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CLASSICAL SOLVABILITY OF THE FIRST BOUNDARY VALUE PROBLEM FOR GILBARG-SERRIN EQUATION

Abstract

In the paper the Dirichlet problem for Gilbarg-Serrin equation is considered. This problem, generally speaking, has no classical solution. In this article the modified Dirichlet problem is put in conformity to given first boundary value problem. The unique classical solvability of the modified problem has been proved.

Introduction.

Let E_n be n- dimensional Euclidean space of points $x=(x_1,...,x_n)$, $n\geq 3$, and D be a bounded domain in E_n with boundary ∂D, $O\in D$. Let's consider in D the following first boundary value problem for Gilbarg-Serrin equation

$$Lu=\Delta u+\mu(r)\sum_{i,j=1}^{n}\frac{x_i x_j}{r^2}\frac{\partial^2 u}{\partial x_i \partial x_j}=f(x);\quad x\in D;\ u|_{\partial D}=\varphi(x), \tag{1}$$

where $r=|x|$, $f(x)\in C^{\alpha}(D)$, $\alpha\in(0,1)$, $\varphi(x)\in C(\partial D)$, and relative to function $\mu(r)$ the following conditions are satisfied

$$\mu(r)\in C^{\alpha}(D), \tag{2}$$

$$d_1\leq\mu(r)\leq d_2;\ d_1>n-2,\ \ d_2<\infty. \tag{3}$$

Here by $C^{\alpha}(D)$, $0<\alpha<1$, denote Banach space of functions $u(x)$, defined on D, with finite norm

$$\|u\|_{C^{\alpha}(D)}=\sup_{x\in D}|u(x)|+\sup_{\substack{x,y\in D\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^{\alpha}}.$$

It is well-known that the problem (1) in general case has not neither classical nor general solution in non-weighted Sobolev spaces [1-2]. In case of $\mu\equiv const$ weak generalized solvability of problem (1) in weighted Sobolev spaces was established in [3-5]. For equations with $\mu(r)\not\equiv const$ analogous results were obtained in [6-7]. In present paper the problem (1) is put in conformity to some modified Dirichlet problem and then unique classical solvability of the last is proved.

1^0. Generalized by Wiener solution of the modified problem.

Let $D'=D\setminus\{0\}$, a is any fixed number. We will call modified first boundary value problem for Gilbarg-Serrin equation the following problem

$$Lu=f,\ x\in D';\ \ u|_{\partial D}=\varphi,\quad u(0)=a. \tag{4}$$

In other words the point O be in problem (4) boundary for domain D', and what is more it is the carrier of the Dirichlet's data.

Let's give now the definition of generalized by Wiener solution of problem (4).

Let $\{D_m\}$, $m=1,...,$ be expanding sequence of domains with twice smooth boundaries ∂D_m, approximating domain D from within, i.e. $\overline{D}_m\subset D$, $\overline{D}_m\subset D_{m+1}$,

$\lim_{m\to\infty} D_m = D$. Let's consider further the sequence of spheres $S_m = \left\{x : |x| = \frac{1}{m}\right\}$ and closed balls $Q_m = \left\{x : |x| \le \frac{1}{m}\right\}$, $m = 1,...,$.

Let $D'_m = D_m \setminus Q_m$, $m = 1,...,$. It is obvious that for sufficiently large m for any domain D subdomain D'_m is non-empty. Not losing in a generality, we will suppose that D'_m is non-empty for all natural m. It is not difficult to see that the sequence $\{D'_m\}$ approximates domain D' from within. Let's continue boundary function $\varphi(x)$ as continuous from ∂D to D and let's denote the obtained extension by $\Phi(x)$. Let's consider for natural m the sequence of Dirichlet problems

$$Lu_m = f\ ,\ \ x \in D'_m\ ;\ \ u_m\big|_{\partial D_m} = \Phi\big|_{\partial D_m}\ ,\ \ u_m\big|_{S_m} = a\ . \tag{5}$$

Later we will show that for any natural m classical solution of problem (5) really exists.

If there exists limit $u_\varphi(x) = \lim_{m\to\infty} u_m(x)$, $x \in D'$, independent neither on method of extension of function $\varphi(x)$ from ∂D to D, nor on method of approximation of domain D by sequence of domains $\{D_m\}$, then function $u_\varphi(x)$ is called generalized by Wiener solution of the modified first boundary value problem (4).

Theorem 1. *Let relative to function* μ *conditions (2)-(3) are satisfied. Then for all functions* $\varphi(x) \in C(\partial D)$, $f(x) \in C^\alpha(D)$ *and for any* $a \in E_1$ *there exists generalized by Wiener solution* $u_\varphi(x)$ *of modified first boundary value problem (4). Moreover* $u_\varphi(x) \in C^2(D')$ *and for* $x \in D'$ $Lu_\varphi(x) = f(x)$.

Proof. In proof we will keep to a method stated in the monography [8]. First of all w will show that generalized solution $u_\varphi(x)$ does not depend on method of extension of boundary function $\varphi(x)$ from ∂D to D. Let $\Phi_1(x)$ and $\Phi_2(x)$ are two different continuous extensions of function $\varphi(x)$, and $u^i_m(x)$ are solutions of Dirichlet problems

$$Lu^i_m = f\ ,\ \ \ x \in D'_m\ ;\ \ u^i_m\big|_{\partial D_m} = \Phi_i\big|_{\partial D_m}\ ,\ \ \ u^i_m\big|_{S_m} = a\ .$$

Here $i = 1,2$; and $m = 1,...,$. It is not difficult to see that for every natural m the difference $w_m(x) = u^1_m(x) - u^2_m(x)$ is a solution of Dirichlet problem

$$Lw_m = 0,\ \ x \in D'_m\ ;\ \ w_m\big|_{\partial D_m} = \left(\Phi_1 - \Phi_2\right)\big|_{\partial D_m}\ ,\ \ w_m\big|_{S_m} = 0\ .$$

According to a maximum principle

$$\sup_{D'_m} w_m = \max\left\{\sup_{\partial D_m} w_m\ ,\ \sup_{S_m} w_m\right\} = \max\left\{\sup_{\partial D_m} w_m\ ,\ 0\right\} \le \sup_{\partial D_m}|w_m| = \sup_{\partial D_m}|\Phi_1 - \Phi_2|\ .$$

and analogically

$$\inf_{D'_m} w_m = \min\left\{\inf_{\partial D_m} w_m , \inf_{S_m} w_m\right\} \ge -\sup_{\partial D_m}|\Phi_1 - \Phi_2|\ .$$

Hence we conclude that

$$\sup_{D'_m}|w_m| \le \sup_{\partial D_m}|\Phi_1 - \Phi_2|\ . \tag{6}$$

On the other side

$$\lim_{m\to\infty}\sup_{\partial D_m}\left|\Phi_1-\Phi_2\right|=0, \tag{7}$$

since $\Phi_1(x)$ and $\Phi_2(x)$ are continuous extensions of the same boundary function $\varphi(x)$, and $\partial D_m\to\partial D$ for $m\to\infty$. From (6)-(7) we obtain

$$\lim_{m\to\infty}\sup_{D_m'}\left|w_m\right|=0,$$

from here follows independence of $u_\varphi(x)$ on method of extension of function $\varphi(x)$.

Now let's prove that classical solution $u_m(x)$ of the first boundary value problem (5) exists for any natural m.

Let for $i,j=1,...,n;\ x\in D_m'$

$$a_{ij}(x)=\delta_{ij}+\mu(|x|)\frac{x_ix_j}{|x|^2},$$

where δ_{ij} is Cronecker's symbol. We have for $x,y\in D_m'$

$$\left|a_{ij}(x)-a_{ij}(y)\right|=\left|\mu(|x|)\frac{x_ix_j}{|x|^2}-\mu(|y|)\frac{y_iy_j}{|y|^2}\right|\le$$

$$\le\left|\mu(|x|)\frac{x_ix_j}{|x|^2}-\mu(|y|)\frac{x_ix_j}{|x|^2}\right|+\mu(|y|)\left|\frac{x_ix_j}{|x|^2}-\frac{y_iy_j}{|y|^2}\right|=i_1+i_2. \tag{8}$$

Using a condition (2), we receive

$$i_i\le\left|\mu(|x|)-\mu(|y|)\right|\le C_1|x-y|^\alpha, \tag{9}$$

where a constant C_1 depends only on function μ.

We have further

$$i_2\le d_2\left|\frac{x_ix_j}{|x|^2}-\frac{y_iy_j}{|y|^2}\right|=d_2\left|\sum_{k=1}^n(x_k-y_k)\frac{\partial}{\partial x_k}\left(\frac{x_ix_j}{|x|^2}\right)\Big|_{x=\theta}\right|,$$

where θ is a point located on an interval, connecting points x and y.

Taking into account that

$$\frac{\partial}{\partial x_k}\left(\frac{x_ix_j}{|x|^2}\right)=\frac{\delta_{ik}x_j}{|x|^2}+\frac{\delta_{jk}x_i}{|x|^2}-2\frac{x_ix_jx_k}{|x|^4},$$

we will conclude

$$\left|\frac{\partial}{\partial x_k}\left(\frac{x_ix_j}{|x|^2}\right)\Big|_{x=\theta}\right|\le\frac{4}{|\theta|}.$$

But $\theta\in D_m'$, and consequently $|\theta|\ge\frac{1}{m}$. So

$$i_2\le 4d_2m\sum_{k=1}^n|x_k-y_k|\le 4d_2m\sqrt{n}|x-y|\le C_2m|x-y|^\alpha, \tag{10}$$

where the positive constant C_2 depends only on functions μ, n and domain D.

Taking into account (9)-(10) in (8), we receive

$$\left|a_{ij}(x)-a_{ij}y\right|\le (C_1+C_2m)|x-y|^{\alpha},$$

where $i,j=1,...,n;\ x,y\in D'_m$.

Thus the coefficients of operator in every subdomain D'_m are continuous in Hölder's sense. Moreover $dD'_m\in C^2$, $f(x)\in C^{\alpha}(D)$. Then according to well-known Schauder's theorem, the first boundary value problem (5) for any natural m has unique classical solution $u_m(x)\in C^{2+\alpha}(D'_m)\cap C(\overline{D}'_m)$.

Now we will prove the following fact: if a limit $u_{\varphi}(x)$ exists, hen it does not depend on method of approximation of domain D by sequence of domains $\{D_m\}$, $m=1,...,$.

Let $\{D_{m,1}\}$ and $\{D_{m,2}\}$ $(m=1,...,)$ be two expanding sequences of domains with twice smooth boundaries, approximating domain D from within, $D'_{m,i}=D_{m,i}\setminus Q_m$.

Let's further $u_{m,i}(x)$ are solutions of the first boundary value problems

$$Lu_{m,i}=f,\ \ x\in D'_m,i;\ \ u_{m,i}\big|_{\partial D_{m,i}}=\Phi\big|_{\partial D_{m,i}},\ \ u_{m,i}\big|_{S_m}=a.$$

Here $i=1,2;\ m=1,...,$. Let's denote $\lim\limits_{m\to\infty}u_{m,i}(x)$ by $u^i_{\varphi}(x)$, $i=1,2$. We will fix an arbitrary $\varepsilon>0$ and sufficiently large natural number m. Since $\lim\limits_{m\to\infty}D_{m,2}=D$, then there exists a natural number $m_1=m_1(m)$ such that $D'_{m,1}\subset D'_{m_1,2}$.

Without losing of generality we will assume that $m_1\ge m$.

Let for $x\in D_{m,1}$ $\vartheta_m(x)=u_{m,1}(x)-u_{m_1,2}(x)$. It is not difficult to see that the function $\vartheta_m(x)$ is the solution of the following Dirichlet problem

$$L\vartheta_m=0,\quad x\in D'_{m,1};\quad \vartheta_m\big|_{\partial D_{m,1}}=\Phi\big|_{\partial D_{m,1}}-u_{m,2}\big|_{\partial D_{m,1}},\ \ \vartheta_m\big|_{S_m}=0.$$

Since $\lim\limits_{m\to\infty}D_{m,1}=D$, then there exists a natural number m_2 such that for $m\ge m_2$

$$\left|\Phi\big|_{\partial D_{m,1}}-u_{m,2}\big|_{\partial D_{m,1}}\right|<\frac{\varepsilon}{2}.$$

Thus if $m\ge m_2$, then if $x\in D'_{m,1}$

$$\left|\vartheta_m(x)\right|<\frac{\varepsilon}{2}. \tag{11}$$

On the other side for $x\in D'_{m,1}$

$$\begin{aligned}\left|\vartheta_m(x)\right|&=\left|\left(u_{m,1}(x)-u^1_{\varphi}(x)\right)+\left(u^2_{\varphi}(x)-u_{m_1,2}(x)\right)+\left(u^1_{\varphi}(x)-u^2_{\varphi}(x)\right)\right|\ge\\&\ge\left|u^1_{\varphi}(x)-u^2_{\varphi}(x)\right|-\left|u_{m,1}(x)-u^1_{\varphi}(x)\right|-\left|u_{m_1,2}(x)-u^2_{\varphi}(x)\right|.\end{aligned} \tag{12}$$

There exists a natural number m_3, such that if $m\ge m_3$ and $x\in D'_{m,1}$, then

$$\left|u_{m,1}(x)-u^1_{\varphi}(x)\right|<\frac{\varepsilon}{4}. \tag{13}$$

Similarly, we will conclude about the existence of a natural number m_4, such that if $m\ge m_4$ (and $m_1\ge m_4$) and $x\in D'_{m,1}$, then

$$\left|u_{m_1,2}(x)-u^2_{\varphi}(x)\right|<\frac{\varepsilon}{4}. \tag{14}$$

Let $m_5 = \max\{m_2, m_3, m_4\}$. Then from (11)-(14) we obtain that if $m \ge m_5$ and $x \in D'_{m,1}$

$$\left|u_\varphi^1(x) - u_\varphi^2(x)\right| < \varepsilon .$$

Now taking into account arbitrariness of ε and tending m to infinity, we conclude that $u_\varphi^1(x) \equiv u_\varphi^2(x)$ for $x \in D'$.

Let's prove that a limit $u_\varphi(x)$ exists. For this purpose we consider such continuous extension of boundary function $\varphi(x)$ with ∂D in D, that $\Phi(x) \equiv a$ when $x \in Q_1$. We will fix arbitrary $\sigma > 0$ and let $F(x)$ is twice continuously differentiable in $\overline{D}$ function such that $F(x) \equiv a$ for $x \in Q_1$ and

$$\left|\Phi(x) - F(x)\right| < \sigma , \quad x \in \overline{D} . \tag{15}$$

Consider a function

$$F^+(x) = \frac{F(x)}{2} + A\sum_{k=1}^{n} x_k^2 ,$$

where the positive constant A will be chosen later. We have

$$LF^+ = \frac{1}{2}LF + 2An + 2A\mu(r)\sum_{i=1}^{n}\frac{x_i^2}{r^2} \ge -C_3 + 4A(n-1),$$

where the positive constant C_3 depends only on functions $\mu(r)$, $F(x)$ and n.

Choosing $A = \frac{1}{4(n-1)}\left(C_3 + \frac{1}{2}\sup_D|f|\right)$, we receive

$$LF^+ \ge \frac{1}{2}\sup_D|f| . \tag{16}$$

Completely similarly it is possible to show that if

$$F^-(x) = \frac{F(x)}{2} - A\sum_{k=1}^{n} x_k^2 ,$$

then for $x \in D$

$$LF^- \le -\frac{1}{2}\sup_D|f| . \tag{17}$$

Thus it is easy to see

$$F^+(x) + F^-(x) = F(x), \quad x \in D . \tag{18}$$

Let's introduce into consideration two sequences $\{u_m^+(x)\}$ and $\{u_m^-(x)\}$, $m = 1,\ldots,$ of solutions of the first boundary value problems

$$Lu_m^+ = \frac{f}{2}, \quad x \in D'_m \; ; \; u_m^+\big|_{\partial D_m} = F^+\big|_{\partial D_m} , \; u_m^+\big|_{S_m} = \frac{a}{2}$$

and

$$Lu_m^- = \frac{f}{2}, \quad x \in D'_m \; ; \; u_m^-\big|_{\partial D_m} = F^-\big|_{\partial D_m} , \; u_m^-\big|_{S_m} = \frac{a}{2}$$

correspondingly.

Let further for natural m $w_m^+(x) = u_m^+(x) - F^+(x)$. According to (16)

$$Lw_m^+ \le 0 \quad \text{for} \quad x \in D'_m .$$

Besides that $w_m^+\big|_{\partial D_m}=0$ and $w_m^+\Big|_{S_m}=\frac{a}{2}-F^+\Big|_{S_m}=\frac{a}{2}-\frac{1}{2}F\Big|_{S_m}-\frac{A}{m^2}=-\frac{A}{m^2}$, since $F\Big|_{S_m}=a$. By maximum principle we conclude that $w_m^+(x)\geq-\frac{A}{m^2}$ for $x\in D_m'$, i.e.

$$u_m^+(x)\geq F^+(x)-\frac{A}{m^2}. \tag{19}$$

By analogy, if for natural numbers m $w_m^-(x)=u_m^-(x)-F^-(x)$, then according to (17) $Lw_m^-\geq 0$ for $x\in D_m'$ and $w_m^-(x)\leq\frac{A}{m^2}$, i.e.

$$u_m^-(x)\leq F^-(x)+\frac{A}{m^2}. \tag{20}$$

For natural numbers $m>1$ we consider the domain D_{m-1}' and in it function $z_m^+(x)=u_m^+(x)-u_{m-1}^+(x)$.

It is clear that $Lz_m^+=0$ for $x\in D_{m-1}'$ and besides that by virtue of (19)

$$z_m^+\Big|_{\partial D_{m-1}}\geq F^+\Big|_{\partial D_{m-1}}-\frac{A}{m^2}-u_{m-1}^+\Big|_{\partial D_{m-1}}=-\frac{A}{m^2}$$

and

$$z_m^+\Big|_{S_{m-1}}\geq F^+\Big|_{S_{m-1}}-\frac{A}{m^2}-u_{m-1}^+\Big|_{S_{m-1}}=\frac{1}{2}F\Big|_{S_{m-1}}+\frac{A}{(m-1)^2}-\frac{A}{m^2}-\frac{a}{2}=$$
$$=\frac{A}{(m-1)^2}-\frac{A}{m^2}\geq-\frac{A}{m^2}$$

Again we apply a maximum principle. We will receive

$$z_m^+(x)\geq-\frac{A}{m^2},\quad x\in D_{m-1}',$$

i.e.

$$u_m^+(x)-u_{m-1}^+(x)\geq-\frac{A}{m^2}. \tag{21}$$

Now for natural numbers $m>1$ and $x\in D_{m-1}'$, we consider the function $g_m^+(x)=u_m^+(x)-\frac{A}{m}$. By virtue of (21)

$$g_m^+-g_{m-1}^+=u_m^+-u_{m-1}^++A\left(\frac{1}{m-1}-\frac{1}{m}\right)\geq-\frac{A}{m^2}+\frac{A}{m(m-1)}>0.$$

Thus sequence $\{g_m^+(x)\}$ increases. Besides that it is bounded. Really for $x\in D_{m-1}'$

$$g_m^+(x)\leq u_m^+(x)\leq\max\left\{\frac{1}{2}\sup_D\left(|F(x)|+2A\sum_{k=1}^n x_k^2\right),\ \frac{a}{2}\right\}+C_4\sup_D|f|,$$

where the positive constant C_4 depends on function μ, domain D and n.

So, sequence $\{g_m^+(x)\}$ converges when $m\to\infty$. The convergence of a sequence $\{u_m^+(x)\}$, $m\to\infty$ is proved.

Similarly, if for naturals $m>1$ in domain D'_{m-1} we consider function $z_m^-(x)=u_m^-(x)-u_{m-1}^-(x)$, then using (20) we can show that

$$z_m^-(x)\le\frac{A}{m^2}\ ,\quad x\in D'_{m-1}\,.$$

After that the sequence of functions $g_m^-(x)=u_m^-(x)+\frac{A}{m}$; $m=2,...,$; $x\in D'_{m-1}$ are introduced into consideration and is proved that

$$g_m^-(x)-g_{m-1}^-(x)\le 0\,.$$

Thus taking into account boundedness of sequence $\{g_m^-(x)\}$ we conclude about the existence of its limit when $m\to\infty$. So we have shown that the sequence $\{u_m^-(x)\}$ converges, when $m\to\infty$.

Now we will consider for natural m in domain D'_m the sequence $\vartheta_m(x)=u_m^+(x)+u_m^-(x)$. From above proved follows that this sequence converges, when $m\to\infty$.

Let at last $\{u_m(x)\}$ be the sequence of the solutions of the first boundary value problems (5), $m=1,...,$; $x\in D'_m$. According to (5) and (18) the difference $u_m(x)-\vartheta_m(x)$ for any natural m is a solution of the following Dirichlet problem

$$L(u_m-\vartheta_m)=0\ ,\ x\in D'_m\ ;\ (u_m-\vartheta_m)\big|_{\partial D_m}=(\Phi-F)\big|_{\partial D_m}\ ,\quad (u_m-\vartheta_m)\big|_{S_m}=0\,.$$

Using now the maximum principle, we receive

$$\sup_{D'_m}|u_m-\vartheta_m|\le\sup_{D'_m}|\Phi-F|\le\sup_{\overline{D}}|\Phi-F|\,.$$

Last inequality with (15) entails

$$\sup_{D'_m}|u_m-\vartheta_m|<\sigma\,,$$

that by virtue of arbitrariness σ, allows to make a conclusion about convergence of sequence $\{u_m(x)\}$, when $m\to\infty$. So existence of a limit $u_\varphi(x)$ is proved.

Now we will show that $u_\varphi(x)\in C^2(D')$. With this purpose we will consider any strictly interior subdomain $\widetilde{D}$ of domain D'.

It is obvious that there exists a natural number m^*, such that $\widetilde{D}\subset D'_m$ for all $m\ge m^*$. Let's consider subdomain D'_{m^*+1}. It is clear that

$$\nu=dist\left(\partial\widetilde{D},\,\partial D'_{m^*+1}\right)>0\,.$$

Let for natural m $u_m(x)$ be a solution of Dirichlet problem (5). According to a condition on function $f(x)$, (9) and (10)

$$\|f\|_{C^\alpha\left(D'_{m^*+1}\right)}\le C_5\,,$$

$$\|a_{ij}\|_{C^\alpha\left(D'_{m^*+1}\right)}\le C_6\ ,\quad i,j=1,...,n\,,$$

where a constant C_5 depends only on function f, and a constant C_6 - only on function μ, domain D, n and number m^*. Here $a_{ij}(x)=\delta_{ij}+\mu(|x|)\frac{x_ix_j}{|x|^2}$; $i,j=1,...,n$. Let's apply Schauder's interior appriori estimation ([8]).

We obtain that for all naturals m

$$\|u_m\|_{C^{2+\alpha}(\widetilde{D})} \le C_7, \tag{22}$$

where the positive constant C_7 depends only on functions f, μ, domain D, n, m^* and ν. Here

$$\|u\|_{C^{2+\alpha}(\widetilde{D})} = \|u\|_{C^2(\widetilde{D})} + \sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{C^\alpha(\widetilde{D})}.$$

Thus, it is easy to see that estimation (22) implies compactness of sequence $\{u_m(x)\}$ in norm $C^2(\widetilde{D})$. The last by virtue of uniform convergence of $u_m(x)$ to $u_\varphi(x)$, when $m \to \infty$ allows to conclude that $u_\varphi(x) \in C^2(\widetilde{D})$. Now it is enough to take into account arbitrariness of subdomain $\widetilde{D}$, and we receive $u_\varphi(x) \in C^2(D')$.

Now we will show that

$$Lu_\varphi(x) = f(x), \quad x \in D'. \tag{23}$$

Again we will consider an arbitrary strictly interior subdomain $\widetilde{D}$ of domain D'. For all natural $m > m^*$

$$Lu_m(x) = f(x), \quad x \in \widetilde{D},$$

that together with estimation (22) and by a continuity of coefficients $a_{ij}(x)$ in $\widetilde{D}$ implies equality

$$Lu_\varphi(x) = f(x), \quad x \in \widetilde{D}. \tag{24}$$

Now it is enough to take into account an arbitrariness of subdomain $\widetilde{D}$ and from (24) follows required equality (23). The theorem is proved.

2^0. Unique classical solvability of the modified problem.

Let for B- set $H \subset E_n$ $cap(H)$ means its Wiener's capacity [8]. Consider any point $x^0 \in \partial D$ and let for $z > 0$

$$\beta(z) = cap\left\{\left\{x : \left|x - x^0\right| < z\right\} \setminus D\right\}.$$

According to [9] it is possible to show that if

$$\int_0^{diam D} \frac{\beta(z)}{z^{n-2}} \frac{dz}{z} = \infty, \tag{25}$$

then a point x^0 is regular relative to Dirichlet problem, and for any function $\varphi(x) \in C(\partial D)$ the following equality is true

$$\lim_{\substack{x \to x^0 \\ x \in D'}} u_\varphi(x) = \varphi(x^0).$$

Theorem 2. *Let at every point $x^0 \in \partial D$ condition (25) holds, and function $\mu(r)$ satisfies the conditions (2)-(3). Then the modified first boundary value problem (4) has unique classical solution $u(x) \in C^2(D') \cap C(\overline{D}')$ for arbitrary $f(x) \in C^\alpha(D)$, $\varphi(x) \in C(\partial D)$ and $a \in E_1$. Moreover*

$$|u(x) - a| \le C_8 |x|^\delta, \quad x \in D', \tag{26}$$

where the positive constant δ depends only on function μ and n, and a constant C_8- also on a, functions φ, f and domain D.

Proof. Without losing of generality we may assume that $Q_1 \subset D$. We will show that a solution $u(x)$ of the modified Dirichlet problem (4) is its generalized by Wiener solution $u_\varphi(x)$. According to theorem 1 this function is classical solution of equation $Lu = f$ in D'. Besides that if at every point $x^0 \in \partial D$ Wiener's condition (25) is satisfied then $u_\varphi(x)$ is continuous up to boundary ∂D. At last the uniqueness of the classical solution directly follows from a maximum principle. Thus, we have to prove the estimation (26). It is enough to show its truth for $x \in Q_1 \setminus \{0\}$.

Let's find at first radial symmetrical solution $g(|x|)$ of equation $Lg = 0$ in $Q_1 \setminus \{0\}$. We have for $i, j = 1,...,n$

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = g''(r)\frac{x_i x_j}{r^2} + g'(r)\left[\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\right],$$

where δ_{ij} is Cronecker's symbol.

From here follows that

$$Lg = g''(r)(1 + \mu(r)) + \frac{n-1}{r} g'(r).$$

Let's suppose for $r \in (0,1)$ solution of ordinary differential equation

$$g''(r)(1 + \mu(r)) + \frac{n-1}{r} g'(r) = 0$$

as

$$g(r) = \int_0^r \exp\left[(n-1)\int_t^1 \frac{d\tau}{\tau(1 + \mu(\tau))}\right] dt.$$

Then $Lg = 0$ for $x \in Q_1 \setminus \{0\}$.

But on the other side by virtue of a condition (3)

$$\mu(r) \ge d_1 = n - 2 + \delta_1,$$

where $\delta_1 > 0$. So

$$\int_t^1 \frac{d\tau}{\tau(1 + \mu(\tau))} \le \frac{1}{n - 1 + \delta_1}\int_t^1 \frac{d\tau}{\tau} = \frac{1}{n - 1 + \delta_1} \ln\frac{1}{t},$$

and hence

$$g(|x|) \le \int_0^{|x|} \exp\left[\frac{n-1}{n-1+\delta_1} \ln\frac{1}{t}\right] dt = C_9 |x|^{\frac{\delta_1}{n-1+\delta_1}},$$

where $C_9 = \frac{n-1+\delta_1}{\delta_1}$. We will take now $\delta = \frac{\delta_1}{n-1+\delta_1}$. It is clear that $\delta \in (0,1)$. So we have shown that

$$0 < g(|x|) \le C_9 |x|^\delta, \ x \in Q_1 \setminus \{0\}. \tag{27}$$

Let's consider in $Q_1 \setminus \{0\}$ an auxiliary function

$$w_1(x) = u_\varphi(x) + A_1|x|^2,$$

where the positive constant A_1 will be chosen later. We have

$$Lw_1 = f + 2A_1 n + 2A_1\mu(r)\sum_{i=1}^{n}\frac{x_i^2}{r^2} \ge -C_{10} + 4A_1(n-1).$$

Here the positive constant C_{10} depends only on function f. Thus, the function $w_1(x)$ is L-subelliptic in $Q_1\setminus\{0\}$, if only $A_1 = \frac{C_{10}}{4(n-1)}$.

Let's introduce into consideration a function

$$P(x) = w_1(x) - a - C_{11}g(|x|), \quad x \in Q_1 \setminus \{0\},$$

where the positive constant C_{11} will be chosen later.

According to above proved the function $P(x)$ also is L-subelliptic in $Q_1\setminus\{0\}$. Besides that $P(0)=0$, and

$$P\big|_{\partial Q_1} \le \sup_{D'}|u_\varphi| + A_1 + |a| - C_{11}g(1) \le C_{12} + A_1 + |a| - C_{11}g(1), \tag{28}$$

where constant $C_{12} > 0$ depends only on functions φ, f, n and D.

On the other side

$$g(1) = C_{13} > 0, \tag{29}$$

where a constant C_{13} depends only on functions μ and n.

Using (29) in (28) we receive

$$P\big|_{\partial Q_1} \le C_{12} + A_1 + |a| - C_{11}C_{13}.$$

Let's choose and fix

$$C_{11} = \frac{C_{12} + A_2 + |a|}{C_{13}}.$$

Then $P\big|_{\partial Q_1} \le 0$. According to a maximum principle we conclude that $P(x)\le 0$ for $x\in Q_1\setminus\{0\}$, i.e.

$$u_\varphi(x) - a \le w_1(x) - a \le C_{11}g(|x|).$$

Taking into account (27), from the last inequality we obtain

$$u_\varphi(x) - a \le C_9 C_{11}|x|^\delta, \quad x \in Q_1\setminus\{0\}. \tag{30}$$

Completely similarly it is possible to show that function

$$w_2(x) = u_\varphi(x) - A_1|x|^2.$$

is L-superelliptic in $Q_1\setminus\{0\}$.

Further, introducing into consideration a function

$$P_1(x) = w_2(x) - a + C_{11}g(|x|),$$

it is not difficult to prove that

$$LP_1 \le 0, \ \ x\in Q_1\setminus\{0\};$$

$$P_1(0) = 0, \ \ P_1\big|_{\partial Q_1} \ge 0.$$

Thus according to a maximum principal follows that $P_1(x)\ge 0$ in $Q_1\setminus\{0\}$, i.e.

$$u_\varphi(x) - a \ge w_2(x) - a \ge -C_{11}g(|x|).$$

Thus

$$u_\varphi(x) - a \ge -C_9 C_{11}|x|^\delta, \quad x\in Q_1\setminus\{0\}. \tag{31}$$

From (30)-(31) we conclude

$$\left|u_{\varphi}(x)-a\right| \le C_8 |x|^{\delta}, \quad x \in Q_1 \setminus \{0\},$$

where $C_8 = C_9 C_{11}$. Last inequality coincides with required estimation (26).

The theorem is proved.

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