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EXACT ANNIHILATORS AND THEIR APPLICATIONS IN APPROXIMATION THEORY

Abstract

A brief review on applications of the notion of exact annihilator of a class of functions the solutions of different approximation theory problems is reduced, and by means of a exact annihilator of a class of approximating functions exact constants in bilateral estimates of the best approximation by means of the sums of functions of fewer variables are estimated.

At the beginning of eighties the author has introduced the notion of exact annihilator (EA) of a set of functions that turned out to be very fruitful in applications to solutions of different basic problems of approximation theory. EA of a series of approximation devices by means of which we have solved such important problems as 1) finding the best approximation (b.a), b) construction of formulas for the calculation of the exact meaning of b.a., 3) finding extremal functions, d) establishment of characteristic properties of a class of approximate function in the form of formulas for the calculation of the b.a. were constructed.

A brief review of these results are given in §1. Exact constants in two-sided estimates of the best approximation are established in §2.

§1. The notion of exact annihilator and its applications.

Consider a normed space X and a set $H\subset X$. A set of continuous operators $\{\nabla_\theta\}_{\theta\in T}$, $\nabla_\theta:X\to X$ depending on some parameter $\theta\in T\subset R^n$ we shall call an exact annihilator (EA) of the set H, if

$$f\in H\Leftrightarrow \nabla_\theta f=0\quad \forall\ \theta\in T$$

(Sometimes instead of «a set of operators» $\{\nabla_\theta\}$ we shall simply say «the operator» ∇_θ). The simplest instance of EA is a class of mixed finite differences $\{\Delta_{h_1h_2}f\}_{(h_1,h_2)}\in R^2$ being the EA of set of functions of the form $\varphi(x)+\psi(y)$:

$$f(x,y)=\varphi(x)+\psi(y)\Leftrightarrow \Delta_{h_1h_2}f=0\quad \forall(h_1,h_2)\in R^2 .$$

Let T_n be n- dimensional unique cube $K=I^n$, $I=[0,1]$, $n-$dimensional torus $\pi^n,\pi=[0,2\overline{\pi}]$ or n-dimensional Euclidean space. Let $x=(x_1,\dots x_n)\in T$, $f:T\to\aleph$, where $\aleph=R$ or C;

$$\theta=(\theta_1,\dots\theta_n)\in T;\quad \theta^j=(0,\dots,\theta_j,\dots,0),$$

$$\Delta_j f\overset{df}{=}\Delta_{\theta_j}f=f(x+\theta^j)-f(x),\ \Delta_{ij}=\Delta_i(\Delta_j f),\dots$$

Let D be a set of subsets $\overline{n}=\{1,\dots n\}$, $D=\{\delta_1,\dots,\delta_m\}$, $\delta_i\subset\overline{n}$, $i=\overline{1,m}$; $\delta_i\not\subset\delta_j$, $i\neq j$; $\overline{D}=\bigcup\limits_{\delta\in D}\delta$.

For each $\delta=\left\{\delta^{(1)},\ldots,\delta^{(|\delta|)}\right\}$, where $|\delta|$ - is the number of elements δ, we shall denote the mixed difference $\Delta_{\theta_{\delta^{(1)}}\ldots\theta_{\delta^{(|\delta|)}}}f$ by $\Delta_\delta f=\Delta_{\theta_\delta}f$. Let D be a set of all subsets $\overline{n}$, being the subset of none of $\delta_1,\ldots,\delta_m$: $D=\left\{\delta\subset\overline{n}\,\middle|\,\delta\not\subset\delta_i,\ i=\overline{1,m}\right\}$.

Let $Q=R^n$ or π^n. We associate to the set D the totality of linear operators: for the functions f measurable in Q,

$$\nabla_{D_\theta}f\overset{df}{=}\nabla^Q_{D_\theta}f=\sum_{\delta\in D}\Delta_\delta f\quad,\qquad\theta\in Q$$

and for the functions f, measurable in K

$$\nabla^K_{D_\theta}f=\begin{cases}\sum\limits_{\delta\in D}\Delta_\delta f, & if\quad x,x+\theta\in K\\ 0, & if\quad x\ \ or\ \ x+\theta\notin K\end{cases}$$

Later, by $C(T)$ we shall denote a space of continuous functions f with the norm

$$\|f(x)\|_{C(T)}=\max_{x\in T}|f(x)|,$$

and by $L_p(T)$, $0<p\le\infty$ - a space of functions with the norm

$$\|f\|_p=\left(\int_T|f(x)|^p\,dx\right)^{1/p},\quad 0<p<\infty\ ,$$

$$\|f\|_\infty=vrai\sup_{x\in T}|f(x)|,\qquad p=\infty\ .$$

Consider a group of variables $U_i=x_{\delta_i}=\left\{x_j\,\middle|\,j\in\delta_i\right\}$, $i=\overline{1,m}$ and denote $U=\{u_1,\ldots,u_m\}$. Let $\Sigma_U=\left\{\sum\limits_{i=1}^m\varphi_i(u_i)\right\}$ be a set of sums of real or complex functions of real variables. Consider the classes of functions

$$\Sigma^C_U=\Sigma_U\bigcap C(T);\ \ \Sigma^p_U=\Sigma_U\bigcap L_p(T).$$

The EA of a class of sums of the functions of the fewer number of variables is in the next theorem, and that the same the criterium of representability of many variable functions is established in the form of a sum of functions of fewer number variables in spaces C and L_p, $\quad 0<p\le\infty$

Theorem 1. [2]: *1) Let* $f\in C(T)$. *Then* $f\in\Sigma^C_U\ \Leftrightarrow\nabla^T_{D_\theta}f=0\ \ \forall\theta\in T$;

2) if $f\in L_p(T)$, $0<p\le\infty$, *then*

$f\in\Sigma^p_U\ \Leftrightarrow\ \nabla^T_{D_\theta}f=0$ *for almost all* $(x,\theta)\in T^2$, *where* T^2 *means* I^{2n},π^{2n} *or* R^{2n}.

2. Consider b.a. of the function f on the set T by the class of Σ^Y_U functions of the form $\varphi\overset{df}{=}\sum\limits_{i=1}^m\varphi_i(u_i)$

$$E\left(f,\Sigma^Y_U\right)_{Y(T)}=\inf_{\varphi\in\Sigma^Y_U}\|f-g\|_{Y(T)},$$

where $Y(T)$ - is one the spaces $C(T)$ or $L_p(T)$, $0<p\le\infty$. Define D - continuity module of the function $f\in Y$ on the set T as follows $\omega_D(f)_{Y(T)}=\sup_{\theta\in T}\left\|\nabla^T_{D_\theta}f\right\|_{Y(T)}$

Theorem 2. [2]. *For any function* $f\in Y(T)$

$$E\left(f,\Sigma^Y_U\right)_{Y(T)}=\omega_D(f)_{Y(T)}.$$

Remark. EA is constructed and for the classes of polynomials that admitted to establish the order of corresponding b.a. A quasipolynomial represents finite sum, each summand of which is a polynomial on some group of variables with coefficients depending on other polynomials.

3. Let $C(T)$ be a space of continuous functions on the set $T=X\times Y$, $x\subset R^d$, $Y\subset R^{l-d}$. Consider a class of bilinear forms

$$G=G_C^{m-1}(T)=\left\{g\left|g=\sum_{i=1}^{m}\varphi_i(x)\psi_i(y),\ \varphi_i\in C(X),\ \psi_i\in C(Y)\right.\right\}\text{ and b.a. }E(f,G)_C.$$

Definition. *Let* $\Theta\subset\left(R^d_X R^{l-d}\right)$. *Exact annihilator of the set* $G_C^{M-1}(T)$ *is called any continuous operator* $\nabla=\nabla_\theta$ *from* $C(T)$ *to* $C(\Theta)$ *go that it is valid the relation* $f\in G_C^{M-1}(T)\Leftrightarrow\nabla_\theta f=0\quad\forall_\theta\in\Theta$.

Take such $\Theta=T^M$ that $\theta\in\Theta\Leftrightarrow\theta=(x_1,...,x_M,y_1,...,y_M)$, where

$$x_i\in X\subset R^d,\quad y_i\in Y\subset R^{l-d}$$

and definite the operator $\overset{M}{\nabla}$ on $C(T)$ by the formula

$$\overset{M}{\nabla}f=\left(\overset{M}{\nabla}f\right)_\theta=\det\left\|f(x_i,y_j)\right\|_{i,j=1}^{M}.$$

Theorem 3. [4]: *For each* $M\ge2$ *the operator* $\overset{M}{\nabla}$ *is EA of the set of bilinear forms* $G_C^{M-1}(T)$. *Consider the quantity*

$$\left(\overset{M}{\nabla}\cdot f\right)_\theta=\begin{cases}\dfrac{\left(\overset{M}{\nabla}f\right)_\theta}{\left\|\overset{M-1}{\nabla}f\right\|_{C\left(T^{M-1}\right)}}, & \left(\overset{M}{\nabla}f\right)_\theta\ne0\\[2ex] 0, & \left(\overset{M}{\nabla}f\right)_\theta=0\end{cases}$$

that differs from $\overset{M}{\Delta}f$ *for the factor differing from zero and therefore it is also a EA of a class of bilinear forms* G.

Theorem 4 [4]: *For any function* $f\in C(T)$ *the estimates*

$$\frac{1}{M^2}\left\|\overset{M}{\nabla}\cdot f\right\|_{C\left(T^M\right)}\le E(f,G)_{C(T)}\le\left\|\overset{M}{\nabla}\cdot f\right\|_{C\left(T^M\right)}$$

are valid.

For the lover bound we need the establishment of a geometric substance that is of great interest. To arbitrary fixed $f \in C(T)$ and $\Theta = (x_1,...,x_M, y_1,...,y_M) \in T^M$ we correlate the $\mathcal{M} = \mathcal{M}_{\theta,f}$ in the space

$$l_\infty^M = \left\{ \bar{\xi} = (\xi_1,\dots,\xi_M) \middle| \xi_i \in R,\ i = \overline{1,m} \right\}$$

with the norm $\|\bar{\xi}\|_\infty = \max_{i=\overline{1,m}} |\xi_i|$, by putting

$$\mathcal{M}_{\theta,f} = \left\{ \bar{\xi} = \sum_{i=1}^{M} \eta_i \bar{f}_i \in l_\infty^M \middle| \sum_{i=1}^{M} |\eta_i| \le 1 \right\}.$$

Note that if vectors $\bar{f}_i \in l_\infty^M$ are linearly independent (i.e. $\left(\overset{M}{\nabla} f\right)_\theta \ne 0$), then $\mathcal{M}$ is a body in l_∞^M. We denote by $\nabla_{ij} = \left(\overset{M-1}{\nabla}_{ij} f\right)_\theta$ an algebraic complement of the element $f(x_i, y_j)$ in the determinant $\nabla f = \left(\overset{M}{\nabla} f\right)_\theta$. Note that if we denote by θ_{ij} the element T^{M-1} obtained from $\theta \in T^M$ by eliminating x_i and y_j, then

$$\left(\overset{M-1}{\nabla} f\right)_\theta = (-1)^{i+j} \left(\overset{M-1}{\nabla} f\right)_{\theta_{ij}}.$$

Lemma 1. *For any function $f \in C(T)$ and for any $\theta \in T^M$ a closed ball V_α in l_∞^M of radius*

$$\alpha = \alpha_f(\theta) = \begin{cases} \dfrac{\left|\left(\overset{M}{\nabla} f\right)_\theta\right|}{\sum\limits_{i,j=1}^{M} \left|\left(\overset{M-1}{\nabla}_{ij} f\right)_{\theta_{ij}}\right|}, & \left(\overset{M}{\nabla} f\right)_\theta \ne 0 \\ 0, & \left(\overset{M}{\nabla} f\right)_\theta = 0 \end{cases}$$

with a center in the origin of coordinates is contained in $\mathcal{M}_{\theta,f}$.

4. Cite the result on the calculation of the b.a. by means of EA. Let

$$\Pi = \Pi(a,h) = \left\{ x \in R^n \middle| a_i \le x_i \le a_i + h_i,\ \ i = \overline{1,n} \right\}$$

is n- dimensional parallelepiped. Choosing the numbers $0 = k_0 < k_1 < k_m = n$, denote $K = (k_0,\dots,k_m)$, $|K| = m$. Put $t = (t_1,\dots,t_m)$, $t_j = \left(x_{k_{j-1}+1},\dots,x_{k_j}\right)$, $j = \overline{1,m}$.

Later, let $\mathcal{D}^m = \left\{ \varepsilon = (\varepsilon_1,\dots,\varepsilon_m),\ \ \varepsilon_j = 0,1;\ j = \overline{1,m} \right\}$ be a set of vertices of m dimensional unique cube, denote $\delta(\varepsilon) = \sum_{j=1}^{m} (1 - \varepsilon_j)$. Consider the mopping $g_{(\xi,\tau)}: \mathcal{D}^m \to \Pi(\xi,\tau)$ of the set $\mathcal{D}^m$ into the set of vertices of m- dimensional parallelepiped $\Pi(\xi,\tau)$

$$g_{(\xi,\tau)}(\varepsilon) = \left(\xi_1 + \varepsilon_1\tau_1, \ldots, \xi_{k_1} + \varepsilon_1\tau_{k_1}, \ldots, \xi_{k_{m-1}+1} + \varepsilon_m\tau_{k_{m-1}+1}, \ldots, \xi_{k_m} + \varepsilon_m\tau_{k_m}\right).$$

By $M_K = M_K(\Pi(a,h))$ denote a class of functions $f = f(x): R^n \to R,\ x \in \Pi(a,h)$ for a parallelepiped $\Pi(\xi,\tau) \subset \Pi(a,h)$, satisfying the condition

$$\mathcal{L}_K(f, \Pi(\xi,\tau)) \overset{df}{=} 2^{-|K|} \sum_{\varepsilon \in \mathcal{D}^{|K|}} (-1)^{\delta(\varepsilon)} f\left(g_{(\xi,\tau)}(\varepsilon)\right) \geq 0.$$

The functional $\mathcal{L}_K(\cdot, \Pi(\xi,\tau))$ is a EA of a class of sums of functions depending on $m-1$ group of variables $N_K = \left\{\varphi = \sum_{\nu=1}^{m} \varphi_\nu,\ \varphi_\nu = \varphi_\nu(t \setminus t_\nu)\right.$.

Theorem 5. [1]: $\varphi \in N_K \Leftrightarrow \mathcal{L}_K(\varphi, \Pi(\xi,\tau)) = 0 \quad \forall\ (\xi,\tau) \in T^2$. *Consider a parallelepiped* $Q = [x_1', x_1''; \ldots; x_n', x_n''] = \Pi(x', x'' - x')$ *and introduce notations*

$$\mathcal{L}_K(f, Q) = \begin{bmatrix} t_1'' \ldots t_m'' \\ t_1' \ldots t_m' \end{bmatrix},\ \sum\nolimits_r^{(t)} = \sum_{k=1}^{[\frac{m+1}{2}]} \sum_{I_{2k} \subset \overline{m}} \begin{bmatrix} d_1 \ldots t_{i_1} \ldots t_{i_{2k}} \ldots d_m \\ t_1 \ldots c_{i_1} \ldots c_{i_{2k}} \ldots t_m \end{bmatrix},$$

$$\sum\nolimits_H^{t} = \sum_{k=1}^{[\frac{m+1}{2}]} \sum_{I_{2k} \subset \overline{m}} \begin{bmatrix} d_1 \ldots t_{i_1} \ldots t_{i_{2k-1}} \ldots d_m \\ t_1 \ldots c_{i_1} \ldots c_{i_{2k-1}} \ldots t_m \end{bmatrix},$$

where

$$I_p = \{i_1, \ldots i_p\},\ c_j = \left(a_{k_{j-1}+1}, \ldots, a_{k_j}\right),\ d_j = \left(a_{k_{j-1}+1} + h_{k_{j-1}+1}, \ldots, a_{k_j} + h_{k_j}\right),$$

$[r]$ *is an entire part of* r.

Introduce a class $W_K = W_K(\Pi(a,h))$ of functions $f,\ \forall t \in \Pi(a,h)$ satisfying the conditions $\sum_r^{(t)} \geq 0,\ \ \sum_H^{(t)} \geq 0$.

Lemma 2. *The class* M_K *is an eigen-subset of* W_K.

A calculation example of the b.a. by means of EA is

Theorem 6. [1]: $f \in W_K \Rightarrow E(f, N_K, \Pi(a,h))_C = Z_K(f, (a,h))$.

5. The application of EA turned out to be unexpected for the characterization of a class of approximate functions by means of formulas for the calculation of b.a.

Theorem 7. [1]: $f \in M_K \Leftrightarrow E[f, N_K, \Pi(\xi,\tau)] = \mathcal{L}_K(f, \Pi(\xi,\tau))$

$$\forall \Pi(\xi,\tau) \subset \Pi(a,h).$$

6. Finally it is proved to be that EA is applicable and for the construction of extremal function in approximation theory. Let $\overline{m} = \{1, \ldots, m\}$, and by Q_p we denote «parallelepiped» $Q_p \overset{df}{=} \left[t_1, d_1; \ldots; c_{i_1}, t_{i_1}; \ldots; c_{i_p}, t_{i_p}; \ldots; t_m, d_m\right]$, where $I_p = \{i_1, \ldots, i_p\} \subset \overline{m}$.

Theorem 8. [3]: $f \in W_K \Rightarrow$ *the function*

$$\sum\nolimits_f^0 \overset{df}{=} f - \sum_{p=0}^{m} (-1)^{p+m} \sum_{I_p \subset \overline{m}} \mathcal{L}_K(f, Q_p)$$

is best approximate function in approximation of $E\left[f,N_K,\Pi(a,h)\right]_C$.

§2. Exact constants in two-sided estimates of the best approximation.

The following representability criterion of the function of n variables the sums of functions of $(1\le k<n)$ variables may be obtained from general theorem 1.

Theorem 9. *1) Let* $f\in C(T)$. *In order that the function* f *be presented in the form of*

$$f\equiv\sum_{1\le i_1<\ldots<i_k\le n}\varphi_{i_1\ldots i_k}\left(x_{i_1},\ldots,x_{i_k}\right) \tag{1}$$

it is necessary and sufficient that $\forall\,\theta\in T$ *it is fulfilled the equality*

$$\overset{K}{\nabla}_\theta f\overset{df}{=}\sum_{s=k+1}^{n}\sum_{1\le j_1<\ldots<j_s\le n}\Delta_{\theta_{j_1}}\ldots\theta_{j_s}f=0. \tag{2}$$

2) If $f\in L_p(T)$, $0<p\le\infty$, *then in order that the representation (1) hold, it is necessary and sufficient that (2) is fulfilled almost for all* $(x,\theta)\in T^2$.

Denote

$$f\left(x_i\overset{\vee}{+}\theta_i\right)=f\left(x\setminus x_i,x_i+\theta_i\right)=f\left(x_1,\ldots,x_{i-1},x_i+\theta_i,x_{i+1},\ldots,x_n\right).$$

Analogously define $f\left(x_\delta\overset{\vee}{+}\theta_\delta\right)\overset{df}{=}f\left(x\setminus x_\delta,x_\delta+\theta_\delta\right)$.

Theorem 10. *The operator* $\overset{K}{\nabla}_\theta f$ *being the EA of a class* $\sum_k$ *may be written in the form*

$$\overset{K}{\nabla}_\theta f=f(x+\theta)-\sum_{\substack{\delta\subset\bar n\\|\delta|\le k}}(-1)^{k-|\delta|}\binom{n-|\delta|-1}{k-|\delta|}f\left(x_\delta\overset{\vee}{+}\theta_\delta\right). \tag{3}$$

Proof. It follows from the definition of a mixed difference

$$\Delta_\delta f\overset{df}{=}\Delta_{\theta_\delta}f=\sum_{\nu\subset\delta}(-1)^{|\delta|-|\nu|}f\left(x_\nu\overset{\vee}{+}\theta_\nu\right),$$

and hence in particular it follows that for each set $\nu\subset\delta$ the summoned of the form $f\left(x_\nu\overset{\vee}{+}\theta_\nu\right)$ in $\Delta_\delta f$ only for once with signs +1 or -1. Note that since

$$\delta=\left(\delta^{(1)},\ldots,\delta^{(|\delta|)}\right),\ \Delta_\delta f=\Delta_{\theta_\delta(1)\ldots\theta_\delta(|\delta|)}f,$$

then equalities (1) and (2) may be written respectively in the form

$$f=\sum_{\substack{\delta\subset\bar n\\|\delta|\le k}}\varphi_\delta(x_\delta)\ \text{ and }\ \overset{K}{\nabla}_\theta f=\sum_{\delta\subset\bar n}\Delta_\delta f=0\ . \tag{4}$$

Considering this and by using (4) we get

$$\overset{K}{\nabla}_\theta f=\sum_{\substack{\delta\subset\bar n\\|\delta|>k}}\sum_{\nu\subset\delta}(-1)^{|\delta|-|\nu|}f\left(x_\nu\overset{\vee}{+}\theta_\nu\right). \tag{5}$$

Show that each $f\left(x_\nu \overset{\vee}{+} \theta_\nu\right)$ participates in (3) and (5) by the same way. At the right hand side of (5) the summand $f(x+\theta)$ participates only for once (for $\delta = \nu = \overline{n}$): $(-1)^{n-n} f(x+\theta) = f(x+\theta)$, that agrees with (3). Let $k < s < n$, $\mu = \mu(s) = \{1,\dots,s\}$. The summand $f\left(x_\mu + \theta_\mu\right)$ for $\mu < \delta$ participates in (5) for

$\delta = \mu = \{1,\dots,s\}$, $\delta = \{1,\dots,s,i\}$, $i = \overline{s+1,n}$; $\delta = \mu = \{1,\dots,s,i,j\}$, $s+1 \le i < j \le n$ and etc. After citing these terms we get

$$f\left(x_\mu \overset{\vee}{+} \theta_\mu\right)\left[(-1)^{s-s} + (-1)^{s+1-s}\binom{n-s}{1} + \dots + (-1)^{n-s}\binom{n-s}{n-s}\right] = f\left(x_\mu \overset{\vee}{+} \theta_\mu\right)(1-1)^{n-s} = 0,$$

i.e. the summands of the form $f\left(x_\delta + \theta_\delta\right)$, $\delta = \{1,\dots,|\delta|\}$, $k < |\delta| < n$ in (5) are annihilated, and they are not in (3) at all. Now let $0 \le \nu \le k$, $\eta = \eta(\nu) = \{1,\dots,\nu\}$. The function $f\left(x_\eta \overset{\vee}{+} \theta_\eta\right)$ participates in (3) only in summand corresponding to $\delta = \eta(\nu)$, and the form $(-1)^{k-\nu+1}\binom{n-\nu-1}{k-\nu} f\left(x_\eta \overset{\vee}{+} \theta_\eta\right)$.

It participates also in (5) for each $\delta = \{1,\dots\nu,i_1,\dots,i_l\}$, $1 \le l \le n-\nu$ and the number of such δ is $\binom{n-\nu}{l}$. Thus, $f\left(x_\eta \overset{\vee}{+} \theta_\eta\right)$ participates in (5) as follows

$$\sum_{\substack{\delta\in\overline{n}\\ |\delta|>k}} (-1)^{|\delta|-\nu} f\left(x_\eta \overset{\vee}{+} \theta_\eta\right) = f\left(x_\eta \overset{\vee}{+} \theta_\eta\right)\left[(-1)^{k+1-\nu}\binom{n-\nu}{k+1-\nu} + (-1)^{k+2-\nu}\binom{n-\nu}{k+2-\nu} + \right.$$

$$\left. + \dots + (-1)^{n-\nu}\binom{n-\nu}{n-\nu}\right] = (-1)^{n-\nu} f\left(x_\eta \overset{\vee}{+} \theta_\eta\right) \sum_{s=0}^{n-(k+1)} (-1)^s \binom{n-\nu}{s}.$$

Using the known formula $\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$ continue the equality

$$= (-1)^{n-\nu}(-1)^{n-(k+1)}\binom{n-\nu-1}{n-k-1} f\left(x_\eta \overset{\vee}{+} \theta_\eta\right) = (-1)^{k-\nu+1}\binom{n-\nu-1}{k-\nu} f\left(x_\eta \overset{\vee}{+} \theta_\eta\right).$$

Since all variables x_i, $i = \overline{1,n}$ in (3) and (5) are equivalent, then theorem 10 is proved.

Remark. Equality (3) may be written in the form

$$\overset{K}{\nabla}_\theta f = f(x+\theta) - \sum_{l=0}^{k} (-1)^{k-l}\binom{n-l-1}{k-l} \sum_{1\le i_1<\dots<i_l\le n} f\left(x_1,\dots,x_{i_1}+\theta_{i_1},\dots,x_{i_l}+\theta_{i_l},\dots,x_n\right)$$

Theorem 11. [1]: *1) For any function* $f \in C(K)$

$$\left[N\left(\nabla_{\mathcal{D}_\theta}\right)\right]^{-1} \omega_{\mathcal{D}}(f)_{C(T)} \le E(f,\Sigma)_{C(T)} \le \omega_{\mathcal{D}}(f)_{C(T)}.$$

2) For any function

$$f \in L_p(T),\ 0 < p \le \infty,\ \left[N\left(\nabla_{\mathcal{D}_\theta}\right)\right]^{-\gamma} \omega_{\mathcal{D}}(f)_{L_p(T)} \le E(f,\Sigma)_{L_p(T)} \le 2^{n/p}\omega_{\mathcal{D}}(f)_{L_p(T)},$$

$\gamma = \max(1, 1/p)$, *where* $N(\nabla_{D_\theta})$ *is the amount of summands* f *(with regard to coefficients* f *) in* $\nabla_{\mathcal{D}_\theta} f$ *after citing similar terms. We can easily be convinced that*

$$N\left(\overset{K}{\nabla}_\theta\right) = 1 + \sum_{s=0}^{k} \binom{n-s-1}{k-s} \binom{n}{s}.$$

So, we get

Theorem 12. *The coefficients* $\left[N\left(\overset{K}{\nabla}_\theta\right)\right]^{-1}$ *below in the best approximate estimate in theorem 10 in case of approximation by sums of functions of one, two or* $n-1$ *variables respectively equal to* $\left[N\left(\overset{1}{\nabla}_\theta\right)\right]^{-1} = (2n)^{-1}$; $\left[N\left(\overset{2}{\nabla}_\theta\right)\right]^{-1} = \left[2(n-1)^2\right]^{-1}$ *and*

$$\left[N\left(\overset{n-1}{\nabla}_\theta\right)\right]^{-1} = 2^{-n}.$$

At present the solution of the following problems are perspective:

1) to construct EA for new classes of approximating functions and to solve for them the abovesaid problems;
2) the application of EA for the solution of approximation problems by the Ride functions and Neural Networks sets is attractive.

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