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ON THE SECOND-ORDER SUBDIFFERENTIAL OF CONVEX FUNCTIONS

Abstract

Investigations on the properties of the second order subdifferential are continued on the paper introduced by the author. The second order subdifferentials of convex functions are considered in detail.

Various definitions of a second-order subdifferential have been suggested and some of their properties have been studied in paper [1] of the author. The investigations of the properties of a second order subdiferential are continued in this paper. Now consider second order subdifferentials of convex functions in detail.

Let X be a Banash space, $f: X \to R$. By $B(X^2, R)$ denote a set of all continuous bilinear functionals from $X \times X$ to R. It is clear that $B(X^2, R)$ is a normed space with respect to the norm

$$\|x^*\| = \sup\{x^*(x_1, x_2): \|x_1\| \le 1, \|x_2\| \le 1,\}, \quad x^* \in B(X^2, R).$$

If there exists such a bilinear symmetric functional $x^* \in B(X^2, R)$ that $Q(x) = x^*(x,x)$ then $Q(x)$ given in the space X is called a quadratic functional. A set of all continuous quadratic functions denote by $B_0(X)$. Denote

$$\overline{B}(X^2, R) = \{x^* \in B(X^*, R): x^* \ summmetrically\},$$
$$\varphi(x) = f(x_0 + x) - 2f(x_0) + f(x_0 - x),$$
$$uep\varphi = \{(x, \alpha) \in X \times R_+ : \varphi(x) \le \alpha^2\}.$$

Assume

$$f^{(2)+}(x_0; x) = \overline{\lim_{t \downarrow 0}} \frac{\varphi(tx)}{t^2}, \; f^{(2)-}(x_0; x) = \underline{\lim_{t \downarrow 0}} \frac{\varphi(tx)}{t^2},$$
$$\check{D}_2^+ f(x_0) = \{Q \in B_0(X): f^{(2)+}(x_0; x) \ge Q(x), x \in X\},$$
$$\gamma^+((0,0); uep\varphi) = \{(x, \alpha) \in X \times R_+ : \exists 0(\lambda), \ that \ (\lambda x, \lambda\alpha + 0(\lambda)) \in uep\varphi$$
$$for \ \lambda \in (0, \eta_{x,\alpha}), \quad \eta_{x,\alpha} > 0, \ \frac{0(\lambda)}{\lambda} \to 0 \ \ for \ \ \lambda \downarrow 0\},$$
$$N^{(2)+}(\varphi(0)) = \{Q \in B_0(X): Q(x) - \alpha^2 \le 0 \ \ for \ \ (x, \alpha) \in \gamma^+((0,0); uep\varphi)\}.$$

Theorem 1. $\gamma^+((0,0); uep\varphi) = uepf^{(2)+}(x_0; \cdot)$. *Besides if* $\varphi(x) \ge 0$, *then* $N^{(2)+}(\varphi(0)) = \check{D}_2^+ f(x_0)$.

Proof. If $(x, \alpha) \in \gamma^+((0,0)\,;\, uep\varphi)$, then $\varphi(\lambda x) \le (\lambda\alpha + 0(\lambda))^2$ for $\lambda \in (0, \eta_{x,\alpha})$. Therefore

$$f^{(2)+}(x_0;x)=\overline{\lim_{\lambda\downarrow 0}}\frac{\varphi(\lambda x)}{\lambda^2}\le\alpha^2,\ i.e.\ (x,\alpha)\in uepf^{(2)+}(x_0;\cdot).$$

If $(x,\alpha)\in uepf^{(2)+}(x_0;\cdot)$, *then* $\overline{\lim_{\lambda\downarrow 0}}\frac{\varphi(\lambda x)}{\lambda^2}\le\alpha^2$. By using from lemma 1.2.10 [1] we obtain that there exist $\eta>0$ and $0(\lambda):[0,\eta]\to R_+$, where $\frac{0(\lambda)}{\lambda}\to 0$ for $\lambda\downarrow 0$, that $(\lambda x,\lambda\alpha+0(\lambda))\in uep\varphi$ *for* $\lambda\in[0,\eta]$. The validity of the second equality follows from the first relation. The theorem is proved.

The function f is called $\{2\}$ - Lipschitzian with a constant K at the point x_0, if for some $\varepsilon>0$ the f satisfies the condition: $|\varphi(x)|\le K\|x\|^2$ *for* $x\in\varepsilon B$, where $B=\{x\in X:\ \|x\|<1\}$.

Assume $\rho_2(x,\alpha)=\inf\{\|x-y\|+(\alpha-\beta)^2:(y,\beta)\in uep\varphi\}$

Lemma 1. *Let f be a $\{2\}$ - Lipschitzian function at the point x_0 with a constant K. Then there exists such a $\varepsilon>0$, that* $|\rho_2(x,\alpha)+\rho_2(-x,-\alpha)|\le 4(k+1)\|(x,\alpha)\|^2$ *for* $x\in\varepsilon B,\alpha\in R.$

Proof. If $\varphi(x)\le\beta^2$, then it is clear that $\rho_2(x,\alpha)\le(\alpha-\beta)^2\le 2(\alpha^2+\beta^2)$.
By condition f is a $\{2\}$-Lipschitian function at the point x_0 with a constant K, therefore, there exists such a $\varepsilon>0$ that $\varphi(x)\le K\|x\|^2$ for $x\in\varepsilon B$. Assuming $\beta^2=K\|x\|^2$ for $x\in\varepsilon B$ we get $\rho_2(x,\alpha)\le 2(\alpha^2+K\|x\|^2)\le 2(K+1)\|(\alpha,x)\|^2$.

Then it is clear that $0\le\rho_2(x,\alpha)+\rho_2(-x,-\alpha)\le 4(K+1)\|(\alpha,x)\|^2$ for $x\in\varepsilon B$. The lemma is proved.

Lemma 2. *If* $\varphi(x)\ge 0$, *then*

$$\left\{Q\in B_0(X):Q(x)+\alpha^2\in 0{,}5\check{D}_2^+\rho_2(0,0)\right\}\subset\check{D}_2^+f(x_0).$$

Proof. Assuming $\varphi(tx)=\beta_t^2$, we get from definition $\rho_2(x,\alpha)$ that

$$\rho_2^{(2)+}((0,0);(x,\alpha))=\overline{\lim_{t\downarrow 0}}\frac{1}{t^2}(\rho_2(tx,t\alpha)+\rho_2(-tx,-t\alpha))\le$$

$$\le\overline{\lim_{t\downarrow 0}}\frac{(t\alpha-\beta_t)^2+(-t\alpha-\beta_t)^2}{t^2}=2\alpha^2+2\overline{\lim_{t\downarrow 0}}\frac{\varphi(tx)}{t^2}=2(\alpha^2+f^{(2)+}(x_0;x)).$$

Therefore, if $2(Q(x)+\alpha^2)\in\check{D}_2^+\rho_2(0,0)$ then

$$2Q(x)+2\alpha^2\le\rho_2^{(2)+}((0,0);(x,\alpha))\le 2\alpha^2+2f^{(2)+}(x_0;x).$$

Hence, it follows that $Q\in\check{D}_2^+f(x_0)$. The lemma is proved.

Lemma 3. *If f is a Lipschitzian function in the vicinity of the point x_0 with a constant K, then $(x,\alpha)\in\gamma^+((0,0);uep\varphi)$ if and only if* $\lim_{t\downarrow 0}\frac{\rho_2(tx,t\alpha)}{t^2}=0.$

Remark 1. It follows from $(x,\alpha)\in\gamma^+((0,0);uep\varphi)$ that

$$(-x,\alpha)\in\gamma^+((0,0);uep\varphi).$$

Proof. Let $\lim\limits_{t\downarrow 0}\frac{\rho_2(tx,t\alpha)}{t^2}=0$. By definition the functions $\rho_2(x,\alpha)$ for any $t>0$ there exists such a $(y_t,\beta_t)\in uep\varphi$ that

$$\|tx-y_t\|+(t\alpha-\beta_t)^2\le\rho_2(tx,t\alpha)+t^3.$$

Assuming $0_1(t)=y_t-tx$, $0_2(t)=\beta_t-t\alpha$ we get $\frac{0_1(t)}{t^2}\to 0$ and $\frac{0_2(t)}{t}\to 0$.

It follows from $(y_t,\beta_t)\in uep\varphi$ that $\varphi(tx+0_1(t))\le(t\alpha+0_2(t))^2$ i.e.

$$f(x_0+tx+0_1(t))-2f(x_0)+f(x_0-tx-0_1(t))\le(t\alpha+0_2(t))^2.$$

If $\|tx+0_1(t)\|<\delta$, $\|tx\|<\delta$, then by condition we get

$$\varphi(tx)\le(t\alpha+0_2(t))^2+2K\|0_1(t)\|\le\left(t\alpha+0_2(t)+\sqrt{2K\|0_1(t)\|}\right)^2.$$

Hence it follows that $(x,\alpha)\in\gamma^+((0,0);uep\varphi)$. Conversely, if $(x,\alpha)\in\gamma^+((0,0);uep\varphi)$, then there exist $0(t)\in R_+$ and $\eta_{x,\alpha}>0$, where $\frac{0(t)}{t}\to 0$ for $t\downarrow 0$, that $(tx,t\alpha+0(t))\in uep\varphi$ for $t\in(0,\eta_{x,\alpha})$. Therefore, $\rho_2(tx,t\alpha)\le(0(t))^2$ for $t\in(0,\eta_{x,\alpha})$. It follows that $\lim\limits_{t\downarrow 0}\frac{\rho_2(tx,t\alpha)}{t^2}=0$. The lemma is proved.

Assume $\overline{\rho}_2(x,\alpha)=\inf\left\{(\alpha-\beta)^2:(x,\beta)\in uep\varphi\right\}$. It is easily verified that $(x,\alpha)\in\gamma^+((0,0);uep\varphi)$ if and only if $\lim\limits_{t\downarrow 0}\frac{\overline{\rho}_2(tx,t\alpha)}{t^2}=0$.

If f is convex then we obtain

$$\varphi(x)=2\left(\frac{1}{2}f(x_0+x)+\frac{1}{2}f(x_0-x)-f(x_0)\right)\ge 2\left(f\left(\frac{x_0+x}{2}+\frac{x_0-x}{2}\right)-f(x_0)\right)=0.$$

Besides, if $\alpha_1\ge 0,\alpha_2\ge 0$, $\alpha_1+\alpha_2=1$, then

$$\varphi(\alpha_1x_1+\alpha_2x_2)\le\alpha_1f(x_0+x_1)+\alpha_2f(x_0+x_2)-2\alpha_1f(x_0)-2\alpha_2f(x_0)+$$
$$+\alpha_1f(x_0-x_1)+\alpha_2f(x_0-x_2)=\alpha_1\varphi(x_1)+\alpha_2\varphi(x_2),$$

i.e. if f is convex, then φ is convex and $\varphi(x)\ge 0$.

Lemma 4. *If $\sqrt{g(x)}$ is a symmetric sublinear lower- semicontinuous function and $\varphi(x,y)=\sqrt{g(x)g(y)}$, then $\varphi(x,y)=\sup\left\{x^*(x,y):x^*\in\overline{\partial}_2\varphi\right\}$, where*

$$\overline{\partial}_2\varphi=\left\{x^*\in B(X^2,R):\varphi(x,y)\ge x^*(x,y),\ x,y\in X\right\}.$$

Proof. We obtain from Hormander's theorem that $\sqrt{g(x)}=\sup\left\{y^*(x):y^*\in\partial\sqrt{g}\right\}$, where $\partial\sqrt{g}=\left\{y^*\in X^*:\sqrt{g(x)}\ge y^*(x),x\in X\right\}$. Since

$\sqrt{g(x)} = \sqrt{g(-x)}$, then it follows from inclusion $y^* \in \partial\sqrt{g}$ that $-y^* \in \partial\sqrt{g}$. Then, it is clear that $\sqrt{g(x)} = \sup\{|y^*(x)|: y^* \in \partial\sqrt{g}\}$. Therefore,

$$\varphi(x,y) = \sqrt{g(x)g(y)} = \sup\{|y^*(x)|: y^* \in \partial\sqrt{g}\} \cdot \sup\{|z^*(y)|: z^* \in \partial\sqrt{g}\} =$$
$$= \sup\{|y^*(x)| \cdot |z^*(y)|: y^* \in \partial\sqrt{g},\, z^* \in \partial\sqrt{g}\} = \sup\{y^*(x)z^*(y): y^*, z^* \in \partial\sqrt{g}\} \le$$
$$\le \sup\{x^*(x,y): x^* \in \bar{\partial}_2\varphi\}.$$

By definition $\varphi(x,y) \ge \sup\{x^*(x,y): x^* \in \bar{\partial}_2\varphi\}$. Therefore,

$\varphi(x,y) = \sup\{x^*(x,y): x^* \in \bar{\partial}_2\varphi\}$. The lemma is proved.

Corollary 1. *According to the condition of lemma 4* $g(x) = \sup\{x^*(x,x): x^* \in \partial_2\varphi\}$, *where* $\partial_2\varphi = \{x^* \in \overline{B}(X^2, R): \varphi(x,y) \ge x^*(x,y), x, y \in X\}$.

Note that corollary 1 follows also from lemmas 1.2.3 [1].

Lemma 5. *If* $\sqrt{g}$ *is a convex function, then* $uep\,g$ *is a convex set.*

Proof. Let $(x_1,\alpha_1) \in uep\,g$, $(x_2,\alpha_2) \in uep\,g$. Since $\sqrt{g}$ is a convex function, then for $\lambda \ge 0, \mu \ge 0, \lambda + \mu = 1$ it is fulfilled the inequality

$$\sqrt{g(\lambda_1 x_1 + \lambda_2 x_2)} = \lambda_1\sqrt{g(x_1)} + \lambda_2\sqrt{g(x_2)}.$$

Therefore,

$$g(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1^2 g(x_1) + 2\lambda_1\lambda_2\sqrt{g(x_1)} \cdot \sqrt{g(x_2)} + \lambda_2^2 g(x_2).$$

Since $g(x_1) \le \alpha_1^2$, $g(x_2) \le \alpha_2^2$, then we obtain

$$g(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1^2\alpha_1^2 + 2\lambda_1\lambda_2\alpha_1\alpha_2 + \lambda_2^2\alpha_2^2 = (\lambda_1\alpha_1 + \lambda_2\alpha_2)^2,$$

i.e. $\lambda_1(x_1,\alpha_1) + \lambda_2(x_2,\alpha_2) \in uepg$. The lemma is proved.

Lemma 6. *If* $uep\varphi$ *is a convex set, then* $\rho_2(x,\alpha)$ *is a convex function.*

Proof. Take (x_1,α_1) and (x_2,α_2) *from* $X \times R$ *and* $\lambda \in (0,1)$. Let $\varepsilon > 0$ and (y_1,β_1), (y_2,β_2) from $uep\varphi$ are so that

$$\|x_1 - y_1\| + (\alpha_1 - \beta_1)^2 \le \rho_2(x_1,\alpha_1) + \varepsilon,\ \|x_2 - y_2\| + (\alpha_2 - \beta_2)^2 \le \rho_2(x_2,\alpha_2) + \varepsilon.$$

Assume $y = \lambda y_1 + (1-\lambda)y_2$, $\beta = \lambda\beta_1 + (1-\lambda)\beta_2$. It is clear that

$$\rho_2(\lambda x_1 + (1-\lambda)x_2, \lambda\alpha_1 + (1-\lambda)\alpha_2) \le \|\lambda x_1 + (1-\lambda)x_2 - y\| +$$
$$+ (\lambda\alpha_1 + (1-\lambda)\alpha_2 - \beta)^2 \le \lambda\|x_1 - y_1\| + (1-\lambda)\|x_2 - y_2\| +$$
$$+ (\lambda(\alpha_1 - \beta_1) + (1-\lambda)(\alpha_2 - \beta_2))^2 = \lambda\|x_1 - y_1\| + (1-\lambda)\|x_2 - y_2\| +$$
$$+ \lambda^2(\alpha_1 - \beta_1)^2 + 2\lambda(1-\lambda)(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) + (1-\lambda)^2(\alpha_2 - \beta_2)^2 \le$$
$$\le \lambda\|x_1 - y_1\| + (1-\lambda)\|x_2 - y_2\| + \lambda^2(\alpha_1 - \beta_1)^2 + \lambda(1-\lambda)((\alpha_1 - \beta_1)^2 + (\alpha_2 - \beta_2)^2) +$$
$$+ (1-\lambda)^2(\alpha_2 - \beta_2)^2 = \lambda\|x_1 - y_1\| + (1-\lambda)\|x_2 - y_2\| + \lambda(\alpha_1 - \beta_1)^2 + (1-\lambda)(\alpha_2 - \beta_2)^2 \le$$
$$\le \lambda\rho_2(x_1,\alpha_1) + (1-\lambda)\rho_2(x_2,\alpha_2) + \varepsilon.$$

Since ε is arbitrary then the lemma is proved.

Let f be convex. Assume

$$\Psi(x)=2\big(f(x_0+x)-f(x_0)-f'(x_0;x)\big),$$

$$f'(x_0;x)=\lim_{\lambda\downarrow 0}\frac{f(x_0+\lambda x)-f(x_0)}{\lambda},\quad f^{\{2\}+}(x_0;x)=\overline{\lim_{\lambda\downarrow 0}}\frac{\Psi(\lambda x)}{\lambda^2},$$

$$\partial^{\{2\}+}f(x_0)=\left\{Q\in B_0(X): f^{\{2\}+}(x_0;x)\ge Q(x),\ x\in X\right\}.$$

The following theorem is proved similar to theorem 1.

Theorem 2. *If f is a convex function, then*

$$\gamma^+\big((0,0);uep\Psi\big)=uepf^{\{2\}+}(x_0;\cdot),\ \ N^{(2)+}\big(\Psi(0)\big)=\partial^{\{2\}+}f(x_0).$$

Let f be convex. The set

$$\partial^2 f(x_0)=\left\{x^*\in\overline{B}\big(X^2,R\big): \sqrt{f^{(2)+}(x_0;x)f^{(2)+}(x_0;y)}\ge x^*(x,y),\ x,y\in X\right\}$$

we call a $\{2\}$ - subdiferential of the function f at the point x_0.

Note that a bilinear symmetric functional x^* is determined on $Q(x)$ uniquely. Therefore, we shall identify the symmetric bilinear functional x^* and corresponding quadratic functional $Q(x)$.

It follows from definition that $0\in\partial^2 f(x_0)\subset \check{D}_2^+ f(x_0)$. If X is a Hilbert space, f is a convex function, and $f''(x_0)$ exists, then it follows from the Cauchy-Schwartz inequality (see [2]) that $\overline{co}\{f''(x_0)(x,y);-f''(x_0)(x,y);0\}\subset\partial^2 f(x_0)$.

Lemma 7. *If f is a convex and $\{2\}$ - Lipschitzian function at the point x_0, then $\partial^2 f(x_0)$ are non-empty convex and compact with respect to topology $\sigma\big(\overline{B}(X^2,R),X\times X\big)$ subset in $\overline{B}\big(X^2,R\big)$.*

Proof. It is clear that $0\in\partial^2 f(x_0)$. The convexity of the set $\partial^2 f(x_0)$ is directly verified. Show the compactness of the set $\partial^2 f(x_0)$. By condition f is a $\{2\}$-Lipschitzian function at the point x_0 with a constant K. Therefore, $f^{(2)+}(x_0;x)\le K\|x\|^2$. Hence we have that $\sqrt{f^{(2)+}(x_0;x)f^{(2)+}(x_0;y)}\le K\|x\|\|y\|$, i.e. $|x^*(x,y)|\le K\|x\|\|y\|$. This means $\partial^2 f(x_0)$ is bounded. For each pair $(x,y)\in X\times Y$ the mapping $x^*\to x^*(x,y)$ is a linear functional on $\overline{B}\big(X^2,R\big)$. Therefore, by using the Banach - Alaoglu theorem (see [3]) we can easily verify that $\partial^2 f(x_0)$ is compact with respect to the topology $\sigma\big(\overline{B}(X^2,R),X\times X\big)$. The lemma is proved.

Assume $f^{(2)}(x_0;x)=\lim_{t\downarrow 0}\dfrac{\varphi(tx)}{t^2}$.

Lemma 8. *If f_1 and f_2 are convex functions from X to R and $f_1^{(2)}(x_0;x)$ exist, then $\partial^2(f_1+f_2)(x_0)\supset\partial^2 f_1(x_0)+\partial^2 f_2(x_0)$.*

Proof. By condition of lemma 8 we easily get

$$(f_1+f_2)^{(2)+}(x_0;x)=f_1^{(2)}(x_0;x)+f_2^{(2)+}(x_0;x).$$

Therefore we easily obtain

$$\left((f_1+f_2)^{(2)+}(x_0;x)\cdot(f_1+f_2)^{(2)+}(x_0;y)\right)^{\frac{1}{2}}=\Big(f_1^{(2)}(x_0;x)f_1^{(2)}(x;y)+$$

$$+f_1^{(2)}(x_0;y)f_2^{(2)+}(x_0;x)+f_1^{(2)}(x_0;x)f_2^{(2)+}(x_0;y)+f_2^{(2)+}(x_0;x)f_2^{(2)+}(x_0;y)\Big)^{\frac{1}{2}}\geq$$

$$\geq\sqrt{f_1^{(2)}(x_0;x)f_1^{(2)}(x_0;y)}+\sqrt{f_2^{(2)+}(x_0;x)f_2^{(2)+}(x_0;y)}.$$

Hence, it follows that if $x_1^*\in\partial^2 f_1(x_0)$ and $x_2^*\in\partial^2 f_2(x_0)$, then $x_1^*+x_2^*\in\partial^2(f_1+f_2)(x_0)$. The lemma is proved.

Let $P(X)=\{A\subset X: A\neq 0\}$, $\underline{0}:(0,+\infty)\to P(X)$ is a multivalued mapping. Denote

$$d(x,M)=\inf\{\|x-y\|: y\in M\},\quad for\quad M\subset X.$$

Assume

$$\underline{\lim_{t\downarrow 0}}\,\underline{0}_t=\left\{y:\overline{\lim_{t\downarrow 0}}\,d(y,\underline{0}_t)=0\right\},$$

$$\overline{\lim_{t\downarrow 0}}\,\underline{0}_t=\left\{y:\underline{\lim_{t\downarrow 0}}\,d(y,\underline{0}_t)=0\right\}.$$

It is clear that

$$\underline{\lim_{t\downarrow 0}}\,\underline{0}_t=\left\{v:\forall t_k\downarrow 0,\ \exists(v_k)\to v,\ v_k\in\underline{0}(t_k)\right\},$$

$$\overline{\lim_{t\downarrow 0}}\,\underline{0}_t=\left\{v:\exists t_k\downarrow 0,\ \exists(v_k)\to v,\ v_k\in\underline{0}(t_k)\right\}.$$

If $\underline{\lim_{t\downarrow 0}}\,\underline{0}_t=\overline{\lim_{t\downarrow 0}}\,\underline{0}_t=\underline{0}$, then we call $\underline{0}$ the boundary of mappings $\underline{0}_t$ for $t\to 0$.

It is clear that $\underline{0}=\lim_{t\downarrow 0}\underline{0}_t=\left\{y\in X:\lim_{t\downarrow 0}d(y,\underline{0}_t)=0\right\}$.

Similar to works [4] we introduce the lower and upper Duping indicatrices of the function f at the point x_0. Let $S_t(f(x_0))=\{x\in X:\varphi(x)\leq t\}$. Assume

$$\underline{Ind}f(x_0)=\underline{\lim_{t\downarrow 0}}\frac{1}{\sqrt{t}}S_t(f(x_0)),\ \overline{Ind}f(x_0)=\overline{\lim_{t\downarrow 0}}\frac{1}{\sqrt{t}}S_t(f(x_0)).$$

Theorem 3. *The following equalities are valid:*

$$\underline{Ind}f(x_0)=\underline{\lim_{t\downarrow 0}}\left\{x\in X:\frac{1}{t^2}\big(f(x_0+tx)-2f(x_0)+f(x_0-tx)\big)\leq 1\right\},$$

$$\overline{Ind}f(x_0)=\overline{\lim_{t\downarrow 0}}\left\{x\in X:\frac{1}{t^2}\big(f(x_0+tx)-2f(x_0)+f(x_0-tx)\big)\leq 1\right\}.$$

Proof. It is clear that

$$\underline{Ind}f(x_0)=\underline{\lim_{t\downarrow 0}}\frac{1}{\sqrt{t}}\{x\in X:\varphi(x)\leq t\}=$$

$$=\varliminf_{t\downarrow 0}\left\{x\in X:\varphi\left(\sqrt{t}\,x\right)\le t\right\}=\varliminf_{t\downarrow 0}\left\{x\in X:\frac{\varphi\left(\sqrt{t}\,x\right)}{t}\le 1\right\}=$$

$$=\varliminf_{t\downarrow 0}\left\{x\in X:\frac{\varphi(t\,x)}{t^2}\le 1\right\}.$$

The second formula is proved analogously. The theorem is proved.

It follows from the definition of lower and upper indicatrices that

$$\underline{Ind}f(x_0)\subset\overline{Ind}f(x_0).$$

If $\underline{Ind}f(x_0)=\overline{Ind}f(x_0)$, then we call these set Dupin's indicatrice of the function f at the point x_0 and denote by $Indf(x_0)$.

Lemma 9. It the function $f:X\to R$ is convex, then the set $\underline{Ind}f(x_0)$ is also convex.

Proof. It is clear that

$$\underline{Ind}f(x_0)=\varliminf_{t\downarrow 0}\frac{S_t}{\sqrt{t}}=\left\{y\in X:\varlimsup_{t\downarrow 0}d\left(y,\frac{S_t}{\sqrt{t}}\right)=0\right\}.$$

It is known that (see [5]), if $M\subset X$ is a convex set, then $y\to d(y,M)$ is a convex function. Therefore, assuming $F(t)=\frac{S_t}{\sqrt{t}}$, we get

$$\varlimsup_{t\downarrow 0}d\left(\alpha_1y_1+\alpha_2y_2,F(t)\right)\le\varlimsup_{t\downarrow 0}\left[\alpha_1d\left(y_1,F(t)\right)+\alpha_2d\left(y_2,F(t)\right)\right]\le$$
$$\le\alpha_1\varlimsup_{t\downarrow 0}d\left(y_1,F(t)\right)+\alpha_2\varlimsup_{t\downarrow 0}d\left(y_2,F(t)\right).$$

Hence it follows that if $y_1,y_2\in\underline{Ind}f(x_0)$, then $\alpha_1y_1+\alpha_2y_2\in\underline{Ind}f(x_0)$. The lemma is proved.

The function f we call θ - two lipschitzian with a constant K at the point x_0, if for some $\varepsilon>0$ f satisfies the condition

$$\left|f(x_0+x)+f(x_0-x)-f(x_0+y)-f(x_0-y)\right|\le K\|x-y\|^{\theta}\left(\|x\|+\|y\|\right)^{2-\theta}\text{ for}$$
$x,y\in\varepsilon B,\ 0<\theta<2$.

Theorem 4. *If the function f satisfies the θ-two lipschitzian condition at the point x_0, then*

$$\left\{x:f^{(2)+}(x_0:x)\le 1\right\}\subset\underline{Ind}f(x_0)\subset\overline{Ind}f(x_0)\subset\left\{x:f^{(2)-}(x_0:x)\le 1\right\}.$$

Proof. If $f^{(2)+}(x_0;x)<1$, then for any $t_k\downarrow 0$ there exists such N, that for $k\ge N$

$$\frac{1}{t_k^2}\left(f(x_0+t_kx)-2f(x_0)+f(x_0-t_kx)\right)\le 1. \qquad (1)$$

We obtain from the first relation of theorem 3 that $x\in\underline{Ind}f(x_0)$ if and only if for any $t_k\downarrow 0$ there exists such $\{x_k\}$, where $x_k\to x$ that

$$\frac{1}{t_k^2}\left(f(x_0+t_kx_k)-2f(x_0)+f(x_0-t_kx_k)\right)\le 1.$$

Therefore, by relation (1) we get that if $f^{(2)+}(x_0;x)<1$, then

$$x\in \underline{Ind}f(x_0),\ \ i.e.\ \left\{x: f^{(2)+}(x_0;x)<1\right\}\subset \underline{Ind}f(x_0).$$

Let $f^{(2)+}(x_0;x)\le 1,\ 0<\alpha_k<1$ *and* $\alpha_k\to 1$. Then it is clear that $\alpha_k\bar{x}\to\bar{x}$ and $f^{(2)+}(x_0;\alpha_k\bar{x})=\alpha_k^2 f^{(2)+}(x_0;\bar{x})<1$. Hence it follows that $\alpha_k\bar{x}\in\underline{Ind}f(x_0)$. Therefore $\bar{x}\in\underline{Ind}\,f(x_0)$.

Show that $\overline{Ind}f(x_0)\subset\left\{x: f^{(2)-}(x_0;x)\le 1\right\}$. If $x\in\overline{Ind}f(x_0)$, then we get from the second relation of theorem 3 that there exist such $t_k\downarrow 0$ and $x_k\to x$, that

$$\frac{1}{t_k^2}\left(f(x_0+t_kx_k)-2f(x_0)+f(x_0-t_kx_k)\right)\le 1.$$

Therefore $f^{(2)-}(x_0;x)=\varliminf_{t\downarrow 0,y\to x}\frac{\varphi(ty)}{t^2}\le 1$. Since f satisfies the θ- twolipschitzian condition with a constant K at the point x_0, then

$$f^{(2)-}(x_0;x)\ge\varliminf_{\substack{t\downarrow 0\\ y\to x}}\frac{1}{t^2}\left(f(x_0+tx)-2f(x_0)+f(x_0-tx)-K\|ty-tx\|^{\theta}\left(\|ty\|+\|tx\|\right)^{2-\theta}\right)=$$

$$=\varliminf_{t\downarrow 0}\frac{1}{t^2}\varphi(tx)=f^{(2)-}(x_0;x).$$

We obtain that $\overline{Ind}f(x_0)\subset\left\{x: f^{(2)-}(x_0;x)\le 1\right\}$. Since $\underline{Ind}f(x_0)\subset\overline{Ind}f(x_0)$, then we get the validity of theorem 4. The theorem is proved.

Corollary 2. If the condition of theorem 4 is satisfied, and $f^{(2)}(x_0;x)$ exists, then $\underline{Ind}f(x_0)=\overline{Ind}f(x_0)$.

Assume

$$\left[\check{D}_2^+f(x_0)\right]^0=\left\{x\in X: Q(x)\le 1,\ \ \forall Q\in\check{D}_2^+f(x_0)\right\},$$

$$\left[\partial^2 f(x_0)\right]^0=\left\{x\in X: x^*(x,x)\le 1,\ \ \forall x^*\in\partial^2 f(x_0)\right\}.$$

The next corollary follows from corollary 1.2.2. [1] and theorem 4.

Corollary 3. *It X is a Hilbert space, and f is a θ- twolipschitzian at the point x_0, then* $\left[\check{D}_2^+f(x_0)\right]^0\subset\underline{Ind}f(x_0)$.

Corollary 4. *It the condition of corollary 3 is satisfied, and f is a convex function, then* $\left[\check{D}_2^+f(x_0)\right]^0=\left[\partial^2 f(x_0)\right]^0$.

A set of linear continuous operators from X to X^* denote by $L(X,X^*)$. If $x^*\in B(X^2,R)$, then there exists such an operator $A\in L(X,X^*)$, that $x^*(x,y)=(Ax)y$. Assume

$$L_0(X,X^*)=\{A\in L(X,X^*): A \text{ is a symmetric operator}\}.$$

Example 1. Let X be a Hilbert space, $\underline{0}\subset B_0(X)$ and it follows from $Q\in\underline{0}$ that $Q(x)\geq 0$. Assume $g_1(x)=\sup\{Q(x): Q\in\underline{0}\}$ and $g(x)=\sqrt{g_1(x)}$. If $Q\in B_0(X)$, then exists such an operator $A\in L_0(X,X^*)$, that $Q(x)=\langle Ax,x\rangle$. Assume $\underline{0}_0=\{A:\langle Ax,x\rangle\in\underline{0}\}$. It is clear that $g(\lambda x)=\lambda g(x)$ for $\lambda>0$ and if $A\in\underline{0}_0$, then according to the Cauchy-Schwartz inequality we get $\langle Ax,y\rangle\leq\sqrt{\langle Ax,x\rangle\langle Ay,y\rangle}$.
Therefore

$$g(x+y)=\left\{\sup_{A\in\underline{0}_0}\left(\langle Ax,x\rangle+\langle Ay,x\rangle+\langle Ax,y\rangle+\langle Ay,y\rangle\right)\right\}^{\frac{1}{2}}\leq$$

$$\leq\sup_{A\in\underline{0}_0}\left\{\langle Ax,x\rangle+2\sqrt{\langle Ax,x\rangle\langle Ay,y\rangle}+\langle Ay,y\rangle\right\}^{\frac{1}{2}}=$$

$$=\sup_{A\in\underline{0}_0}\left\{\sqrt{\langle Ax,x\rangle}+\sqrt{\langle Ay,y\rangle}\right\}\leq\sup_{A\in\underline{0}_0}\sqrt{\langle Ax,x\rangle}+\sup_{A\in\underline{0}_0}\sqrt{\langle Ay,y\rangle}=g(x)+g(y),$$

i.e. g - is a convex function.
Since

$$\frac{1}{t^2}\left(g_1(tx)+g_1(-tx)\right)=\frac{1}{t^2}\left(\sup_{Q\in\underline{0}}Q(tx)+\sup_{Q\in\underline{0}}Q(-tx)\right)=2\sup_{Q\in\underline{0}}Q(x).$$

Therefore from theorem 3 we get $Indg_1(0)=\left\{x:\sup_{Q\in\underline{0}}Q(x)\leq 0{,}5\right\}$.

Let $q: X\to R$, , where X is a Hillbert space. Assume

$$q^*(A)=\sup_x\{\langle Ax,x\rangle-q(x)\},\quad A\in L_0(X,X^*),$$

$$q^{**}(x)=\sup\left\{\langle Ax,x\rangle-q^*(A): A\in L_0^+(X,X^*)\right\},$$

where $L_0^+(X,X^*)=\{A\in L_0(X,X^*):\langle Ax,x\rangle\geq 0,\ x\in X\}$.

It is clear that $q^*(A)\geq\langle Ax,x\rangle-q(x)$, for $A\in L_0(X,X^*)$ and $q^{**}(x)\leq q(x)$.

Lemma 10. *The functions* $q^*: L_0(X,X^*)\to R$, $q^{**}: X\to R$ *are convex.*

Proof. It $\alpha_1+\alpha_2=1$, $\alpha_1\geq 0$, $\alpha_2\geq 0$, $A_1,A_2\in L_0(X,X^*)$ then

$$q^*(\alpha_1A_1+\alpha_2A_2)=\sup_x\{\alpha_1\langle A_1x,x\rangle+\alpha_2\langle A_2x,x\rangle-\alpha_1q(x)-\alpha_2q(x)\}\leq$$

$$\leq\alpha_1\sup_x\{\langle A_1x,x\rangle-q(x)\}+\alpha_2\sup_x\{\langle A_2x,x\rangle-q(x)\}=\alpha_1q^*(A)+\alpha_2q^*(A_2).$$

Show that q^{**} is a convex function. It is clear that

$$q^{**}(\alpha_1x_1+\alpha_2x_2)=\sup_{A\in L_0^+(X,X^*)}\left\{\langle A(\alpha_1x_1+\alpha_2x_2),\alpha_1x_1+\alpha_2x_2\rangle-q^*(A)\right\}=$$

$$=\sup_{A\in L_0^+(X,X^*)}\left\{\alpha_1^2\langle Ax_1,x_1\rangle+2\alpha_1\alpha_2\langle Ax_1,x_2\rangle+\alpha_2^2\langle Ax_2,x_2\rangle-q^*(A)\right\}.$$

We have from the Cauchy-Schwartz inequality that

$$\langle Ax_1, x_2\rangle \le \sqrt{\langle Ax_1, x_1\rangle\langle Ax_2, x_2\rangle} \le \frac{1}{2}\left(\langle Ax_1, x_1\rangle + \langle Ax_2, x_2\rangle\right).$$

Therefore

$$g^{**}(\alpha_1 x_1 + \alpha_2 x_2) \le \sup_{A\in L_0^+(X,X^*)}\left\{\alpha_1^2\langle Ax_1, x_1\rangle + \alpha_1\alpha_2\langle Ax_1, x_1\rangle + \alpha_1\alpha_2\langle Ax_2, x_2\rangle + \right.$$

$$\left. + \alpha_2^2\langle Ax_2, x_2\rangle - q^*(A)\right\} = \sup_{A\in L_0^+(X,X^*)}\left\{\alpha_1\langle Ax_1, x_1\rangle + \alpha_2\langle Ax_2, x_2\rangle - q^*(A)\right\} \le$$

$$\le \alpha_1 \sup_{A\in L_0^+(X,X^*)}\left\{\langle Ax_1, x_1\rangle - q^*(A)\right\} + \alpha_2 \sup_{A\in L_0^+(X,X^*)}\left\{\langle Ax_2, x_2\rangle - q^*(A)\right\} =$$

$$= \alpha_1 q^{**}(x_1) + \alpha_2 q^{**}(x_2).$$

The lemma is proved.

Lemma 11. *If* $\varphi^*(A) = 0$, *then* $\langle Ax, x\rangle \in \check{D}_2^+ f(x_0)$.

Proof. Since $\varphi^*(A) = \sup_x\{\langle Ax, x\rangle - \varphi(x)\} = 0$, then $\langle Ax, x\rangle - \varphi(x) \le 0$. Hence it follows that $\varphi(x) \ge \langle Ax, x\rangle$. Then, it is clear that $f^{(2)+}(x_0; x) = \overline{\lim_{t\downarrow 0}}\frac{\varphi(tx)}{t^2} \ge \langle Ax, x\rangle$, i.e. $\langle Ax, x\rangle \in \check{D}_2^+ f(x_0)$. The lemma is proved.

Assume

$$g_z(x, y) = f(z + x + y) - f(z + x) - f(z + y) + f(z),$$

$$2 - epg_z = \left\{((x,\alpha),(y,\beta)) \in (X\times R)^2 : g_z(x,y) \le \alpha\beta\right\},$$

$$\rho_z((x,\alpha),(y,\beta)) = \inf\left\{|\alpha - \alpha_1|(|\beta| + |\beta_1|) + |\beta - \beta_1|(|\alpha| + |\alpha_1|) : ((x,\alpha_1),(y,\beta_1)) \in 2 - ep\, g_z\right\},$$

$$\Gamma_2(f(x_0)) = \left\{((x,\alpha),(y,\beta)) \in (X\times R)^2 : \forall z_i \to x_0, \forall\lambda_i \downarrow 0, \forall\mu_i \downarrow 0 \text{ exists}\right.$$

$$\left.\exists\alpha_i \to \alpha, \exists\beta_i \to \beta, \text{ that } ((\lambda_i x, \lambda_i\alpha_i),(\mu_i y, \mu_i\beta_i)) \in 2 - epg_{z_i}\right\},$$

$$N_2(f(x_0)) = \left\{x^* \in \overline{B}(X^2, R) : x^*(x,y) - \alpha\beta \le 0,\ \forall((x,\alpha),(y,\beta)) \in \Gamma_2(f(x_0))\right\}.$$

Lemma 12. $((x,\alpha),(y,\beta)) \in \Gamma_2(f(x_0))$ *if and only if*

$$\lim_{\substack{z\to x_0\\ \lambda\downarrow 0,\mu\downarrow 0}} \frac{\rho_z(\lambda(x,\alpha), \mu(y,\beta))}{\lambda\mu} = 0.$$

Proof. Let $\lim_{\substack{z\to x_0\\ \lambda\downarrow 0,\mu\downarrow 0}} \frac{\rho_z(\lambda(x,\alpha), \mu(y,\beta))}{\lambda\mu} = 0$ and $z_i \to x_0$, $\lambda_i \downarrow 0$, $\mu_i \downarrow 0$. Then we have $\lim_{i\to\infty}\frac{\rho_{z_i}(\lambda_i(x,\alpha), \mu_i(y,\beta))}{\lambda_i\mu_i} = 0$. Therefore there exist

$$((\lambda_i x, a_i),(\mu_i y, b_i)) \in 2 - epg_{z_i}$$

such, that $|a_i - \lambda_i\alpha|(|b_i| + |\mu_i\beta|) + |b_i - \mu_i\beta|(|\lambda_i\alpha| + |a_i|) \le \rho_{z_i}(\lambda_i(x,\alpha), \mu_i(y,\beta)) + \frac{\lambda_i\mu_i}{i}$.

By denoting $\alpha_i = \frac{a_i}{\lambda_i}$, $\beta_i = \frac{b_i}{\mu_i}$, we get that $a_i = \lambda_i \alpha_i, b_i = \mu_i \beta_i$. Besides $\alpha_i \to \alpha, \beta_i \to \beta$ and $\left(\lambda_i (x, \alpha_i), \mu_i (y, \beta_i)\right) \in 2 - epg_{z_i}$.

Conversely, if $\left((x,\alpha),(y,\beta)\right) \in \Gamma_2\left(f(x_0)\right)$, then for any $z_i \to x_0$, $\lambda_i \downarrow 0$ *and* $\mu_i \downarrow 0$ there exist such $\alpha_i \to \alpha, \beta_i \to \beta$, that $\left(\lambda_i (x, \alpha_i), \mu_i (y, \beta_i)\right) \in 2 - epg_{z_i}$. Therefore

$\overline{\lim_{i\to\infty}} \frac{\rho_{z_i}\left(\lambda_i (x,\alpha), \mu_i (y,\beta)\right)}{\lambda_i \mu_i} \le \lim_{i\to\infty}\left(|\alpha - \alpha_i|\left(|\beta| + |\beta_i|\right) + |\beta - \beta_i|\left(|\alpha| + |\alpha_i|\right)\right)$. Thus, we get

$\lim_{\substack{z\to x_0 \\ \lambda\downarrow 0, \mu\downarrow 0}} \frac{\rho_z\left(\lambda (x,\alpha), \mu (y,\beta)\right)}{\lambda \mu} = 0$. The theorem is proved.

Assume

$$f^{[2]}(x_0; x, y) = \overline{\lim_{\substack{z\to x_0 \\ \lambda\downarrow 0, \mu\downarrow 0}}} \frac{g_z(\lambda x, \mu y)}{\lambda \mu},$$

$$\partial_2 f(x_0) = \left\{x^* \in \overline{B}\left(X^2, R\right): f^{[2]}(x_0; x, y) \ge x^*(x,y), \forall x, y \in X\right\}.$$

Lemma 13. $\Gamma_2\left(f(x_0)\right) = 2 - epf^{[2]}(x_0, \cdot)$.

Proof. Let $\left((x,\alpha),(y,\beta)\right) \in \Gamma_2\left(f(x_0)\right)$. Then for any $z_i \to x_0, \lambda_i \downarrow 0$ and $\mu_i \downarrow 0$ there exist such $\{\alpha_i\}$ *and* $\{\beta_i\}$, where $\alpha_i \to \alpha$, $\beta_i \to \beta$ that $g_{z_i}(\lambda_i x, \mu_i y) \le \lambda_i \alpha_i \mu_i \beta_i$. Therefore

$$\overline{\lim_{i\to\infty}} \frac{g_{z_i}(\lambda_i x, \mu_i y)}{\lambda_i \mu_i} \le \lim_{i\to\infty} \alpha_i \beta_i .$$

Hence, it follows that $f^{[2]}(x_0; x, y) \le \alpha\beta$, *i.e.* $\left((x,\alpha),(y,\beta)\right) \in 2 - epf^{[2]}(x_0; \cdot)$.

If $\left((x,\alpha),(y,\beta)\right) \in 2 - epf^{[2]}(x_0; \cdot)$, then $\overline{\lim_{\substack{z\to x_0 \\ \lambda\downarrow 0, \mu\downarrow 0}}} \frac{g_z(\lambda x, \mu y)}{\lambda \mu} \le \alpha\beta$.

Let $\{z_i\}, \{\lambda_i\}, \{\mu_i\}$ be such that $z_i \to x_0, \lambda_i \downarrow 0$ *and* $\mu_i \downarrow 0$. *Then* $\overline{\lim_{i\to\infty}} \frac{g_{z_i}(\lambda_i x, \mu_i y)}{\lambda_i \mu_i} \le \alpha\beta$.

Assume

$$v_i = \begin{cases} 0 & ; \ \frac{g_{z_i}(\lambda_i x, \mu_i y)}{\lambda_i \mu_i} \le \alpha\beta, \\ \frac{g_{z_i}(\lambda_i x, \mu_i y)}{\lambda_i \mu_i} - \alpha\beta; & \frac{g_{z_i}(\lambda_i x, \mu_i y)}{\lambda_i \mu_i} > \alpha\beta. \end{cases}$$

It is clear that $v_i \to 0$ for $i \to \infty$. Let $\alpha_i \to \alpha, \beta_i \to \beta$ *and* $\alpha_i \beta_i = \alpha\beta + v_i$. Then it is clear that $g_{z_i}(\lambda_i x, \mu_i y) \le \lambda_i \alpha_i \mu_i \beta$, *i.e.* $\left((x,\alpha),(y,\beta)\right) \in \Gamma_2\left(f(x_0)\right)$. The lemma is proved.

Theorem 5. $N_2\left(f(x_0)\right) = \partial_2 f(x_0)$.

The proof of lemma follows from lemma 13 and definition $N_2(f(x_0))$ *and* $\partial_2 f(x_0)$.

Assume

$$d_z((x,\alpha),(y,\beta)) = \inf\{(\|y\| + \|y_1\| + |\beta| + |\beta_1|)(\|x - x_1\| + |\alpha - \alpha_1|) + (\|x\| + \|x_1\| + |\alpha| + |\alpha_1|) \times$$
$$\times (\|y - y_1\| + |\beta - \beta_1|) : ((x_1,\alpha_1),(y_1,\beta_1)) \in 2 - epg_z\},$$
$$\tilde{\Gamma}_2(f(x_0)) = \{((x,\alpha),(y,\beta)) \in (X \times R)^2 : \forall z_i \to x_0, \forall \lambda_i \downarrow 0, \forall \mu_i \downarrow 0,$$

$\exists (x_i,\alpha_i) \to (x,\alpha), \exists (y_i,\beta_i) \to (y,\beta)$, that $((\lambda_i x_i, \lambda_i \alpha_i),(\mu_i y_i, \mu_i \beta_i)) \in 2 - epg_{z_i}\}$, it is proved that

$$\tilde{\Gamma}_2(f(x_0)) = \left\{((x,\alpha),(y,\beta)) \in (X \times R)^2 : \lim_{\substack{z \to x_0 \\ \lambda \downarrow 0, \mu \downarrow 0}} \frac{d_z(\lambda(x,\alpha), \mu(y,\beta))}{\lambda \mu} = 0\right\}$$

and $\tilde{N}_2(f(x_0))$ is considered.

Assume

$$\varphi_z(x) = f(z+x) - 2f(z) + f(z-x),$$
$$lep\,\varphi_z = \{(x,\beta) \in X \times R_+ : \varphi_z(x) \ge -\beta^2\},$$
$$f^{[2]+}(x_0;x) = \overline{\lim_{\substack{z \to x_0 \\ t \downarrow 0}}} \frac{\varphi_z(tx)}{t^2}, \quad f^{[2]-}(x_0;x) = \underline{\lim_{\substack{z \to x_0 \\ t \downarrow 0}}} \frac{\varphi_z(tx)}{t^2},$$
$$D_2^+ f(x_0) = \{Q \in B_0(X) : f^{[2]+}(x_0;x) \ge Q(x), x \in X\},$$
$$D_2^- f(x_0) = \{Q \in B_0(X) : f^{[2]-}(x_0;x) \le Q(x),\ x \in X\},$$
$$D_2 f(x_0) = D_2^+ f(x_0) \cap D_2^- f(x_0).$$

Consider the following cones

$$K_0^{[2]+}(f(x_0)) = \{(x,\alpha) \in X \times R_+ : \forall z_i \to x_0, \forall \lambda_i \downarrow 0, \exists (x_i,\alpha_i) \to (x,\alpha),$$
$$\textit{that}\ (\lambda_i x_i, \lambda_i \alpha_i) \in uep\varphi_{z_i}\},$$
$$K_1^{[2]+}(f(x_0)) = \{(x,\alpha) \in X \times R_+ : \forall z_i \to x_0, \forall \lambda_i \downarrow 0, \exists \alpha_i \to \alpha,$$
$$\textit{that}\ (\lambda_i x, \lambda_i \alpha_i) \in uep\varphi_{z_i}\},$$
$$K_0^{[2]-}(f(x_0)) = \{(x,\beta) \in X \times R_+ : \forall z_i \to x_0, \forall \lambda_i \downarrow 0, \exists (x_i,\beta_i) \to (x,\beta),$$
$$\textit{that}\ (\lambda_i x_i, \lambda_i \beta_i) \in lep\varphi_z\},$$
$$K_1^{[2]-}(f(x_0)) = \{(x,\beta) \in X \times R_+ : \forall z_i \to x_0, \forall \lambda_i \downarrow 0, \exists \beta_i \to \beta,$$
$$\textit{that}\ (\lambda_i x, \lambda_i \beta_i) \in lep\varphi_z\}.$$

Assume

$$N_i^{[2]+}(f(x_0)) = \{Q \in B_0(X) : Q(x) - \alpha^2 \le 0, \forall (x,\alpha) \in K_i^{[2]+}(f(x_0))\},\ i = 0,1,$$
$$N_i^{[2]-}(f(x_0)) = \{Q \in B_0(X) : Q(x) + \beta^2 \ge 0\ \ \textit{for}\ (x,\beta) \in K_i^{[2]-}(f(x_0))\},\ i = 0,1,$$
$$N_i^{[2]}(f(x_0)) = N_i^{[2]+}(f(x_0)) \cap N_i^{[2]-}(f(x_0)),\ i = 0,1,\ N_i^{[2]\pm}(\varphi(0)) = N_i^{[2]\pm}(f(x_0)).$$

To study the properties of introduced cones we give the following functions

$$\overline{\rho}_z(x,\alpha)=\inf\left\{|\alpha-\beta|:(x,\beta)\in uep\varphi_z\right\},$$
$$d_z(x,\alpha)=\inf\left\{\|x-y\|+|\alpha-\beta|:(y,\beta)\in uep\varphi_z\right\},$$
$$r_z(x,\alpha)=\inf\left\{\|x-y\|^2+(\alpha-\beta)^2:(y,\beta)\in uep\varphi_z\right\}.$$

Lemma 14.

$$K_1^{[2]+}(f(x_0))=uepf^{[2]+}(x_0;\cdot)=\left\{(x,\alpha)\in X\times R_+:\lim_{\substack{z\to x_0\\ t\downarrow 0}}\frac{\overline{\rho}_z(tx,t\alpha)}{t}=0\right\},$$

$$K_0^{[2]+}(f(x_0))=\left\{(x,\alpha)\in X\times R_+:\lim_{\substack{z\to x_0\\ t\downarrow 0}}\frac{d_z(tx,t\alpha)}{t}=0\right\}=$$

$$=\left\{(x,\alpha)\in X\times R_+:\lim_{\substack{z\to x_0\\ t\downarrow 0}}\frac{r_z(tx,t\alpha)}{t^2}=0\right\}.$$

Theorem 6. *If X is a Hilbert space, f is a $\{2\}$-Lipschitzian function in the vicinity of the point x_0 with a constant K (see [1]), then*

$$D_2f(x_0)=\left(N_1^{[2]+}(g_1(0))-K\|x\|^2\right)\bigcap\left(N_1^{[2]-}(g_2(0))+K\|x\|^2\right),$$

where $g_{1z}(x)=\varphi_z(x)+K\|x\|^2,\ g_{2z}(x)=\varphi_z(x)-K\|x\|^2$.

Theorem 7.

$$D_2^+f(x_0)\subset N_1^{[2]+}(f(x_0)),\ D_2^-f(x_0)\subset N_1^{[2]-}(f(x_0)),\ D_2f(x_0)\subset N_1^{[2]}(f(x_0)).$$

Theorem 8. *If*

$$\check{D}_2 f(x_0)=\left\{Q\in B_0(X): f^{(2)-}(x_0;x)\le Q(x)\le f^{(2)+}(x_0;x), x\in X\right\}\neq\varnothing,$$

then in order the function achieve at the point x_0 the local minimum it is necessary that $Q(x)\ge 0$ for any $Q\in\check{D}_2 f(x_0)$.

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