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BEHAVIOR IN UNBOUNDED DOMAINS OF SOLUTION OF DEGENERATE ELLIPTIC EQUATIONS OF THE SECOND ORDER IN DIVERGENCE FORM

Abstract

The class of elliptic equations of the second order of divergent structure with non-uniform power degeneration at the infinity was considered in present paper. For the solution of mentioned equations theorems of the Phragmen-Lindelöf type are proved.

Introduction. Let D be unbounded domain in n-dimensional Euclidian space E_n of points $x=(x_1,\dots,x_n)$, $n\geq 3$, and ∂D be a boundary of D. Consider in D the following equation

$$Lu=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)=0 \tag{1}$$

with assumption, that $\|a_{ij}(x)\|$ is real symmetric matrix with measurable in D elements and for $x\in D$, $\xi\in E_n$ the following condition holds

$$\mu\sum_{i=1}^{n}\lambda_i(x)\xi_i^2\leq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leq\mu^{-1}\sum_{i=1}^{n}\lambda_i(x)\xi_i^2, \tag{2}$$

where $\mu\in(0,1]$ is constant, $\lambda_i(x)=\left(1+|x|_\alpha\right)^{-\alpha_i}$, $|x|_\alpha=\sum_{i=1}^{n}|x_i|^{\frac{2}{2-\alpha_i}}$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_i\in$ $\in\left[0,\frac{2}{n-1}\right)$, $(i=1,\dots,n)$, moreover, if $n=3$, then $|\alpha|=\alpha_1+\alpha_2+\alpha_3<2$.

The aim of present article is to prove theorems of the Phragmen-Lindelöf type for solution of equation (1). For this direction we could mention papers of E.M. Landis [1-2] and V.G. Maz'ya. Later investigations of this subject were described at papers [4-9]. It must be noted, that analogous by statement of problem of present article result for elliptic equation of non-divergent structure was obtained in [10].

1^0. Denotations, definitions and auxiliary statements. Let G be some bounded domain in E_n. Denote by $W_{2,\alpha}^1(G)$ the Banach space of functions $u(x)$, defined on G, with finite norm

$$\|u\|_{W_{2,\alpha}^1(G)}=\left(\int_G\left[u^2+\sum_{i=1}^{n}\lambda_i(x)\left(\frac{\partial u}{\partial x_i}\right)^2\right]dx\right)^{1/2},$$

and by $\overset{\circ}{W}{}_{2,\alpha}^1(G)$ we denote subspace of $W_{2,\alpha}^1(G)$, the dense set in which is aggregate of all functions from $C_0^\infty(G)$.

Let $\lambda(x)$ be almost everywhere positive and finite in G function. By $L_{2,\lambda}(G)$ we will denote Banach space of functions $u(x)$, defined on G, with finite norm

$$\|u\|_{L_{2,\lambda}(G)}=\left(\int_G u^2\lambda(x)dxdx\right)^{1/2}.$$

We will say that function $u(x)$, defined on D, belongs to space $W_{2,\alpha}^{1,loc}(D)$, if $u(x)\in W_{2,\alpha}^{1}(D\bigcap Q_R)$ for each $R>0$. Here Q_R is open ball with radius R and centered at the origin.

Function $u(x)\in W_{2,\alpha}^{1,loc}(D)$ is called a weak solution of equation (1) in D, if for arbitrary function $\varphi(x)\in C_0^\infty(D)$ the following integral identity holds

$$\int_D \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}dx=0.$$

Let B_1 and B_2 be bounded sets in E_n, $B_1\subset \overline{B}_2$ and d be some constant. We will say, that function $u(x)\in W_{2,\alpha}^1(B_2)$ is more than or equal to d on B_1 in the sense of $W_{2,\alpha}^1$, if there exists the sequence of functions $\{\vartheta_j(x)\}$, $j=1,2,\dots$, such that $\vartheta_j(x)\in C^\infty(\overline{B}_2)$, $\vartheta_j(x)\ge d$ for $x\in B_1$ and $\lim_{j\to\infty}\|\vartheta_j-u\|_{W_{2,\alpha}^1(B_2)}=0$.

If B_2 and, maybe, B_1 are unbounded sets, then we will say, that function $u(x)\in W_{2,\alpha}^{1,loc}(B_2)$ is more than or equal to d on B_1 in the sense of $W_{2,\alpha}^1$, if for any $R>0$ there exists sequence of functions $\{\vartheta_j^{(R)}(x)\}$, $j=1,2,\dots$, such that $\vartheta_j^{(R)}(x)\in C^\infty\left(\overline{B_2\bigcap Q_R}\right)$, $\vartheta_j^{(R)}(x)\ge d$ for $x\in B_1\bigcap Q_R$ and $\lim_{j\to\infty}\|\vartheta_j^{(R)}-u\|_{W_{2,\alpha}^1(B_2\bigcap Q_R)}=0$.

By the same way we determine notions $u(x)\le d$ and $u(x)=d$ in the sense of $W_{2,\alpha}^1$.

For $y\in E_n$, $R>0$, $k>0$ by $\mathcal{E}_R^y(k)$ we will denote closed ellipsoid $\left\{x:\sum_{i=1}^n\frac{(x_i-y_i)^2}{R^{-\alpha_i}}\le (kR)^2\right\}$. Let Σ be some ellipsoid, K be strict interior with respect to Σ compact,

$$V_\Sigma(K)=\left\{u\in \overset{\circ}{W}{}_{2,\alpha}^1(\Sigma):u\ge 1 \text{ on } K \text{ in the sense } W_{2,\alpha}^1\right\},\ J_\Sigma(u)=\int_\Sigma\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx.$$

The number $cap_\Sigma(K)=\inf_{u\in V_\Sigma(K)}J_\Sigma(u)$ is called a capacity of compact K with respect to Σ, generated by operator L.

There exists unique function $U_\Sigma(x)$ giving the minimum to functional $J_\Sigma(u)$ (see [11]). It calls capacitary potential K with respect to Σ. For this there exists unique measure μ with support on external boundary of the compact K, such that $U_\Sigma(x)=\int_\Sigma g(x,y)d\mu(y)$, $\mu(K)=cap_\Sigma(K)$ and $U_\Sigma(x)=1$ on K in the sense of $W_{2,\alpha}^1$. Here $g(x,y)$ is the Green's function of operator L ([11]). Measure μ is called capacitary distribution of K with respect to Σ.

If capacity of compact K is generated by operator $L_0=\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\lambda_i(x)\frac{\partial}{\partial x_i}\right)$, then we will denote it by $Cap_\Sigma(K)$. If $\Sigma=E_n$, then we will denote the corresponding capacities by $cap(K)$ and $Cap(K)$.

Lemma 1 ([12]). *Let $u(x)\in W_{2,\alpha}^{1,loc}(D)$ be a weak solution of equation (1) in D, with respect to coefficients of which condition (2) holds. Then function $u(x)$ is continuous in the Hölder's sense at each bounded strictly interior subdomain of domain D.*

Lemma 2 ([13]). *Let $u(x)\in W_{2,\alpha}^{1,loc}(D)$ be non-negative in the sense of $W_{2,\alpha}^{1}$ weak solution of equation (1) in D, moreover, with respect to coefficients of operator L condition (2) holds. Then, if $\mathcal{E}_R^{\gamma}(k)\subset D$, then*

$$\sup_{\partial\mathcal{E}_R^{\gamma}(k)} u\le C_1(\mu,\alpha,n)\inf_{\partial\mathcal{E}_R^{\gamma}(k)} u .$$

Here and further the record $C(\cdots)$ means that positive constant C depends only on contents of the brackets.

For $h>0$ by $u^h(x)$ we denote the average by Friedrichs of function $u(x)$, and let

$$L_h=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}^h(x)\frac{\partial}{\partial x_j}\right).$$

Lemma 3. *Let G be bounded domain in E_n, and coefficients of operator L are defined in G, satisfying to condition (2). Then, if $\varphi(x)\in W_{2,\alpha}^{1}(G)$ is defined function, and $u(x)$ and $u_h(x)$ are weak solutions of boundary value problems $Lu=0$ in G, $u-\varphi\in\mathring{W}_{2,\alpha}^{1}$ and $L_hu_h=0$ in G, $u_h-\varphi^h\in\mathring{W}_{2,\alpha}^{1}(G)$ correspondingly, then*

$$\lim_{h\to 0}\|u-u_h\|_{W_{2,\alpha}^1(G)}=0. \qquad (3)$$

Proof. We will use the following fact, that has been proven in [14]: if $\xi(x)\in W_{2,\alpha}^{1}(G)$, and $\vartheta(x)$ is a weak solution of boundary value problem

$$L\vartheta=f+\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}\ \text{ in }\ G;\quad \vartheta-\zeta\in\mathring{W}_{2,\alpha}^{1}(G), \qquad (4)$$

where $f\in L_2(G)$, $f_i\in L_{2,\lambda_i^{-1}}(G)$, $i=1,\dots,n$, then

$$\|\vartheta\|_{W_{2,\alpha}^1(G)}\le C_2(\mu,\alpha,n,G)\left(\|\zeta\|_{W_{2,\alpha}^1(G)}+\|f\|_{L_2(G)}+\sum_{i=1}^{n}\|f_i\|_{L_{2,\lambda_i^{-1}}(G)}\right). \qquad (5)$$

For this under weak solution of equation of boundary value problem (4) we understand function $\vartheta(x)\in W_{2,\alpha}^{1}(G)$ such that for any $\psi(x)\in C_0^{\infty}(G)$ the following identity holds

$$\int_G\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial\vartheta}{\partial x_i}\frac{\partial\psi}{\partial x_j}dx=\int_G\left[-f\psi+\sum_{i=1}^{n}f_i\frac{\partial\psi}{\partial x_i}\right]dx .$$

Suppose $\vartheta=u-u_h$. It is easy to see that function ϑ is a weak solution of boundary value problem (4) with $\zeta=\varphi-\varphi^h$, $f=0$, $f_i=-\sum_{j=1}^{n}\left(a_{ij}-a_{ij}^h\right)\frac{\partial u_h}{\partial x_j}$. According to estimate (5)

$$\|\vartheta\|_{W_{2,\alpha}^1(G)}\le C_2\left(\left\|\varphi-\varphi^h\right\|_{W_{2,\alpha}^1(G)}+\sum_{i=1}^{n}\left\|\sum_{j=1}^{n}\left(a_{ij}-a_{ij}^h\right)\frac{\partial u_h}{\partial x_j}\right\|_{L_{2,\lambda_i^{-1}}(G)}\right). \qquad (6)$$

Moreover, according to the average property

$$\lim_{h\to 0}\left\|\varphi-\varphi^h\right\|_{W_{2,\alpha}^1(G)}=0. \qquad (7)$$

From the other side

$$\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n}\left(a_{ij}-a_{ij}^{h}\right)\frac{\partial u_h}{\partial x_j}\right\|_{L_{2,\lambda_i^{-1}}(G)}\right)^2\le n^2\sum_{i,j=1}^{n}\int_G\left(a_{ij}-a_{ij}^{h}\right)^2\left(\frac{\partial u_h}{\partial x_j}\right)^2\frac{dx}{\lambda_i(x)}=$$

$$=n^2\sum_{i,j=1}^{n}\int_G\left(\frac{a_{ij}-a_{ij}^{h}}{\sqrt{\lambda_i(x)\lambda_j(x)}}\right)^2\lambda_j(x)\left(\frac{\partial u_h}{\partial x_j}\right)^2dx. \qquad (8)$$

From condition (2) follows, that matrix $\left\|\frac{a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}}\right\|$ is uniformly positively defined in G, i.e. for $i,j=1,\dots,n$ function $\frac{a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}}$ are bounded almost every where in G.

We have

$$\left|\frac{a_{ij}-a_{ij}^{h}}{\sqrt{\lambda_i(x)\lambda_j(x)}}\right|\le\left|\frac{a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}}-\left(\frac{a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}}\right)^h\right|+\int_{Q_h^x}\left|\frac{a_{ij}(y)}{\sqrt{\lambda_i(y)\lambda_j(y)}}\right|\times$$

$$\times\left|1-\sqrt{\frac{\sqrt{\lambda_i(y)\lambda_j(y)}}{\sqrt{\lambda_i(x)\lambda_j(x)}}}\right|\omega_n(x,y)dy;\quad (i,j=1,\dots,n), \qquad (9)$$

where $Q_h^x=\{y:|x-y|<h\}$, and ω_h is a kernel of average. Let $i=1,\dots,n$, $x\in G$, $y\in Q_h^x$. Then

$$\left|\frac{\lambda_i(y)}{\lambda_i(x)}-1\right|=\left(1+\frac{|x|_\alpha-|y|_\alpha}{1+|y|_\alpha}\right)^{\alpha_i}-1\le a_G^i(h),$$

where $\lim_{h\to 0}a_G^i(h)=0$. Therefore from (9)we obtain, that for $i,j=1,\dots,n$ almost everywhere in G

$$\lim_{h\to 0}\left|\frac{a_{ij}(x)-a_{ij}^{h}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}}\right|=0. \qquad (10)$$

Taking into account of (10) in (8), we conclude

$$\sum_{i=1}^{n}\left\|\sum_{j=1}^{n}\left(a_{ij}-a_{ij}^{h}\right)\frac{\partial u_h}{\partial x_j}\right\|_{L_{2,\lambda_i^{-1}}(G)}\le d_G(h)\|u_h\|_{W_{2,\alpha}^1(G)}, \qquad (11)$$

where $\lim_{h\to 0}d_G(h)=0$.

We again use estimate (5). We have for sufficiently small h

$$\|u_h\|_{W_{2,\alpha}^1(G)}\le C_2\left(\left\|\varphi^h-\varphi\right\|_{W_{2,\alpha}^1(G)}+\|\varphi\|_{W_{2,\alpha}^1(G)}\right)\le C_2\left(1+\|\varphi\|_{W_{2,\alpha}^1(G)}\right). \qquad (12)$$

Now from (6), (7), (11) and (12) follows required equality (3). Lemma is proved.

2^0. Some capacitary estimates.

Lemma 4. *Let $R\ge 1$, $|\alpha|=\alpha_1+\dots+\alpha_n$. Then there exist constants $C_3(\alpha,n)$ and $C_4(\alpha,n)$ such, that*

$$C_3 R^{n-2-\frac{|\alpha|}{2}} \le Cap\left(\mathcal{E}_R^0(1)\right) \le C_4 R^{n-2-\frac{|\alpha|}{2}}. \tag{13}$$

Proof. Let $\prod_R=\left\{x:|x_i|\le R^{1-\alpha_i/2}; i=1,\dots,n\right\}$, $\prod_R^1=\left\{x:|x_i|\le 2R^{1-\alpha_i/2}; i=1,\dots,n\right\}$. So as $\mathcal{E}_R^0(1)\subset\prod_R$, then

$$Cap\left(\mathcal{E}_R^0(1)\right)\le Cap\left(\prod_R\right). \tag{14}$$

Consider functions $f_i(t)\in C_0^\infty(E_1)$ such that $f_i(t)=1$ for $|t|\le R^{1-\alpha_i/2}$, $f_i(t)=0$ for $|t|\ge 2R^{1-\alpha_i/2}$, $0\le f_i(t)\le 1$, and

$$\left|\frac{df_i}{dt}\right|\le\frac{C_5}{R^{1-\alpha_i/2}};\quad i=1,\dots,n. \tag{15}$$

Then, if $u(x)=\prod_{j=1}^n f_j(x_j)$, then $u(x)=1$ in $\prod_R$, $u(x)=0$ out of $\prod_R^1$, $u(x)\in C_0^\infty(E_n)$, and, by virtue of (15),

$$\left|\frac{\partial u}{\partial x_i}\right|\le\frac{C_5}{R^{1-\alpha_i/2}};\quad i=1,\dots,n. \tag{16}$$

Therefore,

$$Cap\left(\prod_R\right)\le\sum_{i=1}^n\int_{\prod_R^1\setminus\prod_R}\lambda_i(x)\left(\frac{\partial u}{\partial x_i}\right)^2dx. \tag{17}$$

From the other side for $x\in\prod_R^1\setminus\prod_R$ $|x_j|\ge R^{1-\alpha_j/2}$, $j=1,\dots,n$, i.e. $|x|_\alpha=\sum_{j=1}^n|x_j|^{\frac{2}{2-\alpha_j}}\ge nR$, and

$$\lambda_i(x)\le(1+nR)^{-\alpha_i}; i=1,\dots,n. \tag{18}$$

Taking into account (16) and (18) in (17), we obtain

$$Cap\left(\prod_R\right)\le C_5\sum_{i=1}^n\int_{\prod_R^1\setminus\prod_R}\left(\frac{R}{1+nR}\right)^{\alpha_i}R^{-2}dx\le C_5 nR^{-2}mes\left(\prod_R^1\right)=C_5\cdot 2^n nR^{n-2-\frac{|\alpha|}{2}},$$

and this inequality with inequality (14) give the upper estimate in (13) with $C_4=C_5\times\times 2^n n$. For the proof of the estimate from below we make space transformation $y_i=\left(\frac{1}{R}\right)^{1-\alpha_i/2}x_i$, $i=1,\dots,n$. Let $\vartheta(x)\in C_0^\infty(E_n)$, $\vartheta(x)\ge 1$ on $\prod_R$. Then, if $\tilde\vartheta(y)$ is an image of function $\vartheta(x)$, then $\tilde\vartheta(y)\in C_0^\infty(E_n)$, $\tilde\vartheta(y)\ge 1$ on $\prod_1$. We have

$$\sum_{i=1}^n\int_{E_n}\lambda_i(x)\left(\frac{\partial\vartheta}{\partial x_i}\right)^2dx=\sum_{i=1}^n\int_{E_n}\left(1+\sum_{j=1}^n R^{\left(1-\frac{\alpha_j}{2}\right)\frac{2}{2-\alpha_j}}|y_j|^{\frac{2}{2-\alpha_j}}\right)^{-\alpha_i}\times$$

$$\times R^{\alpha_i-2}\cdot R^{n-\frac{|\alpha|}{2}}\left(\frac{\partial\tilde\vartheta}{\partial y_i}\right)^2dy=R^{n-2-\frac{|\alpha|}{2}}\sum_{i=1}^n\int_{E_n}\left(\frac{1}{R}+|y|_\alpha\right)^{-\alpha_i}\left(\frac{\partial\tilde\vartheta}{\partial y_i}\right)^2dy\ge$$

$$\geq R^{n-2-\frac{|\alpha|}{2}}\sum_{i=1}^{n}\int_{E_n}\lambda_i(y)\left(\frac{\partial\tilde{\vartheta}}{\partial y_i}\right)^2 dy.$$

Condition $R\geq 1$ have been used here. Thus,

$$Cap(\Pi_R)\geq R^{n-2-\frac{|\alpha|}{2}}Cap(\Pi_1). \tag{19}$$

Let $U(x)$ and μ are capacitary potential and capacitive distributions of Π_1 correspondingly, and $g(x,y)$ is the Green's function of operator L_0. We have

$$U(x)=\int_{\partial\Pi_1}g(x,y)d\mu(y);\mu(\partial\Pi_1)=Cap(\Pi_1).$$

So as $0\notin\partial\Pi_1$, then function $U(x)$ is continuous at the point 0, and $U(0)=1$. Moreover, for $y\in\partial\Pi_1, g(0,y)\leq C_6(n,\alpha)$. Therefore

$$1=U(0)\leq C_6\mu(\partial\Pi_1)=C_6 Cap(\Pi_1),$$

and with (19) it gives

$$Cap(\Pi_R)\geq C_6^{-1}R^{n-2-\frac{|\alpha|}{2}}. \tag{20}$$

Now it is enough to take into account, that $\Pi_{pR}\subset\mathcal{E}_R^0(1)$, where $p=n^{-\frac{1}{2-\alpha^+}}$, $\alpha^+=$ $=\max\{\alpha_1,\dots\alpha_n\}$, and from (20) follows the estimate from below (13) with $C_3=C_6^{-1}p^{n-2-\frac{|\alpha|}{2}}$. Lemma is proved.

Lemma 5. *Let* $R\geq 1$, $x^R=\left(0,\dots,0,\frac{1}{2}R^{1-\alpha_n/2}\right)$, $\gamma\in\left(0,\frac{1}{4}\right]$ *is constant. Then*

$$Cap\left(\mathcal{E}_{\gamma R}^{x^R}(1)\right)\geq C_7(\alpha,n)(\gamma R)^{n-2-\frac{|\alpha|}{2}}.$$

This Lemma can be proved quite similar to the previous one.

3^0. Theorem on increasing of positive solutions.

Lemma 6. *Let* $R\geq 1$, $\Sigma=\mathcal{E}_{eR}^0(1)$, $H\subset\mathcal{E}_R^0(1)$ *is compact,* $U_\Sigma(x)$ *is capacitary potential of* H *with respect to* Σ. *Then*

$$\inf_{\partial\mathcal{E}_{2R}^0(1)}U_\Sigma(x)\geq C_8(\alpha,n,\mu)\frac{Cap(H)}{R^{n-2-\frac{|\alpha|}{2}}}. \tag{21}$$

Proof. Suppose, at first, that ∂H is sufficiently smooth surface. Let for $h>0$ operator L_h have the same sense as it was above. Consider the Dirichlet problem

$$L_h U_h=0;\quad x\in\Sigma\backslash H;\quad U_h\big|_{\partial H}=1;\quad U_h\big|_{\partial\Sigma}=0. \tag{22}$$

For each $h>0$ the solution U_h of problem (22) exists, and converges to U_Σ in norm $W_{2,\alpha}^1(\Sigma\backslash H)$. By the Green's formula we have

$$\int_{\Sigma\backslash H}L_H(U_h)^2dx=\int_{\partial(\Sigma\backslash H)}\frac{\partial}{\partial\nu}(U_h)^2dx, \tag{23}$$

where $\frac{\partial}{\partial\nu}$ is a derivative in direction of outward, relative to $\Sigma\backslash H$, conormal, generated by operator L_h. From (23) according to (2), we get

$$\int_{\Sigma\setminus H}\sum_{i,j=1}^{n}a_{ij}^{h}(x)\frac{\partial U_h}{\partial x_i}\frac{\partial U_h}{\partial x_j}dx=\int_{\partial H}\frac{\partial U_h}{\partial \nu}ds.$$

So as U_h is capacitary potential of H, generated by operator L_h, then from here it follows that

$$cap_{\Sigma}^{h}(H)=\int_{\partial H}\frac{\partial U_h}{\partial \nu}ds, \tag{24}$$

where cap_{Σ}^{h} is capacity, generated by operator L_h. We denote $\sup\limits_{\partial\mathcal{E}_{2R}^{0}(1)}U_h$ by b. It is obvious, that $b<1$. Let

$$E_b=\{x:x\in\Sigma, U_h(x)=b\},\quad A_b=\{x:x\in\Sigma, U_h(x)>b\}.$$

So as in $\Sigma\setminus\mathcal{E}_{2R}^{0}(1)$ $U_h(x)<b$, then $A_b\subset\mathcal{E}_{2R}^{0}(1)$. Moreover, $\partial A_b=E_b$, $H\subset A_b$ (so as $U_h\big|_H=1>b$). Then, if $V_h=\dfrac{U_h}{b}$, then from (24) we obtain

$$cap_{\Sigma}^{h}(A_b)=\int_{\partial A_b}\frac{\partial V_h}{\partial \nu'}ds=\int_{E_b}\frac{\partial V_h}{\partial \nu'}ds=\frac{1}{b}\int_{E_b}\frac{\partial U_h}{\partial \nu'}ds, \tag{25}$$

where $\dfrac{\partial}{\partial \nu'}$ is a derivative in direction of outward, relative to $\Sigma\setminus A_b$, conormal, generated by operator L_h. We will again apply the Green's formula for the function U_h in the domain $A_b\setminus H$. Denoting by $\dfrac{\partial}{\partial \nu''}$ the derivative in direction of outward, relative to $A_b\setminus H$, conormal, generated by operator L_h, we obtain

$$0=\int_{A_b\setminus H}L_hU_h dx=\int_{\partial(A_b\setminus H)}\frac{\partial U_h}{\partial \nu''}ds,$$

i.e.

$$\int_{\partial H}\frac{\partial U_h}{\partial \nu''}ds=\int_{\partial H}\frac{\partial U_h}{\partial \nu}ds=-\int_{E_b}\frac{\partial U_h}{\partial \nu''}ds=\int_{E_b}\frac{\partial U_h}{\partial \nu'}ds. \tag{26}$$

Using (25) and (26) in (24), we conclude

$$b=\frac{cap_{\Sigma}^{h}(H)}{cap_{\Sigma}^{h}(A_b)}\geq\frac{cap_{\Sigma}^{h}(H)}{cap_{\Sigma}^{h}\left(\mathcal{E}_{2R}^{0}(1)\right)}. \tag{27}$$

But, according to Lemma 2, $\inf\limits_{\partial\mathcal{E}_{2R}^{0}(1)}U_h\geq C_1^{-1}b$. From last inequality and (27), we have

$$\inf_{\partial\mathcal{E}_{2R}^{0}(1)}U_h\geq C_1^{-1}\frac{cap_{\Sigma}^{h}(H)}{cap_{\Sigma}^{h}\left(\mathcal{E}_{2R}^{0}(1)\right)}.$$

Tending h to zero, we obtain

$$\inf_{\partial\mathcal{E}_{2R}^{0}(1)}U_{\Sigma}\geq C_1^{-1}\frac{cap_{\Sigma}(H)}{cap_{\Sigma}\left(\mathcal{E}_{2R}^{0}(1)\right)}. \tag{28}$$

It is easy to see, that if Σ' is ellipsoid, which contain Σ, then for this ellipsoid inequality of (28) type is valid, i.e.

$$\inf_{\partial\mathcal{E}_{2R}^{0}(1)}U_{\Sigma}\geq C_8(\alpha,n,\mu)\frac{cap_{\Sigma'}(H)}{cap_{\Sigma'}\left(\mathcal{E}_{2R}^{0}(1)\right)}.$$

If now the length of the least halfaxis of ellipsoid Σ' we tend to infinity, then from the last inequality follows, that

$$\inf_{\partial\mathcal{E}_{2R}^0(1)} U_\Sigma \geq C_8 \frac{cap(H)}{cap\left(\mathcal{E}_{2R}^0(1)\right)}.$$

Using Lemma 4 and fact, that for any compact $B \subset E_n$ $\mu Cap(B) \leq cap(B) \leq \mu^{-1} Cap(B)$, we conclude

$$\inf_{\partial\mathcal{E}_{2R}^0(1)} U_\Sigma \geq C_9(\alpha, n, \mu) \frac{Cap(H)}{R^{n-2-\frac{|\alpha|}{2}}}. \tag{29}$$

From the proof it is seen that inequality (29) is valid and in that case, when $H \subset \mathcal{E}_{\frac{3}{2}R}^0(1)$.

Let now $H \subset \mathcal{E}_R^0(1)$ be an arbitrary compact. Then there exists sequence of compacts $\{H_m\}$, $m = 1, 2, \ldots$; $H_m \in \mathcal{E}_{\frac{3}{2}R}^0(1)$ with sufficiently smooth boundaries ∂H_m, which approximates H from outside. If U_Σ^m is capacitary potential of H_m with respect to Σ, then, according to (29)

$$\inf_{\partial\mathcal{E}_{2R}^0(1)} U_\Sigma^m \geq C_9 \frac{Cap(H_m)}{R^{n-2-\frac{|\alpha|}{2}}}.$$

Now, it is enough to take into account, that for $m \to \infty$ $Cap(H_m) \to Cap(H)$, $U_\Sigma^m(x) \to U_\Sigma(x)$ uniformly on $\partial\mathcal{E}_R^0(1)$ (see [11] and Lemma 3), then we get, and that inequality (29) is valid for arbitrary compact $H \subset \mathcal{E}_R^0(1)$. Lemma is proved.

Theorem 1. *Let G be domain in $\mathcal{E}_{eR}^0(1)$, which has limit points on $\partial\mathcal{E}_{eR}^0(1)$ and which intersects $\mathcal{E}_R^0(1)$. Let also $u(x)$ be a weak solution of equation (1) in G, non-negative in G, and vanishes on that part Γ of boundary of ∂G, which lies strictly in $\mathcal{E}_{eR}^0(1)$ (in the sense of $W_{2,\alpha}^1$), moreover, with respect to coefficients of operator L condition (2) holds. Then, if $R \geq 1$ and $H = \mathcal{E}_R^0(1) \setminus G$, then*

$$\sup_G u \geq \left(1 + C_{10}(\alpha, n, \mu) \frac{Cap(H)}{R^{n-2-\frac{|\alpha|}{2}}}\right) \sup_{G \cap \mathcal{E}_R^0(1)}. \tag{30}$$

Proof. Let $U(x)$ be capacitary potential of H with respect to $\mathcal{E}_{eR}^0(1)$, $M = \sup_G u$. Consider auxiliary function $W(x) = M[1 - U(x)] - u(x)$. It is clear, that $W(x)$ is a weak solution of equation (1) in G, moreover

$$W\big|_{\partial G \cap \partial\mathcal{E}_{eR}^1(1)} = M - u\big|_{\partial G \cap \partial\mathcal{E}_{eR}^1(1)} \geq 0; \; W\big|_\Gamma = M\left(1 - U\big|_\Gamma\right) \geq 0.$$

According to maximum principle $W(x) \geq 0$ in G, i.e. $u(x) \leq M\left(1 - U\big|_\Gamma\right)$, and, in particular,

$$\sup_{G \cap \mathcal{E}_R^0(1)} u \leq \sup_{G \cap \partial\mathcal{E}_{2R}^0(1)} u \leq M\left[1 - \inf_{\partial\mathcal{E}_{2R}^0(1)} U\right]. \tag{31}$$

Now it is enough to use Lemma 6, and from (31) follows required estimate (30) with $C_{10} = C_9$. Theorem is proved.

4^0. Theorems of the Phragmen-Lindelöf type.

Theorem 2. *Let $u(x)\in W_{2,\alpha}^{1,loc}(D)$ be a weak solution of equation (1) in D, $u|_{\partial D}\le 0$ in the sense of $W_{2,\alpha}^1$, moreover, with respect to coefficients of operator L condition (2) holds. Then*

1) either $u(x)\le 0$ in D,

2) or $\underline{\lim}_{R\to\infty} M(R)\Big/\exp\left[\eta_0(\alpha,n,\mu)\sum_{m=1}^{[\ln R]} e^{-m\left(n-2-\frac{|\alpha|}{2}\right)}Cap(H^m)\right]>0$,

where $M(R)=\sup_{D\cap\partial\mathcal{E}_R^0(1)} u$, $H^m=\mathcal{E}_{e^m}^0(1)\setminus D$, $m=1,2,\ldots,$.

Proof. Let alternative 1) doesn't hold, then there exists point $x^0\in D$, such that $u(x^0)=a>0$. We denote by m_1 the least natural number, for which $x^0\in\mathcal{E}_{e^{m_1}}^0(1)$, and let for $i=1,2,\ldots,$ $M_i=\sup_{D\cap\mathcal{E}_{e^i}^0(1)} u$. According to Theorem 1

$$M_{i+1}\ge\left[1+C_{10}e^{-i\left(n-2-\frac{|\alpha|}{2}\right)}Cap(H^i)\right]M_i. \tag{32}$$

Let $R\ge 1$ is sufficiently large, and $m>m_1$ is natural number, for which $\mathcal{E}_{e^m}^0(1)\subset\mathcal{E}_R^0(1)\subset$ $\subset\mathcal{E}_{e^{m+1}}^0(1)$, i.e.

$$e^m\le R\le e^{m+1}. \tag{33}$$

Applying consequently inequality (32), we obtain

$$M(R)\ge M_m\ge a\prod_{i=m_1}^{m-1}\left(1+C_{10}e^{-i\left(n-2-\frac{|\alpha|}{2}\right)}Cap(H^i)\right),$$

therefore

$$M(R)\ge a\exp\left[\sum_{i=m_1}^{m-1}\ln\left(1+C_{10}e^{-i\left(n-2-\frac{|\alpha|}{2}\right)}Cap(H^i)\right)\right]. \tag{34}$$

According to Lemma 4 $Cap(H^i)\le Cap(\mathcal{E}_{e^i}^0(1))\le C_4 e^{i\left(n-2-\frac{|\alpha|}{2}\right)}$. From the other side for $t\in[0,C_4]$

$$\ln(1+C_{10}t)\ge C_{11}(C_4,C_{10})\,t.$$

Therefore from (34) it follows, that

$$M(R)\ge a\exp\left[C_{11}\sum_{i=m_1}^{m-1}e^{-i\left(n-2-\frac{|\alpha|}{2}\right)}Cap(H^i)\right]=a\exp\left[\frac{C_{11}}{2}\sum_{i=1}^{m-1}e^{-i\left(n-2-\frac{|\alpha|}{2}\right)}Cap(H^i)\right]\times$$
$$\times\exp\left[\frac{C_{11}}{2}\left(\sum_{i=1}^{m-1}e^{-i\left(n-2-\frac{|\alpha|}{2}\right)}Cap(H^i)-2\sum_{i=1}^{m_1-1}e^{-i\left(n-2-\frac{|\alpha|}{2}\right)}Cap(H^i)\right)\right]. \tag{35}$$

It is easy to see, that alternative 2) have contensive sense only in that case, when

$$\sum_{i=1}^{\infty} e^{-i\left(n-2-\frac{|\alpha|}{2}\right)} Cap\left(H^i\right)=\infty . \tag{36}$$

Therefore, taking condition (36) as valid, we conclude, that for sufficiently large m (i.e. R)

$$\sum_{i=1}^{m-1} e^{-i\left(n-2-\frac{|\alpha|}{2}\right)} Cap\left(H^i\right) \geq 2 \sum_{i=1}^{m_1-1} e^{-i\left(n-2-\frac{|\alpha|}{2}\right)} Cap\left(H^i\right).$$

Therefore, from (35) we obtain

$$M(R) \geq a \exp\left[\frac{C_{11}}{2} \sum_{i=1}^{m_1-1} e^{-i\left(n-2-\frac{|\alpha|}{2}\right)} Cap\left(H^i\right)\right]. \tag{37}$$

Now, taking into accounts, that according to (33) $m-1 \geq [\ln R]-2$, from (37) and Lemma 4 we have

$$M(R) \geq a \exp\left[\frac{C_{11}}{2} \sum_{i=1}^{[\ln R]-2} e^{-i\left(n-2-\frac{|\alpha|}{2}\right)} Cap\left(H^i\right)\right] = a \exp\left[\frac{C_{11}}{2} \sum_{i=1}^{[\ln R]} e^{-i\left(n-2-\frac{|\alpha|}{2}\right)} Cap\left(H^i\right)\right] \times$$

$$\times \exp\left[-\frac{C_{11}}{2} \sum_{i=[\ln R]-1}^{[\ln R]} e^{-i\left(n-2-\frac{|\alpha|}{2}\right)} Cap\left(H^i\right)\right] \geq a \exp\left[-C_4 C_{11}\right] \times$$

$$\times \exp\left[\frac{C_{11}}{2} \sum_{i=1}^{[\ln R]} e^{-i\left(n-2-\frac{|\alpha|}{2}\right)} Cap\left(H^i\right)\right].$$

Thus, alternative 2) takes place with $\eta_0 = \dfrac{C_{11}}{2}$. Theorem is proved.

Theorem 3. *Let $h>1$, $\alpha_n = \alpha^- = \min\{\alpha_1,\dots,\alpha_n\}$, $\mathcal{K}_h$ is double cone $\left\{x : x_n^2 \geq \geq h^2 \sum_{i=1}^{n-1} x_i^2\right\}$, $D = E_n \setminus \mathcal{K}_h$. Then, if $u(x) \in W_{2,\alpha}^{1,loc}(D)$ is a weak solution of equation (1) in D, $u|_{\partial D} \leq 0$ in the sense of $W_{2,\alpha}^1$ and with respect to coefficients of operator L condition (2) holds, then*

1) either $u(x) \leq 0$ in D,

2) or $\varliminf\limits_{R\to\infty} M(R) \Big/ R^{\frac{\eta_1}{h^\beta}} > 0$,

where $\eta_1 = \eta_1(\alpha, n, \mu)$, $\beta = \beta(\alpha, n)$.

Proof. Let for $R \geq 1$ Q_R^x and x^R have the same sense as it has in Lemmas 3 and 5 correspondingly, $H(R) = \mathcal{E}_R^0(1) \setminus D$. It is easy to see that $\overline{Q}^{x^R}_{\frac{R^{1-\frac{\alpha_n}{2}}}{4h}} \subset H_R$.

Now we will find such ρ, that $\mathcal{E}_\rho^{x^R}(1)\subset \overline{Q}^{x^R}_{\frac{R^{1-\frac{\alpha_n}{2}}}{4h}}$. For this it is enough that for $i=1,\dots,n$ the inequality $\rho^{1-\frac{\alpha_i}{2}}\le \frac{R^{1-\frac{\alpha_n}{2}}}{4h}$ to be hold. But, so as $\alpha_n=\alpha^-$, then choosing $\rho= \ =\frac{R}{(4h)^{\frac{2}{2-\alpha^-}}}$, we provide validity of required inclusion. From the other side, according to Lemma 5

$$Cap\left(\mathcal{E}^{x^R}_{R/(4h)^{\frac{2}{2-\alpha^-}}}(1)\right)\ge C_7\frac{R^{n-2-\frac{|\alpha|}{2}}}{4^\beta h^\beta},$$

where $\beta=\frac{2n-4-|\alpha|}{2-\alpha^-}$. Therefore, for $m=1,2,\dots$,

$$Cap(H^m)\ge \frac{C_7}{4^\beta h^\beta}e^{m\left(n-2-\frac{|\alpha|}{2}\right)}. \tag{38}$$

Applying now Theorem 2 and taking into account (38), we obtain that if alternative 1) doesn't take place, then

$$\varliminf_{R\to\infty} M(R)\Big/\exp\left[\frac{\eta_0 C_7}{4^\beta h^\beta}[\ln R]\right]>0.$$

Thus, Theorem is proved with $\eta_1=\frac{\eta_0 C_7}{4^\beta}$.

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Received October 6, 1999; Revised December 22, 1999.
Translated by Panarina V.K.