent then the complexes* Kos(E,X) *and* Kos(F,X) *are simultaneously Fredholm and* ind(F,X) = 0.

Proof. We have to repeat the arguments of [7, Proposition 2.6].

2. Now let E be a finite-dimensional nilpotent Lie subalgebra of $L(X)$.

Lemma 9. *Let J be a proper ideal in the algebra L(X) and* $J\cap E\neq 0$. *Then there is nonzero* $k\in J\cap E$ *with* $[k,E]=0$.

Proof. Let $f\in J\cap E$, $f\neq 0$ and $J_n=[\ldots[f,E],\ldots,E]$ (with n brackets). Because E is nilpotent $J_n=0$ and $J_{n-1}\neq 0$ for some n. Take nonzero $k\in J_{n-1}$.

Proposition 10. *If E contains a nonzero finite-dimensional operator and* Kos(E,X) *is Fredholm then* ind(E,X) = 0.

Proof. According to Lemma 9 we may assume that E contains a nonzero finite-dimensional operator k in its center. Let N = Ker$k$, N is invariant for E. We have a short exact sequence of E-modules:

$$0\to N\to X\to X/N\to 0$$

(with the natural inclusion and projection) and the relative exact sequence of complexes:

$$0\to \mathrm{Kos}(E,N)\to \mathrm{Kos}(E,X)\to \mathrm{Kos}(E,X/N)\to 0.$$

Prosition 8 implies that ind(E,N) = 0. Indeed let E_N be the restriction of E to N. Then we have Lie algebra epimorphism $E\to E_N$ which fails to be an isomorphism because k is mapped into zero. Further X/N is finite-dimensional, hence according to Corollary 3 ind(E,X/N) = 0. Hence Lemma 1 implies that ind(E,X) = 0.

Now assume that X is a Banach space and $L(X)$ is the space of all bounded operators. We need some preliminaries on λ- Fredholm essential complexes (cf. [1]).

A sequence (X,α) of Banach spaces X_p and (linear, bounded) operators α_p

$$0\leftarrow X_0\xleftarrow{\alpha_0}X_1\xleftarrow{\alpha_1}\ldots\xleftarrow{\alpha_{n-1}}X_n\leftarrow 0$$

is said to be an essential complex if the products $\alpha_{p-1}\alpha_p$ are compact. There is a natural definition of Fredholmness for essential complexes based on the functor in the category of Banach spaces introduced by B. N. Sadovskii *([10], see also [1, 6]). Remind for* completeness that the Sadovskii functor of a Banach space X is the space $\widetilde{X}=m(X)/k(X)$ where $m(X)$ is the space of all bounded X-valued sequences endowed with the sup-norm, $k(X)$ is its subspace of all precompact sequences. It is proved [6] that an (ordinary) complex of Banach spaces (X,α) is Fredholm if and only if its Sadovskii functor $(\widetilde{X},\widetilde{\alpha})$ is exact. It seems to be natural to call an essential complex (X,α) to be Fredholm if its Sadovskii functor $(\widetilde{X},\widetilde{\alpha})$ is exact. Unfortunately no definition of index for Fredholm essential complexes in this sense is known. There is, however, a restriction of the notion of Fredholmness for essential complexes allowing to define index. This is the following notion of λ-Fredholmness.

Let (X,α) be an essential complex. It is called [1] to be λ-Fredholm if there exist operators β_p: $X_p\to X_{p+1}$ with the following equalities fulfilled for each p:

$$\beta_{p-1}\alpha_{p-1}+\alpha_p\beta_p=1+k_p,$$

where 1 means the identity operator and operators k_p are compact. Note that for ordinary complexes the definition of λ-Fredholmness leads to the condition that the kernels of boundary operators have complements as subspaces of Banach spaces. It is proved in [1] that an essential complex of Hilbert spaces is Fredholm (in the sense of Sadovskii mentioned above) if and only if it is λ-Fredholm.

The following characterization of λ-Fredholmness in terms of exactness given in [1] is essential for us. Let X, Y be Banach spaces, $L(Y,X)$, $K(Y,X)$ be the spaces of all bounded and compact operators from X to Y respectively, $C_Y(X) = L(Y,X)/K(Y,X)$. For a fixed Y $C_Y(.)$ is a functor in the category of Banach spaces transforming bounded operators $a: X_1 \to X_2$ into left multiplication operators:

$$L_a : C_Y(X_1) \to C_Y(X_2),\ L_a(u + K(Y,X_1)) = au + K(Y,X_2),\ u \in L(Y,X_1).$$

It is clear that essential complexes are transformed into ordinary complexes. It is proved in [1] that an essential complex (X,α) is λ-Fredholm if and only if $C_Y(X,\alpha)$ is exact complex for every Y.

The index of λ-Fredholm essential complex is defined in the following way [1]. Define the following operators:

$$\Delta : \bigoplus_p X_{2p} \to \bigoplus_p X_{2p+1},\ \Delta x = (\alpha_{2p-1}x, \beta_{2p}x);$$

$$\Delta' : \bigoplus_p X_{2p+1} \to \bigoplus_p X_{2p},\ \Delta' x = (\alpha_{2p}x, \beta_{2p+1}x).$$

It is easy to verify that $\Delta'\Delta$ and $\Delta\Delta'$ are compact perturbations of triangular operators with identities in the diagonals. Hence these operators are Fredholm with zero indices. Hence Δ and Δ' are Fredholm operators. Note that ind Δ = - ind Δ'. Now define ind(X,α) = ind Δ.

Our main result is a consequence of the following

Proposition 11. *Let $X_0,\dots,X_n$ Banach spaces and there is an essential complex $(X \oplus X, d)$:*

$$0 \leftarrow X_0 \xleftarrow{d_0} X_1 \oplus X_0 \xleftarrow{d_1} X_2 \oplus X_1 \xleftarrow{d_2} \dots \xleftarrow{d_{n-1}} X_n \oplus X_{n-1} \xleftarrow{d_n} X_n \leftarrow 0,$$

where operators d_p are represented by matrices $\begin{pmatrix} \alpha_p & \gamma_p \\ \delta_p & \alpha_{p-1} \end{pmatrix}$ with compact operators γ_p.

If the essential complex $(X \oplus X, d)$ is λ-Fredholm then its index is zero.

Proof. Note that operators $\alpha_p : X_{p+1} \to X_p$ form essential complex (X,α) (take the product of two matrices $d_{p-1}d_p$ and use compactness of it and γ_p). We'll show that (X,α) is λ-Fredholm. Consider the following exact sequence of essential complexes:

$$0 \to (X,\alpha) \xrightarrow{i} (X \oplus X, d) \xrightarrow{j} (X,\alpha) \to 0$$

with the inclusion $i: X_{p-1} \to X_p \oplus X_{p-1}$ and the projection $j: X_p \oplus X_{p-1} \to X_p$. Applying the functor C_Y we get the short exact sequence of ordinary complexes with the related sequence of homology spaces:

$$0 \leftarrow H_0 \leftarrow H'_0 \leftarrow 0 \leftarrow H_1 \leftarrow H'_1 \leftarrow H_0 \leftarrow H_2 \leftarrow H'_2 \leftarrow H_1 \leftarrow \dots \leftarrow H_n \leftarrow H'_n \leftarrow H_{n-1}$$
$$\leftarrow 0 \leftarrow H'_{n+1} \leftarrow H_n \leftarrow 0,$$

where H_i and H_i' are homology spaces of $C_Y(X,\alpha)$ and $C_Y(X\oplus X,d)$ respectively. It is clear that triviality of H_i' implies triviality of H_i (and conversely) hence the complex $C_Y(X,\alpha)$ is exact so the essential complex (X,α) is λ-Fredholm.

Now consider the continuous family of essential complexes $(X\oplus X,\ d(t))$ with spaces $X_p\oplus X_{p-1}$ and operators $d_p(t)=\begin{pmatrix}\alpha_p & t\gamma_p\\ t\delta_p & \alpha_{p-1}\end{pmatrix}$, $t\in[0,1]$. Denote $d(0)$ by $\overline{\alpha}$. The arguments deduced λ-Fredholmness of (X,α) from that of $(X\oplus X,\ d)$ work now in the converse way and deduce λ-Fredholmness of $(X\oplus X,\ d(t))$. The continuity of the index of λ-Fredholm essential complexes [1] implies that $\operatorname{ind}(X\oplus X,\ d)=\operatorname{ind}(X\oplus X,\ \overline{\alpha})$. So we have to prove that

$$\operatorname{ind}(X\oplus X,\ \overline{\alpha})=0\,.$$

Consider operators β_p, Δ and Λ' related to (X,α) (see the definition of λ-Fredholm complexes above) and construct the relative operators for $(X\oplus X,\ \overline{\alpha})$ denoted by $\overline{\beta}_p,\overline{\Delta}$. For $(x,y)\in X_p\oplus X_{p-1}$ set $\overline{\beta}_p(x,y)=(\beta_p x,\beta_{p-1}y)$. It is easy to verify that $\overline{\beta}_{p-1}\overline{\alpha}_{p-1}+\overline{\alpha}_p\overline{\beta}_p=1+\overline{k}_p$ with compact $\overline{k}_p$. Further $\overline{\Delta}$ maps $\bigoplus_p(X_{2p}\oplus X_{2p-1})$ into $\bigoplus_p(X_{2p+1}\oplus X_{2p})$ in the following way: $\overline{\Delta}(x,y)=(\alpha_{2p-1}x,\alpha_{2p-2}y)+(\beta_{2p}x,\beta_{2p-1}y)$.

Regrouping spaces X_p we represent $\overline{\Delta}$ as the direct sum of operators:

$$\Delta\oplus\Delta':(\bigoplus_p X_{2p})\oplus(\bigoplus_p X_{2p+1})\to(\bigoplus_p X_{2p+1})\oplus(\bigoplus_p X_{2p})\,.$$

Hence $\operatorname{ind}\overline{\Delta}=\operatorname{ind}\Delta+\operatorname{ind}\Delta'=0$.

Now let E be a finite-dimensional nilpotent Lie subalgebra of $L(X)$, X be a Banach space.

Proposition 12. *If E contains a nonzero compact operator and* Kos(E,X) *is λ-Fredholm then* ind$(E,X)=0$.

Proof. According to Lemma 9 we may assume that there is a nonzero compact operator k in the center of E. Denote by K the subspace of E generated by k and by F its complement in E. Then $E=F\oplus K$ and hence $\Lambda^pE=\Lambda^pF\oplus(\Lambda^{p-1}F\wedge K)$. It is clear that $\Lambda^pF\approx\Lambda^pF\wedge K$ (because dim$K$ = 1 and K has zero intersection with F). So each space of Kos(E,X) has the following decomposition: $X\otimes\Lambda^pE=(X\otimes\Lambda^pF)\oplus(X\otimes\Lambda^{p-1}F\wedge K)$. So we have to represent boundary operators $d_p:X\otimes\Lambda^{p+1}E\to X\otimes\Lambda^pE$ by matrices $\begin{pmatrix}\alpha_p & \gamma_p\\ \delta_p & \alpha'_{p-1}\end{pmatrix}$, show that γ_p are compact and identify α_p with α'_p in view of the isomorphism $X\otimes\Lambda^pF\approx X\otimes\Lambda^pF\wedge K$. The restriction of d_p to $X\otimes\Lambda^{p+1}F$ is decomposed into the sum of α_p and δ_p in the following way:

$$d_p(x\otimes\underline{u})=\sum(-1)^{i-1}u_ix\otimes\underline{u}^{\wedge i}+\sum(-1)^{i+j-1}x\otimes[u_i,u_j]\wedge\underline{u}^{\wedge i,j}\ ;\ u_i\in F\,.$$

So α_p cuts off the summands of $[u_i,u_j]$ containing k and gives them to δ_p. Then consider the restriction of d_{p+1} to $X\otimes\Lambda^{p+1}F\wedge K$. It is decomposed into the sum of γ_{p+1} and α'_p. We have

$$d_{p+1}(x \otimes \underline{u} \wedge k) =$$
$$= \sum (-1)^{i-1} u_i x \otimes \underline{u}^{\wedge i} \wedge k + (-1)^p kx \otimes \underline{u} + \sum (-1)^{i+j-1} x \otimes [u_i, u_j] \wedge \underline{u}^{\wedge i,j} \wedge k .$$

$\gamma_{p+1}(x \otimes \underline{u} \wedge k) = (-1)^p kx \otimes \underline{u}$, hence γ_{p+1} is compact. Further it is easy to see that α'_p acts as α_p because the factor $\wedge k$ also cuts off the summands of $[u_i, u_j]$ containing k. So it remains to apply Proposition 11 to the complex Kos(E,X) . Proposition is proved.

We finish with the following

Question. Is Proposition 12 valid if Kos(E,X) is Fredholm (not λ-Fredholm) complex?

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Received December 10, 1999; Revised April 26, 2000.
Translated by author.