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ON SOLVABILITY OF ONE DEGENERATED SYSTEM OF SINGULAR INTEGRAL EQUATIONS

Abstract

Homogeneous degenerated singular system of three integral equations of special form with Cauchy kernel and closed Lyapunov contour was considered. It was proved, that ternary of functions of definite Hölder smoothness construct a solution of this system if and only if they are bounded values on given contour of solution of Helmholtz equation and its first derivatives. The fact that general solution of given system contains one arbitrary function, i.e. this system is not Nöter, was stated.

Let D^+ be bounded simple-connected domain in plane $R^+=\{x=(x_1;x_2)\}$, Γ is its boundary, i.e. closed Lyapunov line, $\overline{D}^+=D^+\cup\Gamma$, $D^-=R^2\setminus\overline{D}^+$, $\overline{D}^-=D^-\cup\Gamma$. Consider homogeneous system of integral equation

$$P(y)=\int_\Gamma \mathcal{K}(x,y)P(x)d_x\Gamma,\quad y=(y_1,y_2)\in\Gamma, \tag{1}$$

where $P(y)=colon[p_0(y),p_1(y),p_2(y)]$ is desired vector- function,

$$\mathcal{K}(x,y)=\frac{1}{\pi}\begin{pmatrix} -\frac{\partial}{\partial_x n_x}K_0(ar) & K_0(ar)\cos(n_x,x_1) & K_0(ar)\cos(n_x,x_2) \\ a^2K_0(ar)\cos(n_x,x_1) & -\frac{\partial}{\partial_x n_x}K_0(ar) & \frac{\partial}{\partial_x \tau_x}K_0(ar) \\ a^2K_0(ar)\cos(n_x,x_2) & -\frac{\partial}{\partial_x \tau_x}K_0(ar) & -\frac{\partial}{\partial_x n_x}K_0(ar) \end{pmatrix} \tag{2}$$

$r=|x-y|$, $K_0(r)$ is McDonald function, τ_x and n_x are, correspondingly, directions of tangent and external normal to Γ at point x, $a>0$ is constant.

In process of investigation of various boundary value problems for following equation arises system (1):

$$\Delta u(x)-a^2u(x)=0. \tag{3}$$

According to condition $\Gamma\in C_\alpha^1$, i.e. it allows parameterization (local) $\tilde{x}(s)\in C_\alpha^1[s_0,s_1]$, where by $\tilde{x},\tilde{y},\ldots$ we denote complex values, which corresponds to vectors $x,y,\ldots$ (for example, $\tilde{x}=x_1+ix_2$), and by C_α^k we denote a space of functions, which have continuous derivatives up to the order k in corresponding set, moreover, k-th derivatives are Hölder one with exponent $\alpha>0$. In particular, $C_\alpha^k(\Gamma)$ ($k=0,1$) is space of functions $f(x)$, given on Γ and such, that $f(x(s))\in C_\alpha^k[s_0,s_1]$, for parameterization $x(s)\in C_\alpha^1$ of Lyapunov neighborhood of each point $x\in\Gamma$.

By virtue of well-known asymptotic representations [5]

$$K_0(r)=-\ln r+O(1),\quad K_0'(r)=-r^{-1}+o(1)\quad (r\to 0) \tag{4}$$

and obvious equalities

$$d_x\Gamma=e^{-i\gamma(x)}d\tilde{x},\quad r=|x-y|=(\tilde{x}-\tilde{y})e^{-i\theta(x,y)},$$

$$\gamma(x)=\left(\overset{\rightarrow}{\tau}_x\ \overset{\wedge}{\ }\ \overset{\rightarrow}{ox_1}\right),\quad \theta(x,y)=\arg(\tilde{x}-\tilde{y}),$$

$$\frac{\partial K_0(ar)}{\partial n}=\left[-r^{-1}+o(1)\right]\cos(\vec{r},\vec{n}),$$

$$\frac{\partial K_0(ar)}{\partial \tau}=\left[-r^{-1}+o(1)\right]\cos(\vec{r},\vec{\tau})=-(\tilde{x}-\tilde{y})^{-1}e^{i\gamma(x)}\left[1+ie^{i\left(\vec{r}\,\hat{,}\,\vec{\tau}\right)}\cos\left(\vec{r}\,\hat{,}\,\vec{n}\right)\right]+o(1), \tag{5}$$

taking into account, that $\cos\left(\vec{r}\,\hat{,}\,\vec{n}\right)\le Cr^{\alpha}$, the system (1) we could rewrite in the form:

$$\mathcal{P}(\tilde{y})=\frac{1}{\pi i}\mathcal{K}_0\int_\Gamma(\tilde{x}-\tilde{y})^{-1}\mathcal{P}(\tilde{x})d\tilde{x}+\int_\Gamma\mathcal{K}_1(\tilde{x},\tilde{y})\mathcal{P}(\tilde{x})d\tilde{x},\quad \tilde{y}\in\Gamma, \tag{6}$$

where

$$\mathcal{P}(\tilde{y})=P(y),\quad \mathcal{K}_0=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \mathcal{K}_1(\tilde{x},\tilde{y})=\mathcal{K}_{10}(\tilde{x},\tilde{y}), \tag{7}$$

$0\le\sigma<1$, an elements of matrix $\mathcal{K}_{10}(\tilde{x},\tilde{y})$ belong to the space $C^0_\alpha(\Gamma)$ by $\tilde{x}$ and by $\tilde{y}$. Consequently, (6) (and therefore (1)) is full homogeneous system of singular integral equations with kernel of the Cauchy type.

According to fact, that

$$\det\left[E\pm\mathcal{K}_0\right]=0, \tag{8}$$

where E is unit matrix of third order, this system is degenerated [1, 3].

The following theorem explains connection between system (1) (or, (6)) and differential equations (3).

Theorem 1. *Let* $p_0(y)\in C^1_\alpha(\Gamma),\ p_j(y)\in C^0_\alpha(\Gamma)\ \ (j=1,2)$. *Then for these functions to be boundary values of some solution* $u(x)\in C^2(D^+)\cap C^1(\overline{D}^+)$ *of equation (3) and its first derivatives* $u_{x_1}(x), u_{x_2}(x)$, *correspondingly, it is necessary and sufficient for vector-function* $P(y)$ *to be a solution of system (1).*

Proof. Let $u(x)\in C^2(D^+)\cap C^1(\overline{D}^+)$ be some solution of equation (3),

$$U(x)=colon\left[u(x),u_{x_1}(x),u_{x_2}(x)\right],$$

$$W(x-\xi)=colon\left[K_0(a|x-\xi|),\frac{\partial}{\partial x_1}K_0(a|x-\xi|),\frac{\partial}{\partial x_2}K_0(a|x-\xi|)\right]$$

and $U(y)=\mathcal{P}(y)$, for $y\in\Gamma$. Applying Green formula for the left-hand side of identity

$$\int_{D^+}W(x-\xi)\left[\Delta u(x)-a^2u(x)\right]dx\equiv colon[0,0,0],\quad \xi\in D^+,$$

and taking into account, that $K_0(a|x|)$ is fundamental solution of equation (3) (see [4]), i.e.

$$\left(\Delta-a^2\right)K_0(a|x|)=\delta(x),\quad x\in R^2,$$

we easily obtain validity of equation (1).

Let us prove sufficiency. Let for some functions $p_0(y)\in C^1_\alpha(\Gamma)$, $p_j(y)\in C^0_\alpha(\Gamma)$ $(\alpha>0,\ j=1,2)$ (1) holds. Compose a function

$$\vartheta(\xi)=\vartheta_1(\xi)+\vartheta_2(\xi),\quad \xi\in D^+\cup D^-, \tag{9}$$

where

$$\vartheta_1(\xi)=-\frac{1}{2\pi}\int_\Gamma\frac{\partial}{\partial_x n_x}K_0(a|x-\xi|)p_0(x)d_x\Gamma, \tag{10}$$

$$\vartheta_2(\xi)=\frac{1}{2\pi}\sum_{j=1}^{2}\int_\Gamma K_0(a|x-\xi|)p_j(x)\cos(n_x,x_j)d_x\Gamma. \tag{11}$$

According to properties of potentials of simple and double layers [4, 5]

$$\vartheta_1(\xi)\in C^\infty(R^2\setminus\Gamma),\quad \vartheta_2(\xi)\in C^\infty(R^2\setminus\Gamma)\cap C(R^2),$$

$$C(\overline{D}^\pm)\ni\begin{cases}\vartheta_1(\xi), & \text{for}\quad \xi\in D^\pm\\ \pm\frac{1}{2}p_0(y)+\vartheta_{10}(y), & \text{for}\ \xi=y\in\Gamma\end{cases}$$

$$C(\overline{D}^\pm)\ni\begin{cases}\dfrac{\partial\vartheta_2(\xi)}{\partial n_y}, & \text{for}\ \xi\in D^\pm\\ \mp\frac{1}{2}\sum_{j=1}^{2}p_j(y)\cos(n_y,y_j)+\left(\dfrac{\partial\vartheta_2}{\partial n_y}\right)_0, & \text{for}\ \xi=y\in\Gamma,\end{cases} \tag{12}$$

where $\vartheta_{10}(y)$ and $\left(\dfrac{\partial\vartheta_2}{\partial n_y}\right)_0$ are strict values of potential of double layer and derivative by normal potential of simple layer , correspondingly,

$$\vartheta_{10}(y)=-\frac{1}{2\pi}\int_\Gamma\left[\frac{\partial}{\partial_x n_x}K_0(a|x-y|)\right]p_0(x)d_x\Gamma,\quad y\in\Gamma,$$

$$\left(\frac{\partial\vartheta_2(y)}{\partial n_y}\right)_0=\frac{1}{2\pi}\sum_{j=1}^{2}\int_\Gamma\left[\frac{\partial}{\partial_y n_y}K_0(a|x-y|)\right]p_j(x)\cos(n_x,x_j)d_x\Gamma,\quad y\in\Gamma. \tag{13}$$

Moreover, ϑ_1 and ϑ_2 are solutions of equation (3):

$$\Delta\vartheta_i(\xi)-a^2\vartheta_i(\xi)=0,\quad \xi\in D^\pm. \tag{14}$$

From (9) and (12)-(14) we conclude, that $\vartheta(\xi)\in C^\infty(D^+)$, is solution of equation (3) in D^+ and

$$\lim_{D^+\ni\xi\to y\in\Gamma}\vartheta(\xi)=\frac{1}{2}p_0(y)+\vartheta_{10}(y)+\vartheta_2(y).$$

But by virtue of first equality of system (1), we have:

$$\vartheta_{10}(y)+\vartheta_2(y)=\frac{1}{2}p_0(y).$$

Consequently, function

$$V(\xi)=\begin{cases}\vartheta(\xi) & \text{for}\quad \xi\in D^+\\ p_0(\xi) & \text{for}\quad \xi\in\Gamma\end{cases} \tag{15}$$

is a solution of Dirichlet problem

$$\Delta V(\xi)-a^2V(\xi)=0,\quad \xi\in D^+, \tag{16}$$

$$V(y)=p_0(y),\quad y\in\Gamma \tag{17}$$

and belongs to space $C^2(D^+)\cap C(\overline{D}^+)$. From last, because of $p_0(y)\in C^1_\alpha(\Gamma)$, $\Gamma\in C^1_\alpha$, using inforced Kellog theorem [2], we obtain that $V(\xi)\in C^2(D^+)\cap C^1_\alpha(\overline{D}^+)$. Such smoothness of function $V(\xi)$ allows to apply Green formula for left-hand side of identity

$$\int_{D^+} W(x-\xi)\left[\Delta V(x)-a^2V(x)\right]dx\equiv colon[0,0,0]$$

and it leads to equalities

$$colon\left[V(y),V_{y_1}(y),V_{y_2}(y)\right]=\int_\Gamma \mathcal{K}(x,y)colon\left[V(x),V_{x_1}(x),V_{x_2}(x)\right]d_x\Gamma, \tag{18}$$

where $\mathcal{K}(x,y)$ is matrix, that was defined by formula (2). From first equality of (18), regarding (1) and (17), we obtain

$$\sum_{j=1}^{2}\int_\Gamma K_0(a|x-y|)V^{(j)}(x)\cos(n_x,x_j)d_x\Gamma=0,\quad y\in\Gamma, \tag{19}$$

where

$$V^{(j)}(x)=V_{x_j}(x)-p_j(x)\quad (j=1,2). \tag{20}$$

Analogously, regarding (1), from last two equalities of system (18) we find:

$$V^{(1)}(y)=-\frac{1}{\pi}\int_\Gamma\left[\frac{\partial K_0(a|x-y|)}{\partial_x n_x}V^{(1)}(x)-\frac{\partial K_0(a|x-y|)}{\partial_x\tau_x}V^{(2)}(x)\right]d_x\Gamma,$$

$$V^{(2)}(y)=-\frac{1}{\pi}\int_\Gamma\left[\frac{\partial K_0(a|x-y|)}{\partial_x\tau_x}V^{(1)}(x)+\frac{\partial K_0(a|x-y|)}{\partial_x n_x}V^{(2)}(x)\right]d_x\Gamma. \tag{21}$$

By virtue of (19) and properties of potential of simple layer a function

$$-\frac{1}{2\pi}\int_\Gamma K_0(a|x-y|)\sum_{j=1}^{2}V^{(j)}(x)\cos(n_x,x_j)d_x\Gamma \tag{22}$$

is solution of homogeneous Dirichlet problem for equation (3) as in D^+, as well in D^-, and belongs to space $C^2(R^2\setminus\Gamma)\cap C(R^2)$. Moreover, according to well-known estimation [4, 5] for function $K_0(r)$ for large r, function (22) tends to zero for $|\xi|\to\infty$. Then, it is clear, that this function identically equal to zero, and therefore from last correlation of (12) we obtain

$$\sum_{j=1}^{2}V^{(j)}(x)\cos(n_x,x_j)\equiv 0,\quad x\in\Gamma, \tag{23}$$

and it means, that

$$V^{(j)}(x)=(-1)^{j-1}f(x)\cos\theta_{3-j}(x)\quad (j=1,2), \tag{24}$$

for which functions $f(x)\in C^0_\alpha(\Gamma)$, where $\theta_1(x)$ and $\theta_2(x)$ are angles between vector $\vec{n}_x$ and first and second coordinate axes correspondingly. Substituting (24) into (21), we obtain

$$f(y)\cos\theta_j(y)=-\frac{1}{\pi}\int_\Gamma\left[\frac{\partial K_0(a|x-y|)}{\partial_x n_x}\cos\theta_j(x)+(-1)^j\frac{\partial K_0(a|x-y|)}{\partial_x\tau_x}\cos\theta_{3-j}(x)\right]f(x)d_x\Gamma,$$

$$(j=1,2). \tag{25}$$

Multiplying (25) by matrix

$$\begin{pmatrix} \cos\theta_1(y) & \cos\theta_2(y) \\ -\cos\theta_2(y) & \cos\theta_1(y) \end{pmatrix},$$

we pass to equivalent system

$$f(y) = -\frac{1}{\pi}\int_\Gamma \frac{\partial K_0(a|x-y|)}{\partial_x n_y} f(x) d_x\Gamma,$$

$$0 = -\frac{1}{\pi}\int_\Gamma \frac{\partial K_0(a|x-y|)}{\partial_x \tau_y} f(x) d_x\Gamma. \quad (26)$$

According to equalities

$$\frac{\partial K_0(a|x-y|)}{\partial_x n_y} = -\frac{\partial K_0(a|x-y|)}{\partial_y n_y}, \quad \frac{\partial K_0(a|x-y|)}{\partial_x \tau_y} = -\frac{\partial K_0(a|x-y|)}{\partial_y \tau_y}$$

and fact, that in potential of simple layer the derivative by tangential direction we can replace over the sign of integral [3], and also taking into account jump formula for derivative by normal of potential of simple layer, from (25) we obtain:

$$F(y) = -\frac{1}{2\pi}\int_\Gamma K_0(a|x-y|) f(x) d_x\Gamma \equiv C, \quad y \in \Gamma, \quad (27)$$

$$\lim_{D^+ \ni \xi \to y \in \Gamma} \frac{\partial F(\xi)}{\partial n_y} = 0. \quad (28)$$

From the other side, it is obvious, that

$$\Delta F(\xi) - a^2 F(\xi) \equiv 0, \ \xi \in D^+. \quad (29)$$

From (27)-(29) it follows, that

$$F(\xi) = -\frac{1}{2\pi}\int_\Gamma K_0(a|x-\xi|) f(x) d_x\Gamma \equiv 0, \quad \xi \in R^2,$$

and by the same conclusions, which were done by the help of equality (23), we find:

$$f(y) \equiv 0, \quad y \in \Gamma. \quad (30)$$

From (30), (20) and (24) we obtain, that

$$V_{x_j}(x) = p_j(x) \quad (j = 1,2), \quad x \in \Gamma.$$

The theorem is proved.

From proven theorem follows important corollary on general solution of system (1) (the same, that (6)). Under general solution we understand such solution, from which we obtain any solution $P(y) = colon[p_0(y), p_1(y), p_2(y)]$, which have property: $p_0(y) \in C_\alpha^1(\Gamma)$, $p_j(y) \in C_\alpha^0(\Gamma)$ $(j = 1,2)$.

Corollary. *General solution of system of singular equations (1) contains one arbitrary function.*

Really, if $p_0(y)$ is arbitrary function from $C_\alpha^1(\Gamma)$, then solving Dirichlet problem

$$\Delta u(x) - a^2 u(x) = 0, \ x \in D^+,$$

$$uy = p_0(y), \quad y \in \Gamma$$

we obtain unique function $u(x, p_0) \in C_\alpha^1(\overline{D}^+)$ by the help of which we construct solution of system (1)

$$colon[p_0(y), u_{y_1}(y, p_0), u_{y_2}(y, p_0)], \quad y \in \Gamma \quad (31)$$

which have necessary property. Contrary, if $P^*(y)=colon\left[p_0^*(y), p_1^*(y), p_2^*(y)\right]$ is some solution $\left(p_0^*(y)\in C_\alpha^1(\Gamma),\ p_j^*(y)\in C_\alpha^0(\Gamma),\ j=1,2\right)$ of system (1), then choosing in (31) $p_0(y)=p_0^*(y)$ by theorem we automatically obtain

$$u_{y_j}\left(y,p_0^*\right)=p_j^*(y)\quad (j=1,2).$$

It must be noted, that general solution of system (1) could by constructed by another way. For given functions $l_0(y)>0$, $l_j(y)$ $\left(j=1,2; l_1(y)\cos(n_y,y_1)+l_2(y)\cos(n_y,y_2)>0\right)$ and arbitrary function $q(y)$ from $C_\alpha^1(\Gamma)$ solving problem with oblique derivative

$$\Delta u(x)-a^2u(x)=0,\ x\in D^+,$$

$$l_0(y)u(y)+l_1(y)u_{y_1}(y)+l_2(y)u_{y_2}(y)=q(y),\quad y\in\Gamma,$$

we find unique function $u(x,q)\in C_\alpha^1\left(\overline{D}^+\right)$ [2], by the help of which we construct solution of system (1):

$$colon\left|u(y,q), u_{y_1}(y,q), u_{y_2}(y,q)\right|,\quad y\in\Gamma,$$

containing arbitrary function $q(y)$.

References

[1]. Гахов Ф.Д. *Краевые задачи.* М., «Наука», 1977, 640с.
[2]. Гилбарг Д., Трудингер Н. *Эллиптические дифференциальные уравнения с частными производными второго порядка.* М., "Наука", 1989, 463с.
[3]. Мусхелишвили Н.И. *Сингулярные интегральные уравнения.* М., "Наука", 1968, 511с.
[4]. Расулов М.Л. *Применение метода контурного интеграла.* М., "Наука", 1975, 255с.
[5]. Тихонов А.Н., Самарский А.А. *Уравнения математической физики.* М., "Наука", 1966, 724с.

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