LATIFOV F.S.

REACTION OF THE CYLINDRICAL SHELL AT THE DYNAMIC INTERACTION WITH THE MEDIUM

Abstract

In the paper the reaction of the cylindric shell at the dynamic interaction with the infinite elastic medium and filled by the liquid is investigated. The formulas for the coefficients characterizing the reaction of the cylindric shell contacting with the medium are obtained.

On calculation of thin-walled constructions contacting with the elastic medium, particularly at investigation for the natural oscillations, Vinkler's model is widely used. It can be interpreted as the system of the separate unconnected each other springs with the linear characteristics. Reaction of the elastic medium Q is taken proportional to the deflection of the shell [1]:

$$Q = q \cdot w. \tag{1}$$

Absence of the explicit connection of the coefficient q with the geometrical and elastic parameters of the medium makes difficulties in choose of its numerical value. So the approach is in interest in which q is found from the rigorous mathematical formulation of the problem on interaction of the mediums.

Therefore, for contraction of the elastic medium model let us use the equations of the theory of elasticity. On the surface of the contact by analogy with (1) let us determine the dependence

$$\sigma_{rx} = K_x \cdot u, \quad \sigma_{r\varphi} = K_\varphi \cdot \vartheta, \quad \sigma_{rr} = K_z \cdot w, \tag{2}$$

where K_x, K_φ and K_z are constants subjecting to determination.

The well-known models of the elastic foundation are generalized by the dependence (2). Particularly, for the normal component of the reaction σ_{rr} from (2) Vinkler's model follows, if to consider K_z as constant and is equal to q, or Pasternak's model [1] follows, if $K_z = q + q_0\nabla^2$, where ∇^2 is Laplace two- dimensional operator on the surface $z = 0$.

The principal distinct (2) from the pointed-out models is in that here the connection of the displacements of the shell not only with the normal component of the reaction σ_{rr} but with the tangential components σ_{rx}, $\sigma_{r\varphi}$.

Now we begin to determine the coefficients K_x, K_φ and K_z. Let's consider the natural oscillations of the cylindrical shell, which has the contact with the infinite elastic medium and contains the liquid.

Let us represent the system of equations of the shell motion with respect to the displacements in the form [1]:

$$\sum_{j=1}^{3}\left(L_{ij}(u_j) + h^2 N_{ij}(u_j)\right) = \frac{1-\nu^2}{Eh} q_i, \quad (i = 1,2,3). \tag{3}$$

Here L_{ij} and N_{ij} are the known differential operators of the theory of shells.

$$L_{11} = \frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial \varphi^2}; \quad L_{12} = L_{21} = \frac{1+\nu}{2}\frac{\partial^2}{\partial x \partial \varphi};$$

$$L_{13} = L_{31} = \nu \frac{\partial}{\partial x}; \quad L_{22} = \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varphi^2};$$

$$L_{23} = L_{32} = \frac{\partial}{\partial \varphi}; \quad L_{33} = 1 + b^2\nabla^2\nabla^2; \quad b^2 = \frac{h^2}{12R^2},$$

where ν is Poisson's coefficient of the material of the shell, E is Young's modulus, h is the thickness of the shell, R is the radius of the shell, q_i is the external loading, x, φ are the coordinates.

The motion of the medium is described by the vector equation by Lame in displacement [2]

$$a_l^2 \, grad \, div \vec{s} - a_t^2 \, rot \, rot \vec{s} + \omega^2 \vec{s} = 0. \tag{4}$$

Here a_l, a_t are the spread velocities of the cross and longitudinal waves in the medium. $\vec{s}(s_x, s_\varphi, s_r)$ is the vector of displacement of the medium, ω is the circle frequency of oscillations.

The equation of the motion of the ideal compressible liquid in the cylindric system of coordinates has the form:

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial x^2} + \frac{\omega^2}{a^2}\Phi = 0, \tag{5}$$

where Φ is the potential of the liquid, a is the spread velocity of the small distributions in the liquid.

The equations of the motion of the shell (3), medium (4) and liquid (5) are added with the contact and boundary conditions. At the turn of the medium and the shell the conditions of equality of the displacement components are given:

$$s_x = u_1, \quad s_\varphi = u_2, \quad s_r = u_3 \tag{6}$$

and equality of pressure

$$q_{11} = -\sigma_{rx}, \quad q_{22} = -\sigma_{r\varphi}, \quad q_{33} = -\sigma_{rr} \quad (r = R). \tag{7}$$

The contact stress components $\sigma_{rx}, \sigma_{r\varphi}, \sigma_{rr}$ through the displacement of the medium are determined so [2]

$$\sigma_{rx} = \mu_s\left(\frac{\partial s_x}{\partial r} + \frac{\partial s_r}{\partial x}\right);$$

$$\sigma_{r\varphi} = \mu_s\left[r\frac{\partial}{\partial r}\left(\frac{s_\varphi}{r}\right) + \frac{1}{r}\frac{\partial s_r}{\partial \varphi}\right]; \tag{8}$$

$$\sigma_{rr} = \lambda_s\left(\frac{\partial s_x}{\partial x} + \frac{1}{r}\frac{\partial (r s_r)}{\partial r} + \frac{1}{r}\frac{\partial s_\varphi}{\partial\varphi}\right) + 2\mu_s\frac{\partial s_r}{\partial r}.$$

On the contact surface shell-liquid continuity of the radial velocities and pressures is observed

$$\vartheta_r = \frac{\partial u_3}{\partial t}, \quad \tilde{q}_{33} = -P, \quad \tilde{q}_{11} = \tilde{q}_{22} = 0, \tag{9}$$

where $\tilde{q}_{11}, \tilde{q}_{22}, \tilde{q}_{33}$ are pressures from the shell to the liquid. ϑ_r and P are determined through the potential Φ by the formulas:

$$\vartheta_r = \frac{\partial\Phi}{\partial r}; \quad P = -\rho_0\frac{\partial\Phi}{\partial t}, \tag{10}$$

where ρ_0 is the density of the liquid.

Moreover, in the infinity the vector $\vec{s}$ of displacement of the medium satisfies the condition of the finity of displacements, that is, for $r \to \infty$

$$s_x = s_\varphi = s_r = 0. \qquad (11)$$

Adding the contact conditions (6), (7), (9) to the motion equations (3), (4), (5), we come to the determination of the contact stresses between the shell and the medium and the liquid. Or the problem on finding of the contact stresses is reduced to the joint integration of the equations of the theory of shells, medium and liquid at fulfillment of the pointed out conditions on the surface of their contact.

We take the solution of shell motion equation (3) in the form:

$$\begin{aligned} u_1 &= A\cos n\varphi \cos kx \sin \omega t; \\ u_2 &= B\sin n\varphi \sin kx \sin \omega t; \\ u_3 &= C\cos n\varphi \sin kx \sin \omega t. \end{aligned} \qquad (12)$$

Here n is the number of halfwaves in the circumferencial direction, ω is angular frequency, $\frac{\pi}{k}$ is the length of halfwaves along the generating of cylinder, A, B, C are constants.

The solution of the medium motion equations (4) has the form [1]:

$$\begin{aligned} s_x &= \left[A_s k K_n(\gamma_l r) - \frac{c_s \gamma_t^2}{\mu_t} K_n(\gamma_t r) \right] \cos n\varphi \cos kx \sin \omega t; \\ s_\varphi &= \left[-\frac{A_s n}{r} K_n(\gamma_l r) - \frac{c_s nk}{r\mu_t} K_n(\gamma_t r) - \frac{B_s}{n} \frac{\partial K_n(\gamma_t r)}{\partial r} \right] \sin n\varphi \sin kx \sin \omega t; \\ s_r &= \left[A_s \frac{\partial K_n(\gamma_l r)}{\partial r} - \frac{c_s k}{\mu_t} \frac{\partial K_n(\gamma_t r)}{\partial r} + \frac{B_s n}{r} K_n(\gamma_t r) \right] \cos n\varphi \sin kx \sin \omega t, \end{aligned} \qquad (13)$$

where $\gamma_t^2 = k^2 - \mu_t^2$, $\gamma_l^2 = k^2 - \mu_l^2$, $\mu_t = \frac{\omega}{a_t}$, $\mu_l = \frac{\omega}{a_l}$, A_s, B_s, C_s are the constants, $K_n(x)$ is Bessel's modified function of the n-th order of the second type.

The potential Φ of the liquid has the form [1]:

$$\Phi = \Phi_0 I_n(\gamma_r) \cos n\varphi \sin kx \cos \omega t, \qquad (14)$$

where $I_n(x)$ is Bessel's function, $\gamma^2 = k^2 - \frac{\omega^2}{a^2}$, Φ_0 is the constant. Using (13) for the displacements we can determine the stresses $\sigma_{rx}, \sigma_{r\varphi}, \sigma_{rr}$ by (8). The contact conditions (6) let express the constants A_s, B_s, C_s through A, B, C. Then the components of the contact stresses $\sigma_{rx}, \sigma_{r\varphi}$ and σ_{rr} can be represented in the form:

$$\begin{aligned} \sigma_{rx} &= \widetilde{A}_x \cdot A + \widetilde{B}_x \cdot B + \widetilde{C}_x \cdot C = K_x \cdot A, \\ \sigma_{r\varphi} &= \widetilde{A}_\varphi \cdot A + \widetilde{B}_\varphi \cdot B + \widetilde{C}_\varphi \cdot C = K_\varphi \cdot A, \\ \sigma_{rr} &= \widetilde{A}_r \cdot A + \widetilde{B}_r \cdot B + \widetilde{C}_r \cdot C = K_r \cdot A, \end{aligned} \qquad (15)$$

where

$$K_x = \widetilde{A}_x + \widetilde{B}_x \cdot \frac{B}{A} + \widetilde{C}_x \cdot \frac{C}{A},$$

$$K_\varphi \widetilde{A}_\varphi \cdot \frac{A}{B} + \widetilde{B}_\varphi + \widetilde{C}_\varphi \cdot \frac{C}{B},$$

$$K_r = \widetilde{A}_r \cdot \frac{A}{C} + \widetilde{B}_r \cdot \frac{B}{C} + \widetilde{C}_r \,. \tag{16}$$

The constants $\widetilde{A}_x, \widetilde{B}_x, \widetilde{C}_x, \widetilde{A}_\varphi, \widetilde{B}_\varphi, \widetilde{C}_\varphi, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r$ have the awkward form, so here we don't give them. Fulfillment of (7) using (15), (12) and (3) gives the system of equations with respect to A, B, C. From this system the relations $\frac{A}{C}$ and $\frac{B}{C}$ are determined. Note, that in the expressions for $\widetilde{A}_x, \widetilde{B}_x, \widetilde{C}_x, \widetilde{A}_\varphi, \widetilde{B}_\varphi, \widetilde{C}_\varphi, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r$ there is Bessel's modified function K_n. Using the logarithmic derivative of this function we can simplify these expressions and obtain the approximate formulas for K_x, K_φ, K_r.

In the case when $x << n$ $K_n'(x)/K_n(x)$ has the form [3]:

$$K_n'(x)/K_n(x) = -\frac{n}{x} - \frac{x}{2n} \,. \tag{17}$$

Using (7) gives the following approximate expressions for $\widetilde{A}_x, \widetilde{B}_x, \widetilde{C}_x, \widetilde{A}_\varphi, \widetilde{B}_\varphi, \widetilde{C}_\varphi, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r$:

$$\widetilde{A}_x = \left(n^2 k^{*2} S_l - n^2 \gamma_\tau^{*2} S_t - \varpi_t^{*2} S_l S_t^2\right)\Delta_*^{-1}\mu_s \,;$$

$$\widetilde{A}_\varphi = \mu_s n \alpha^* \gamma_l^{*2} (S_l - S_t)\Delta_*^{-1} \,; \quad \widetilde{A}_r = \left[2n^2 k^* \left(\mu_t^{*2} - \mu_l^{*2}\right) - \frac{\lambda_s}{\mu_s} n^2 k^* \mu_l^{*2} + \right.$$

$$\left. + \left(\frac{\lambda_s}{\mu_s} k^* \mu_l^{*2} - 2k^* \gamma_l^{*2}\right) S_t^{12} + 2k^* \gamma_t^{*2} S_l S_t \right] \Delta_*^{-1}\mu_s \,;$$

$$\widetilde{B}_x = \left(2nk^* \gamma_t^{*2} S_l - k^* \left(\gamma_t^{*2} + k^{*2}\right) S_t^2 + n^2 k^* \mu_t^{*2}\right) \cdot \Delta_*^{-1}\mu_s \,;$$

$$\widetilde{B}_\varphi = \left(2n^2 \gamma_t^{*2} - 2n^3 \mu_t^{*2} - 2n\gamma_t^{*2} S_t - n\mu_t^{*2}\gamma_t^{*2} - 2n^2 \gamma_l^{*2} S_l + 2n\alpha^{*2} S_t^2\right) \cdot \Delta_*^{-1}\mu_s \,;$$

$$\widetilde{B}_r = \left(2n\gamma_t^{*2}\gamma_l^{*2} + 2n^3 \gamma_t^{*2} - \frac{\lambda_s}{\mu_s} n\gamma_t^{*2}\mu_l^{*2} - 2n^2 \mu_t^{*2} + \left(-2n^2 - 2k^{*2}\right)\gamma_l^{*2} S_t - \right.$$

$$\left. - 2n\gamma_t^{*2} S_l + 2k^{*2} S_t^2 \right) \cdot \Delta_*^{-1}\mu_s \,;$$

$$\widetilde{C}_x = \left(2k^* \gamma_t^{*2} S_t S_l - nk^* \gamma_t^{*2} S_t - nk^* \gamma_l^2 S_l\right) \cdot \Delta_*^{-1}\mu_s \,;$$

$$\widetilde{C}_\varphi = \left[\left(2n - k^{*2}\right)\gamma_t^{*2} S_t + \gamma_t^{*2}\left(\gamma_l^{*2} + 2n^2\right) S_l - 2\gamma_t^{*2}(n+1) S_t S_l + \right.$$

$$\left. + 2k^{*2} S_t^2 - 2n^2 k^{*2}\right] \cdot \Delta_*^{-1}\mu_s \,;$$

$$\widetilde{C}_r = \left[- 2nk^{*2} \left(\gamma_t^{*2} + n^2\right) + \gamma_t^{*2} \left(2\gamma_l^{*2} + 2n^2 - \frac{\lambda_s}{\mu_s} \mu_l^{*2} \right) S_t + 2n\gamma_t^{*2} S_l - \right.$$

$$\left. - 2\gamma_t^{*2} (n+1) S_l S_t + 2nk^{*2} S_t^2 \right] \cdot \Delta_*^{-1}\mu_s \,;$$

$$\Delta_* = n\mu_t^{*2} + \gamma_t^{*2} S_l S_t - k^{*2} S_t^2 \,; \quad S_l = -n - \frac{\gamma_l^{*2}}{2n} \,; \tag{18}$$

$$S_t = -n - \frac{\gamma_t^{*2}}{2n} \,; \quad \gamma_l^{*2} = k^{*2} - q_1 z_1 \lambda \,; \quad \gamma_t^{*2} = k^{*2} - z_1 \lambda \,;$$

$z_1 = E/\left(1-\nu^2\right)\rho a_t^2$; $\quad q_1 = \dfrac{a_t^2}{a_l^2}$; $\quad \mu_t^* = \omega R/a_t$; $\quad \mu_l^* = \omega R/a_l$;

$$\lambda = \frac{\left(1-\nu^2\right)\rho\omega^2 R^2}{E}.$$

These formulas let determine K_x, K_φ, K_r by (16).

For illustration in fig. 1 the characteristic dependence of the relative error $q = \dfrac{K_{r\,\text{exact.}} - K_{r\,\text{appr.}} \cdot 100\%}{K_{r\,\text{exact.}}}$ on the number of wave formation k^*, where $K_{r\,\text{exact.}}$, $K_{r\,\text{appr.}}$ have been determine by (16) and (18), correspondingly. The result calculation show that for $n \geq 5$ and any value of k^* the value K_r with error up to 6% can be determined by (18).

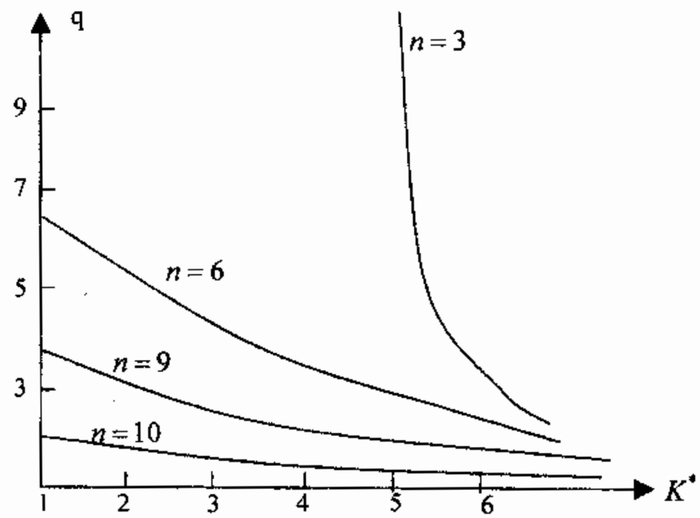

Fig.1. Changing of the relative error q on the dimensionless wave number K^* in direction of the generator-element of the cylinder

In fig. 2 the dependencies of $\dfrac{K_r R}{\mu_s}$ on number of waves n on the circumference is given which (16) and (18) have constructed. The values of $\dfrac{K_r R}{\mu_s}$ calculated by (18) correspond to the dash lines in fig. 2. As it follows from the graphs, the approximate values of $\dfrac{K_r R}{\mu_s}$ from (18) are different from $\dfrac{K_r R}{\mu_s}$ obtained from the mathematically rigorously solution of the problem of the theory of elasticity for the medium (4) for big n.

Using the formulas (10) and (14) we determine the pressure from the liquid to the shell:

$$\widetilde{q}_{33} = -\frac{\rho_0\omega^2 I_n(\gamma R)}{\gamma I_n'(\gamma R)} C \sin kx \sin \omega t = K_G \cdot C \sin kx \sin \omega t,$$

where

$$K_G = -\frac{\rho_0 \omega^2 I_n}{\gamma I_n'(\gamma R)}. \qquad (19)$$

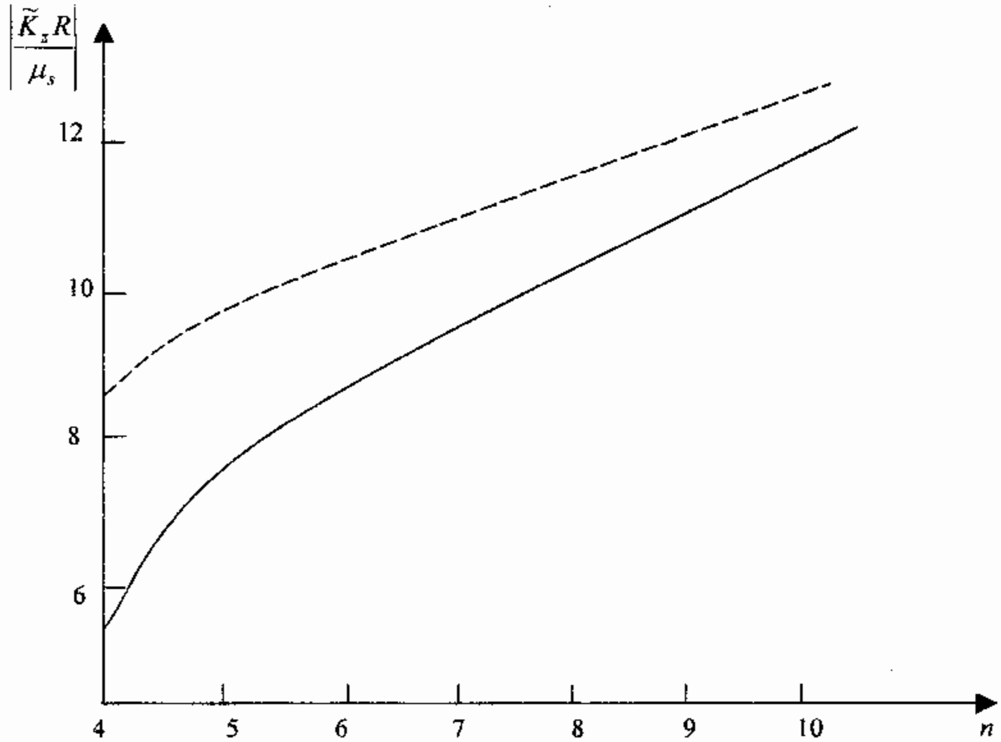

Fig.2. The dependence of the relative amplitude of the reaction of the elastic medium on the number of the waves along the circumference of the cylinder.

Using the asymptotics for the logarithmic derivative of Bessel's function $I_n'(x)/I_n'(x) \approx \frac{n}{x} + \frac{x}{2n}$, which are valid for $x << n$[3], form (19) it is easy to obtain the approximate formula for K_G :

$$K_G = -\rho_0 \omega^2 \left(n + \frac{\gamma^2}{2n} \right).$$

References

[1]. Ильгамов М.А., Иванов В.А., Гулин Б.В. *Расчет оболочек с упругим заполнителем.* Наука, 1987, 260с.
[2]. Аменхаде Ю.А. *Теория упругости.* М., Высшая школа, 1974, 287с.
[3]. Смирнов В.Н. *Курс высшей математики.* М., Наука, т.3, ч.2, 1974, 672с.

Latifov F.S.
Institute of Mathematics and Mechanics of AS Azerbaijan.
9, F. Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

Received January 11, 2000; Revised March 19, 2000.
Translated by Soltanova S.M.