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SOLUTION OF THE PLANE PROBLEMS OF THE NON-HOMOGENEOUS THEORY OF ELASTICITY FOR THE FINITE AREA SLACKENED BY THE CRACKS

Abstract

The method of solution of the plane problems of the non-homogeneous theory of elasticity for the finite body slackened by the system of rectangular cracks. It is supposed that the mechanical characteristics of the p material depend on the coordinates (on temperature). The finding procedure of the coefficients of stress intensivity for the neighbourhood of the ends of the cracks.

At some technological processes particularly at ingot formation, we have to deal with the problems of the non-homogeneous theory of elasticity for the finite area.

Let the crystallizing ingot occupy some area D. On its bound S the external loading are given. There are germ cracks in the ingot which arrange is known from the supplementary metalgraphic analysis.

Thus, let in the plane of the ingot associated with the Cartesian coordinate system XOY, there are N rectangular cracks with the length $2l_k(k=1,2,\dots N)$, the centers O_k of the cracks are determined by the coordinates $z_k^0 = x_k^0 + iy_k^0$. In the points O_k let arrange the origins of the local coordinates $X_k O_k Y_k$. Let joint the axes X_k with the cracks lines and denote by α_k the angles with axis X. The bounds of the cracks are free off the forces.

For visuality of the expressed method let first consider the problem of the theory of elasticity for the ingot with one crack $|x_k| =\leq l_k$, $y_k = 0$.

Note that in the ingot crystallization process because of different temperature conditions in different zones of the casted steel the non-homogeneity of the elastic properties origins (the elastic properties depend on temperature).

Investigation on the stress-strain state is reduced to the solution of the problems of the non-homogeneous theory of elasticity. Moreover, the thermo-visco elasticity, the established creeping and the row of other theories of mechanics of deformable mediums are reduced to the solution of the problems of the non-homogeneous theory of elasticity solving them by the algorithms of differents methods of the subsequental approximations.

Let give the formulation of the problem of the non-homogeneous plane problem of the theory of elasticity written for the stress function F:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2};\ \sigma_y = \frac{\partial^2 F}{\partial x^2};\ \sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y}; \tag{1}$$

$$\Delta(\gamma \Delta F) = \frac{\partial^2 q}{\partial x^2}\cdot\frac{\partial^2 F}{\partial y^2} - 2\frac{\partial^2 q}{\partial x \partial y}\cdot\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 q}{\partial y^2}\cdot\frac{\partial^2 F}{\partial x^2} - \Delta(\beta T); \tag{2}$$

$$F|_L = f_1;\ \frac{\partial F}{\partial n}|_L = f_2, \tag{3}$$

where f_1, f_2 are the given on the contour L of area occupied by the body; the functions $\gamma(x,y), q(x,y), \beta(x,y)$ determine the elastic properties of the ingot material and they are expressed through Young's modulus E and Poisson's coefficient ν by the correlations:

$$\gamma = \frac{1-\nu^2}{E};\ q = \frac{1+\nu}{E};\ \beta = \beta_T(1+\nu). \tag{4}$$

To the problem (2), (3) let use the method of distributions [1]. We introduce the parameter ε by the correlations

$$q = q_0\left[1+\varepsilon\mu_1(x,y)\right], \qquad \gamma = \gamma_0\left[1+\varepsilon\mu_2(x,y)\right];$$
$$\beta = \beta_0\left[1+\varepsilon\mu_3(x,y)\right], \qquad \beta_0 = \beta_T^0(1+\nu_0) = const;$$
$$q_0 = const, \qquad \gamma_0 = const.$$

The boundary-value problem (2), (3) depends on parameter ε and has the form of decomposition by degrees ε.

We will seek the solution of the problem in the form

$$F = \sum_{k=0}^{N} \varepsilon^k F_k \ . \tag{5}$$

In this case we obtain the chain of the boundary-value problems for F_0:

$$F_0\big|_L = f_1; \partial F_0/\partial n/_L = f_2,$$
$$\partial^2 F_0 = \frac{(1-\nu_0)\beta_T^0}{\gamma_0}\Delta T; \tag{6}$$

and the recurrent subsequence of the boundary-value problems for F_k:

$$\Delta^2 F_k = \eta_{k-1};\ F_{k/_L} = 0;\ \partial F_0/\partial n\big|_L = 0;\ k = 1,2,\ldots, \tag{7}$$

where function η_{k-1} has the form

$$\eta_{k-1} = \frac{q_0}{\gamma_0}\left(\frac{\partial^2\mu_1}{\partial x^2}\cdot\frac{\partial^2 F_{k-1}}{\partial y^2} - 2\frac{\partial^2\mu_1}{\partial x\partial y}\cdot\frac{\partial^2 F_{k-1}}{\partial y^2} + \frac{\partial^2\mu_1}{\partial y^2}\cdot\frac{\partial^2 F_{k-1}}{\partial x^2}\right) - \tag{8}$$
$$-\Delta(\mu_2\Delta F_{k-1}) - \Delta(\mu_3 T).$$

The values $\sigma_x^k, \sigma_y^k, \sigma_{xy}^k$ are determined through function F_k by the following formulas

$$\sigma_x^k = \frac{\partial^2 F_k}{\partial};\ \sigma_y^k = \frac{\partial^2 F_k}{\partial x^2};\ \sigma_{xy}^k = -\frac{\partial^2 F_k}{\partial x\partial y};\ k = 0,1,\ldots$$

we seek the solution of the problem of the theory of thermoelasticity in the each approximation in the form of sum:

$$\sigma_x = \sigma_x' + \sigma_x'';\ \sigma_y = \sigma_y' + \sigma_y'';\ \sigma_{xy} = \sigma_{xy}' + \sigma_{xy}'' \ . \tag{9}$$

Here stresses $\sigma_x'', \sigma_y'', \sigma_{xy}''$ are determined by formulas by Kolosov and Muskheleshvily [2]

$$\sigma_x'' + \sigma_y'' = 2\left[\Phi(z) + \overline{\Phi(z)}\right];$$
$$\sigma_y'' - \sigma_x'' + 2i\tau_{xy}'' = 2\left[\bar{z}\Phi'(z) + \Psi(z)\right], \tag{10}$$

where the complex potentials $\Phi(z_k)$ *and* $\Psi(z_k)$ in the coordinate system $X_k O_k Y_k$ have the forms

$$\Phi(z_k)=\frac{1}{2\pi}\int_{-l_k}^{l_k}\frac{g_k(t)}{t-z_k}dt;$$
$$\Psi(z_k)=\frac{1}{2\pi}\int_{-l_k}^{l_k}\left[\frac{\overline{g_k(t)}}{t-z_k}-\frac{tg_k(t)}{(t-z_k)^2}\right]dt; z_k=x_k+iy_k; \tag{11}$$
$$\frac{i(x+1)}{2\mu}g(x)=\frac{\partial}{\partial x}\left[u^+-u^-+i\left(v^+-v^-\right)\right].$$

By the signs «plus» and «minus» the boundary values have been denoted which are taken correspondingly on the upper and below bounds of the crack.

Satisfying the boundary-value conditions on the bound of the body we obtain for determination of $\sigma_x',\sigma_y',\sigma_{xy}'$ the auxiliary problem for the continuous body with the form changed boundary conditions. The stresses $\sigma_x'',\sigma_y'',\sigma_{xy}''$ come into in these boundary conditions and the stresses are expressed through the unknown function $g_k(x)$. Solving the auxiliary problem by the analytical or numerical method, for example, by the Finite Element method. We come to the system of equations for determination of the unknown displacements

$$[K]\{\delta\}=\{F\}, \tag{12}$$

where $\{\delta\}$ is the displacement vector, $\{F\}$ is the loading vector.

There is the unknown function $g_k(x)$ in the right-hand side of (12). For its determination it is necessary to satisfy the boundary conditions on the bound of the crack.

Determining by (10)-(11) the stresses $\sigma_y-i\tau_{xy}$ on the axis X_n and equaling them to zero we will come to the singular integral equation

$$\int_{-l_k}^{l_k}\left[g_k(t)\cdot K_{nk}(t,x)+\overline{g_k(t)}\cdot L_{nk}(t,x)\right]dt=\pi P_n(x);\ |x|<l_k. \tag{13}$$

In our case $n=k$, K_{nk} passes to the Cauchy singular kernel and $L_{nk}=0$. The right-hand side of eq. (13) contains stresses σ_{ij}' expressed by the displacement vector $\{\delta\}$.

It is necessary to join to the singular integral equation the supplementary condition

$$\int_{-l_k}^{l_k}g_k(t)dt=0, \tag{14}$$

which provides the uniqueness of the shifts. Thus, the general equations of the problem (12) and (13) turn out connected and must be solved jointly.

The singular integral equations usually regularized by Carlerman-Bekua with the method of reduction it to Fredholm equation. However, solving the problems representing an interest for applications it is advisable to use one of the methods of the direct solution of singular equations [3].

With help of subtitution of the variable the integral equation can be reduced to the standard form

$$\int_{-1}^{1}\frac{g(t)}{t-x}+\int_{-1}^{1}K(t,x)g(t)dt=\pi P(x),\quad |x|<1.$$

Represent the solution in the form

$$g(x)=g_0(x)\Big/\sqrt{1-x^2}\ .$$

Here $g_0(x)$ is continuous by Holder on [-1,1], moreover the function $g_0(x)$ is substituted by Lagrange interpolation polynomial constructed by Chebishev nodes:

$$g_0(x)=\frac{2}{M}\sum_{m=1}^{M}g_0(t_m)\sum_{r=0}^{M-1}T_r(t_m)T_r(x)-\frac{1}{M}\sum_{m=1}^{M}g_0(t_m);$$
$$t_m=\cos\frac{2m-1}{2M}\pi;\ \ T_r(x)=\cos(r\arccos x);\ \ (m=1,2,\dots,M). \tag{15}$$

Using Gauss quadrature formulas [4] we substitute integral equation and condition (14) by the system M of algebraic equations for determination M unknown $g_0(t_m)$

$$\frac{1}{M}\sum_{m=1}^{M}g_0(t_m)\left[\frac{1}{t_m-x_r}+K(t_m,x_r)\right]=P(x_r);$$
$$\frac{\pi}{M}\sum_{m=1}^{M}g_0(t_m)=0;\ \ x_r=\cos\frac{\pi r}{M};\ \ (r=1,2,\dots,M-1). \tag{16}$$

The systems of equations (12) and (16) must be solved jointly.

The stress intensivity factory are found by the formula

$$K_I^{\pm}+iK_{II}^{\pm}=\pm\sqrt{\pi l}\cdot g_0(\pm 1). \tag{17}$$

Values $g_0(\pm 1)$ are calculated by the correlations

$$g_0(1)=\frac{1}{M}\sum_{m=1}^{M}(-1)^{m+1}\cdot g_0(t_m)\cdot ctg\frac{2m-1}{4M}\pi;$$
$$g_0(-1)=\frac{1}{M}\sum_{m=1}^{M}(-1)^{M+m}\cdot g_0(t_m)\cdot tg\frac{2m-1}{4M}\pi. \tag{18}$$

Let us begin solving the problems on the stress condition of the body with the given system N of arbitrary orientated cracks. By virtue of linearity of the problem the functions

$$\Phi(z)=\frac{1}{2\pi}\sum_{k=1}^{N}\int_{-l_k}^{l_k}\frac{g_k(t)dt}{t-z_k};$$
$$\Psi(z)=\frac{1}{2\pi}\sum_{k=1}^{N}e^{-2i\alpha_k}\int_{-l_k}^{l_k}\left[\frac{\overline{g_k(t)}}{t-z_k}-\frac{\overline{T}_k e^{i\alpha_k}}{(t-z_k)^2}g_k(t)\right]dt; \tag{19}$$
$$T_k=te^{i\alpha_k}+z_k^0;\ \ z_k=e^{-i\alpha_k}\left(z-z_k^0\right).$$

describe the stress condition caused with breaks of the shifts $g_k(x_k)$ on N cracks $|x_k|\le l_k;\ y_k=0,\ (K=1,2,\dots,N)$.

Using the transformation formulas on passing to the new coordinate system [2] we find Kolosov-Muskhelishvily potentials $\Phi_n(z_n)$ and $\Psi_n(z_n)$ in the coordinate system $X_nO_nY_n$

$$\Phi_n(z_{n0})=\frac{1}{2\pi}\sum_{K=1}^{N}\int_{-l_k}^{l_k}\frac{g_k(t)}{t-z_k}dt;$$

$$\Psi_n(z_n)=\frac{1}{2\pi}\sum_{K=1}^{N}e^{2i\alpha_{nk}}\int_{-l_k}^{l_k}\left[\frac{\overline{g_k(t)}}{t-z_k}-\frac{(\overline{T}-\overline{z}_n^0)e^{iak}}{(t-z_k)^2}g_k(t)\right]dt,$$

$$z_k=e^{-iak}\left(z_k e^{iak}+z_n^0-z_k^0\right),\alpha_{nk}=\alpha_n-\alpha_k. \tag{20}$$

Determining by (20) the stresses $\sigma_y - i\tau_{xy}$ on axis X_n and equaling them to zero we will come to the system of the singular integral equations

$$\int_{-l_n}^{l_n}\frac{g_n(t)dt}{t-x}+\sum_{k\neq n}\int_{-l_k}^{l_k}\left[g_k(t)\cdot K_{nk}(t,x)+\overline{g_k(t)}\cdot L_{nk}(t,x)\right]dt=\pi P_n(x);$$

$$|x|<l_n\ (n=1,2,\dots,N),$$

$$K_{nk}(t,x)=\frac{e^{i\alpha_k}}{2}\left(\frac{1}{T_k-X_n}+\frac{e^{-2i\alpha_n}}{\overline{T}_k-\overline{X}_n}\right); \tag{21}$$

$$L_{nk}(t,x)=\frac{e^{-i\alpha_k}}{2}\left[\frac{1}{\overline{T}_k-\overline{X}_n}-\frac{T_k-X_n}{(\overline{T}_k-\overline{X}_n)}\cdot e^{-2i\alpha_n}\right];\ X_n=x\cdot e^{-2i\alpha_n}+z_n^0.$$

Here for convenience the index in X_n has been omitted.

It is necessary to add to the system of integral equations the supplementary conditions

$$\int_{-l_n}^{l_n}g_n(t)dt=0;\ \ (n=1,2,\dots,N). \tag{22}$$

The right-hand side of the system of integral equations contain stresses $\sigma_{ij}^{\prime}$. For their determination as earlier we obtain the auxiliary problem of the theory of elasticity for the continual body with the boundary conditions changed their form and which contain the unknown functions $g_n(x),\ (n=1,2,\dots,N)$.

The auxiliary problem can be solved analytically and numerically. Solving it by the Finite Element Method we obtain the algebraic system (12) where $\{F\}$ is the loading vector containing the unknown functions $g_n(x)$.

Therefore, the system (12) and the system of singular integral equations (21) are connected and must be solved jointly.

Let's substitute the system (21) by the finite system of algebraic equations.

For that let's first transform in the system (21) and in the supplementary conditions (22) the segment of integration to one segment [-1,1].

Using substitution of variables $t=l_n\tau, x=l_n\eta, (|t|<l_n, |x|<l_n)$ let us write the system (21) and the conditions (22)

$$\int_{-1}^{1}\frac{g_n(\tau)d\tau}{\tau-\eta}+\sum_{k\neq n}l_k\int_{-1}^{1}\left[g_k(\tau)\cdot K_{nk}(l_k\tau,l_n\eta)+\right.$$

$$\left.+\overline{g_k(\tau)}\cdot L_{nk}(l_k\tau,l_n\eta)\right]d\tau=\pi P_n(\eta),\ \ |\eta|<1, \tag{23}$$

$$\int_{-1}^{1}g_n(\tau)d\tau=0,\ \ (n=1,2,\dots,N).$$

Represent the solution in the form

$$g_n(\eta) = \frac{u_n(\eta)}{\sqrt{1-\eta^2}}. \tag{24}$$

Here $u_n(\eta)$ are continuous by Holder on [-1,1] moreover the functions $u_n(\eta)$ are substituted by Lagrange interpolation polynomials constructed by Cheibishev nodes.

After transformation of the quadrature formulas we come to the system $N \times M$ of algebraic equations for determination $N \times M$ unknown $u_n(t_m)$:

$$\begin{aligned}
&\frac{1}{M}\sum_{m=1}^{M}\sum_{k=1}^{N} l_k\left[u_k(t_m)\cdot K_{nk}(l_k t_m, l_n x_r) + \overline{u_k(t_m)}\cdot L_{nk}(l_k t_m, l_n x_r)\right] = P_n(x_r);\\
&\sum_{m=1}^{M} u_n(t_m) = 0, \quad (n = 1,2,\dots,N;\ r = 1,2,\dots,M-1).
\end{aligned} \tag{25}$$

Here values t_m and x_r are determined by the former expressions.

The coefficients of stress intensivity are bound by the formulas:

$$K_{In}^{\pm} - iK_{IIn}^{\pm} = \mp\sqrt{\pi d_n}\cdot u_n(\pm 1);$$

$$u_n(1) = \frac{1}{M}\sum_{m=1}^{M}(-1)^{m+1}\cdot u_n(t_m) ctg\frac{2m-1}{4M}\pi;$$

$$u_n(-1) = \frac{1}{M}\sum_{m=1}^{M}(-1)^{M+m}\cdot u_n(t_m) tg\frac{2m-1}{4M}\pi.$$

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