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GEOMETRICAL NON-LINEAR TORSION OF THE ELLIPTIC BAR FABRICATED FROM THE VISCOELASTIC MATERIAL

Abstract

The problem on the geometric nonlinear torsion of the elliptic bar fabricated from the visco-elastic material is solved by the method of sequential approximations.

Let torsional moment $M(t)$ be applied to the prismatic bar with the elliptic cross-section and semiaxis a and b. Let us use the coordinates of the initial condition. Direct the axis OX_3 along the axis of the bar, l is the length of the bar, which we consider rather large; S is the lateral surface; S_1 is the square of the end face of the bar. The components of the tensor of deformation are calculated by the components of the vector of displacement by the formula [1]:

$$2e_{ij} = u_{i,j} + u_{j,i} + u_{S,j} \cdot u_{S,i} \ . \qquad (1)$$

The physical correlations between the components of tensor of deformation and stress are taken in the form [2]:

$$\frac{S_{ij}}{2G_0} = э_{ij} - \int_0^t R(t-\tau)\, э_{ij}(\tau) d\tau , \qquad (2)$$

$$\frac{\sigma}{k} = \theta , \qquad (3)$$

where $э_{ij} = e_{ij} - \delta_{ij} e$ is the deviator of the tensor of

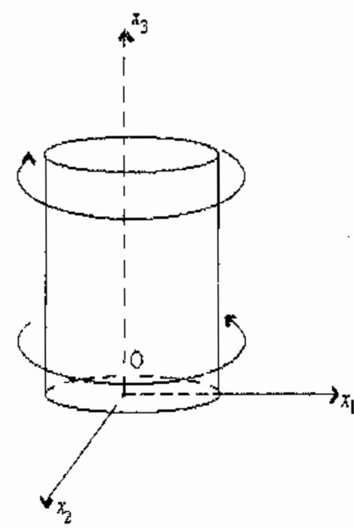


Fig.1.

deformations, $S_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ is the deviator of the tensor of stress; $\theta = e_{kk} = 3e$ is the relative change of volume; $\sigma = \frac{1}{3}\sigma_{kk}$ is mean (hydrostatic) stress; G_0 is the instant elastic modulus of shear; K is the instant modulus of volume deformation, the function (kernel) $R(t-\tau)$ characterizes the rheological properties of the material.

For solving the static problem of the geometric non-linear visco-elasticity we add to the correlations (1)-(3) the equation of equilibrium and the boundary conditions [3]

$$\left[\sigma_{ij}\left(\delta_{ki} + u_{k,j}\right)\right]_{,j} + F_k = 0 , \qquad (4)$$

$$\left[\sigma_{ij}\left(\delta_{ki} + u_{k,j}\right)\right] \cdot l_j = R_{\nu k} \ . \qquad (5)$$

Let us solve the problem in displacements.

According to [4] let us represent the components of the vector of displacement in the form:

$$u_1(x_1, x_2, x_3, t) = -\alpha(t) x_2 x_3 + \alpha^2(t)\left[-\frac{1}{2} x_1^2 x_3^2 + V_1(x_1, x_2, t)\right] ,$$

$$u_2(x_1,x_2,x_3,t)=\alpha(t)x_1x_3+\alpha^2(t)\left[-\frac{1}{2}x_2^2x_3^2+V_2(x_1,x_2,t)\right],$$

$$u_3(x_1,x_2,x_3,t)=\alpha(t)\varphi(x_1,x_2)+\alpha^2(t)c(t)x_3 \quad , \tag{6}$$

where $\alpha(t)$ is the twist, $\varphi(x_1,x_2)$ is the function of torsion corresponding to the linear theory of elasticity; $V_i(x_1,x_2,t)$ and $c(t)$ are the function subjecting to determination.

Using (1)-(3), (6) and (4)-(5) for geometric non-linear function of torsion of the prismatic bar we have the following (at that $F_k=0$, $R_{\nu k}=0$):

equation of equilibrium

$$h_k(u)+F_k'=0 \tag{7}$$

and the boundary conditions on the lateral surface

$$M_k(u)=R_{\nu k}'. \tag{8}$$

Two more integral conditions must be satisfied on the end-faces:

$$\iint_{(S_1)}\sigma_{33}dx_1dx_2=0, \tag{9}$$

$$\iint_{(S_1)}(x_1\sigma_{23}-x_2\sigma_{13})dx_1dx_2=0, \tag{10}$$

where

$$l_k(u)=G_0\nabla^2u_k+a_1\Delta_{ij}+\left(G_0\nabla^2u_k+a_1\Delta_{,i}\right)u_{k,i}+G_0\left(u_{S,k}u_{S,j}+u_{S,k}\cdot u_{S,ij}+\right.$$
$$\left.+u_{i,j}\cdot u_{k,ij}+u_{k,ij}\cdot u_{j,i}\right)+a_3\left(u_{S,S}\cdot u_{k,ij}+u_{r,S}\cdot u_{r,Sk}\right),$$

$$M_k(u)=G_0\left(u_{k,j}+u_{j,k}\right)\cdot l_j+a_3u_{S,S}l_i+a_3\left(u_{r,S}^2+u_{S,k}\cdot u_{k,i}\right)l_i+$$
$$+G_0\left(u_{s,i}\cdot u_{S,k}+u_{k,i}\cdot\left(u_{i,j}+u_{j,i}\right)\right)l_j,$$

$$F_k'=-2G_0\int_0^t\left(R(t-\tau)\ni_{ij}(\tau)\right)_{,j}d\tau+2G_0u_{k,i}\int_0^t\left(R(t-\tau)\ni_{ij}(\tau)\right)_{,j}d\tau,$$

$$R_{\nu k}'=2G_0\int_0^tR(t-\tau)\ni_{ij}(\tau)d\tau\cdot l_j+2G_0w_{k,i}\cdot\int_0^tR(t-\tau)\ni_{ij}(\tau)d\tau\cdot l_j,$$

$$a_1=k+\frac{1}{3}G, \qquad a_3=k-\frac{2}{3}G.$$

In the case of elliptic cross-section we have [5]:

$$\varphi(x_1,x_2)=\chi x_1x_2, \qquad \chi=-\frac{a^2-b^2}{a^2+b^2}.$$

in this case (7), (8) pass to the following correlations:

$$G_0\nabla^2V_1+a_1\Delta_{,1}=L_1,$$
$$G\nabla^2V_2+a_1\Delta_{,2}=-L_2, \tag{11}$$

$$\left[\left(a_2V_{1,1}+a_3V_{2,2}\right)x_1+G\left(V_{1,2}+V_{2,1}\right)\frac{x_2}{\beta^2}\right]_S=-\left[L_3x_1+L_4\frac{x_2}{\beta^2}\right]_S,$$
$$\left[\left(a_2V_{2,2}+a_3V_{1,1}\right)\frac{x_2}{\beta^2}+G\left(V_{1,2}+V_{2,1}\right)x_1\right]_S=-\left[L_4x_1+L_5\frac{x_2}{\beta^2}\right]_S, \tag{12}$$

where

$$L_1=\left(a_1\chi^2+a_3\right)x_1+F_1', \quad L_2=\left(a_3\chi^2+a_3\right)x_2+F_2', \quad \beta=\frac{b}{a},$$

$$L_3 = \frac{1}{2}\left(a_2\chi^2 + a_3\right)x_2^2 + \frac{a_3}{2}\left(\chi^2 + 1\right)x_1^2 + a_3c + R'_\nu \ , \quad L_4 = G_0\chi x_1 x_2 + R'_{\nu 2} \ ,$$

$$L_5 = \frac{1}{2}\left(a_2\chi^2 + a_3\right)x_1^2 + \frac{a_3}{2}\left(\chi^2 + 1\right)x_2^2 + a_3c(t) + R'_{\nu 3}, \tag{13}$$

$$F_1' = -2G_0\int_0^t \left(R(t-\tau)\,\text{э}_{11}(\tau)\right)_{,1} d\tau - 2G_0\int_0^t \left(R(t-\tau)\,\text{э}_{12}(t)\right)_{,2} d\tau - 2G_0\int_0^t \left(R(t-\tau)\,\text{э}_{13}(\tau)\right)_{,3} d\tau,$$

$$F_2' = -2G_0\int_0^t \left(R(t-\tau)\,\text{э}_{21}(\tau)\right)_{,1} d\tau - 2G_0\int_0^t \left(R(t-\tau)\,\text{э}_{22}(t)\right)_{,2} d\tau - 2G_0\int_0^t \left(R(t-\tau)\,\text{э}_{23}(\tau)\right)_{,3} d\tau,$$

$$R'_{\nu 1} = -2G_0\int_0^t R(t-\tau)\,\text{э}_{11}(\tau)d\tau, \quad R'_{\nu 2} = -2G_0\int_0^t R(t-\tau)\,\text{э}_{12}(\tau)d\tau,$$

$$R'_{\nu 3} = -2G_0\int_0^t R(t-\tau)\,\text{э}_{23}(\tau)d\tau.$$

Solving the problem let us use the following approximate method.

Take as the zero approximation the solution of the problem for $R(t-\tau)=0$. Then we have the problem on the geometric non-linear torsion of the elastic elliptic bar. This problem was solved by the author [6] and the solution has the form

$$V_1^{(0)}(x_1,x_2) = b_{11}^{(0)}b^2x_1 + 3b_{12}^{(0)}x_1x_2^2 + b_{13}^{(0)}x_1^3 \ ,$$
$$V_2^{(0)}(x_1,x_2) = b_{11}^{(0)}b^2x_2 + 3b_{22}^{(0)}x_2x_1^2 + b_{23}^{(0)}x_2^3 \ , \tag{14}$$

where $b_{11}^{(0)}, b_{12}^{(0)}, b_{13}^{(0)}, b_{21}^{(0)}, b_{22}^{(0)}, b_{23}^{(0)}$ are the known constants determined through R, G_0, a, b. The constant c and the relation between the angle of twist and the torsional moment are determined from the integral conditions (9), (10):

$$C^{(0)} = \frac{\nu}{1-\nu}b^2\left[b_{111}^{(0)} + b_{211}^{(0)} + \frac{1}{\beta^2}\left(3b_{22}^{(0)} + 3b_{13}^{(0)} + \frac{\chi^2}{2} + \frac{1-\nu}{\nu}\right) + 3b_{23}^{(0)} + \frac{\chi^2}{2} + \frac{1-\nu}{\nu}\right],$$

$$M^{(0)} = \pi G\alpha^{(0)}\frac{a^3b^3}{a^2+b^2}\left[1 + \left(\alpha^{(0)}b\right)^2\gamma\right],$$

where the constant $\gamma = \gamma(k,G,a,b)$.

Substituting $V_1^{(0)}, V_2^{(0)}, c^{(0)}$ in (6) we find the components of displacement $u_i^{(0)}$. Further, substituting them in (1), (2), (3) we determine $e_{ij}^{(0)}, \sigma_{ij}^{(0)}$ *and through them from* (13)- $F_k^{(0)}, R_{\nu k}^{(0)}$.

Substituting all the found expressions in (11), (12) we will obtain the problem of the first approximation which is different from the zero approximation by the existence of the supplementary fixed forces determined by the zero approximation. And in this case the problem is solved similarly to the zero approximation. For any n–th approximation the following problem has place which is also analogous to the zeroth:

$$G_0\nabla^2V_1^{(n)} + a_1\Delta_{,1} = L_1^{(n)} \ ,$$
$$G_0\nabla^2V_2^{(n)} + a_1\Delta_{,2} = L_2^{(n)} \ , \tag{15}$$

$$\left[\left(a_2V_{1,1}^{(n)} + a_3V_{2,2}^{(n)}\right)x_1 + G\left(V_{1,2}^{(n)} + V_{2,1}^{(n)}\right)\frac{x_2}{\beta^2}\right]_S = -\left[L_3^{(n)}x_1 + L_4^{(n)}\frac{x_2}{\beta^2}\right]_S,$$
$$\left[\left(a_2V_{2,2}^{(n)} + a_3V_{1,1}^{(n)}\right)\frac{x_2}{\beta^2} + G\left(V_{1,2}^{(n)} + V_{2,1}^{(n)}\right)x_1\right]_S = -\left[L_4^{(n)}x_1 + L_5^{(n)}\frac{x_2}{\beta^2}\right]_S, \quad (16)$$

$$\int_{(S)}\sigma_{33}^{(n)}dx_1dx_2 = 0, \quad (17)$$

$$\int_{(S)}\left(x_1\sigma_{23}^{(n)} - x_2\sigma_{13}^{(n)}\right)dx_1dx_2 = M. \quad (18)$$

The supplementary fixed forces entering the problem (15)-(18) are known from the solution of the problem of the foregoing approximation.

The solution of the problem (15)-(18) has the form:

$$V_1^{(n)}(x_1,x_2,t) = b_{11}^{(n)}(t)b^2x_1 + 3b_{12}^{(n)}(t)x_1x_2^2 + b_{13}^{(n)}x_1^3,$$
$$V_2^{(n)}(x_1,x_2,t) = b_{21}^{(n)}(t)b^2x_2 + 3b_{22}^{(n)}(t)x_2x_1^2 + b_{23}^{(n)}x_2^3.$$

Here $b_{rs}^{(n)} = b_{rs}^{(n)}(k,G,a,b,t)$, $\left(r=1,2,\ s=\overline{1,3}\right)$.

Beginning from the first approximation the sought functions depend on time.

After determination of the values $V_1^{(n)}, V_2^{(n)}, c^{(n)}(t), \alpha^{(n)}(t)$ we find the components of the displacement by (6). Then by (1)-(3) we determine the components of the tensors of deformation and stress.

The numerical calculation has been carried out. The values of the parameters are taken:

$M=300$ kg cm, $M=450$ kg cm, $M=500$ kg cm, $\alpha(t)-\frac{1}{M}$, $G=3\cdot10^6$ kg cm, $\nu=0{,}3$, $\beta=0{,}6$, $b=0{,}6$ cm.

The graphic of dependence of the twist on time is shown in fig. 2 for the following values of the torsional moment $M=300$ kg m, $M=450$ kg m, $M=500$ kg m.

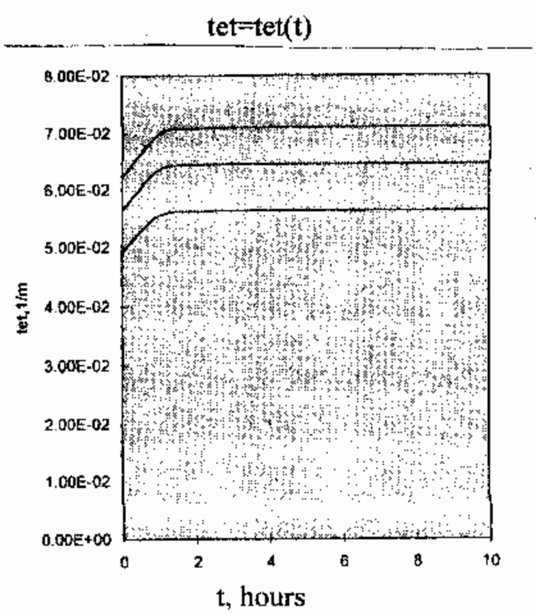


Fig. 2.

Moreover, the graphic of dependence of the twist on the values of the torsional moment for $t = 2\,h$ is shown in fig. 3.

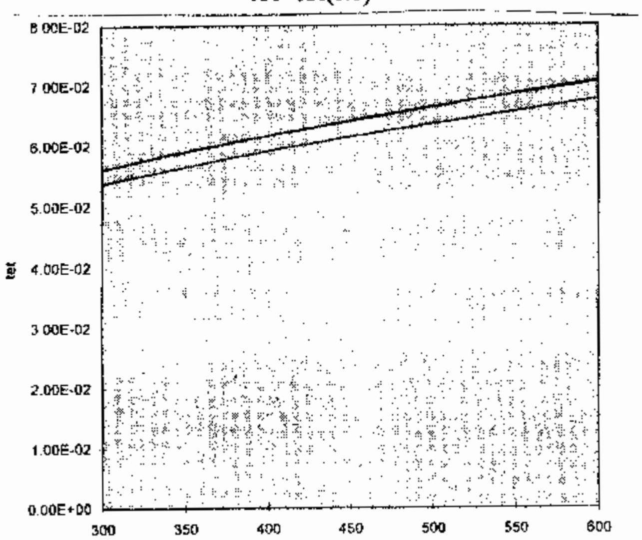
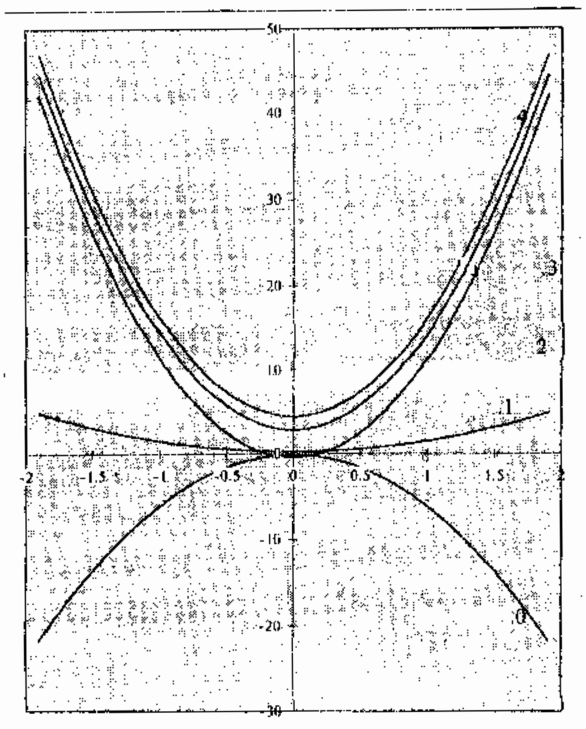


Fig. 3.

As we see from fig.3 the results of the calculation of the second and the third approximation are different a little in the scale of the figure.



Curves 0-4 are problems 0-4

Fig. 4.

In [6] the author made the conclusion on that the solution of the geometric non-linear problem of torsion taking into account the physical linearity of the elastic material does not allow making out the Pounting's effect [7].

However, as we see from fig. 4, Pounting's effect is discovered at solving of the geometric non-linear problem of torsion of the elliptic bar using physical linearity of the visco-elastic material.

As we see by the results and the graphics it follows that the influence of the property of the visco-elastic medium is great on the character of deformation.

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