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EXPECTATION OPERATORS, REDUCING SUBSPACES AND CYCLIC SETS

Abstract

In the paper the conditional expectation operator is applied in the study of reducing subspaces and cyclic sets of some Toeplitz operators in the Hardy space H^2 on the unit disk in the complex plane.

The object of this paper is to apply the conditional expectation operator of probability theory in the study of reducing subspaces and cyclic sets of some Toeplitz operators. In particular, we give strengthening of Nordgren theorem ([1], Theorem 2) on existence of nontrivial reducing subspaces of analytic Toeplitz operators.

1. Let $H^2 = H^2(\mathbf{D})$ denote the Hardy space of functions f analytic in the open disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ which satisfy

$$\sup_{0<r<1} \int_{\mathbf{T}} \left| f\left(re^{it}\right) \right|^2 dt < +\infty,$$

where $\mathbf{T} = \partial\mathbf{D} = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$ - unit circle in the complex plane, let $H^\infty = H^\infty(\mathbf{D})$ be the algebra of bounded analytic functions on $\mathbf{D}$. It is well known that

$$H^2 = \left\{ f \in L^2(\mathbf{T}, m) : \int_{\mathbf{T}} f(\zeta)\zeta^n dm(\zeta) = 0,\ n > 0 \right\},$$

where m is normalized Lebesgue measure on the unit circle $\mathbf{T}$. For φ in $L^\infty = L^\infty(\mathbf{T})$ the Toeplitz operator T_φ is the operator on the Hardy space H^2 given by $T_\varphi f = P_+ \varphi f$ where $P_+ = P_{H^2}$ is the orthogonal projection of $L^2 = L^2(\mathbf{T}, m)$. For φ in H^∞, T_φ is the analytic Toeplitz operator defined by $T_\varphi f = \varphi f$. A function $\theta \in H^\infty$ is said to be inner if $\lim_{r \to 1^-} \left|\theta\left(re^{it}\right)\right| = 1$ for almost every $t \in [0, 2\pi]$. If θ is not a linear fractional transformation, then T_θ is a shift of multiplicity greater than one and has therefore many nontrivial reducing subspaces. Moreover, if $\varphi \in H^\infty$ and $\varphi = \psi \circ \theta$, that is, if $\varphi(z) = \psi(\theta(z))$ for some ψ in H^∞, then any reducing subspace for T_θ also reduces T_φ[1]. Thus, if φ is the composite of a bounded analytic function and an inner function other than a linear fractional transformation of the unit disc $\mathbf{D}$ onto itself, then T_φ has a nontrivial reducing subspace ([1], Theorem 2). We extend that result to the case when $\varphi \in L^\infty$. Throughout this paper θ will be used to denote an inner function other than a linear fractional transformation.

2. **Theorem 1.** *The Toeplitz operator $T_{\varphi\circ\theta}$, where $\varphi \in L^\infty$, has a nontrivial reducing subspace.*

Proof. The proof essentially use Nordgren's argument from [1]. Let $alg(T_\theta, T_{\bar\theta})$ denote the weak closed algebra generated by T_θ and $T_{\bar\theta} = T_\theta^*$. Let

$$\sigma_n(\varphi)\overset{def}{=}\sum_{k=-n}^{n}\left(1-\frac{|k|}{n}\right)\hat{\varphi}(k)\zeta^k \tag{1}$$

be a Fejer means for the function $\varphi\in L^\infty$. Then it is well known that $\|\sigma_n(\varphi)\|_\infty\le\|\varphi\|_\infty$ and $\lim\limits_{n\to+\infty}(\sigma_n(\varphi))(\zeta)=\varphi(\zeta)$ for almost every $\zeta,\zeta\in\mathbf{T}$. Consequently, the sequence $\sigma_n(\varphi)\circ\theta$ is also bounded and since the measure $m\theta^{-1},(m\theta^{-1})(e)\overset{def}{=}m(\theta^{-1}(e)),\ e\subset\mathbf{T}$, is absolutely continuous with respect to m ([1]), then

$$\lim_{n\to+\infty}(\sigma_n(\varphi)\circ\theta)(\zeta)=(\varphi\circ\theta)(\zeta) \tag{2}$$

for almost every $\zeta,\zeta\in\mathbf{T}$. Therefore, since $\sigma_n(\varphi)\circ\theta=\sum\limits_{k=-n}^{n}a_k\theta^k$, where a_k denote the coefficients of the sum (1), one has $T_{\sigma_n(\varphi)\circ\theta}\in alg(T_\theta,T_{\bar\theta})$. We show that $T_{\varphi\circ\theta}\in alg(T_\theta,T_{\bar\theta})$. Really, let $f,g\in H^2$. Then

$$\left|\left\langle\left(T_{\sigma_n(\varphi)\circ\theta}-T_{\varphi\circ\theta}\right)f,g\right\rangle\right|=\left|\left\langle P_+(\sigma_n(\varphi)\circ\theta-\varphi\circ\theta)f,g\right\rangle\right|=$$

$$=\left|\left\langle(\sigma_n(\varphi)\circ\theta-\varphi\circ\theta)f,g\right\rangle\right|=$$

$$=\left|\int_{\mathbf{T}}(\sigma_n(\varphi)\circ\theta-\varphi\circ\theta)(\zeta)f(\zeta)\bar g(\zeta)dm(\zeta)\right|\le$$

$$\le\int_{\mathbf{T}}|(\sigma_n(\varphi)\circ\theta)(\zeta)-(\varphi\circ\theta)(\zeta)||f(\zeta)||g(\zeta)|dm(\zeta).$$

Hence, by (2) and Lebesgue Dominated Convergence Theorem we have

$$T_{\varphi\circ\theta}=w-\lim_n T_{\sigma_n(\varphi)\circ\theta},$$

that is, $T_{\varphi\circ\theta}\in alg(T_\theta,T_{\bar\theta})$. By a similar argument we can show that $T^*_{\varphi\circ\theta}\in alg(T_\theta,T_{\bar\theta})$. Thus,

$$T_{\varphi\circ\theta},T^*_{\varphi\circ\theta}\in alg(T_\theta,T_{\bar\theta}). \tag{3}$$

Since T_θ is completely non-unitary isometry, it is unitarily equivalent to the shift operator S in the vector-valued Hardy space $H^2(E)$ (that is, $T_\theta=USU^{-1}$ for some unitary operator $U,U:H^2(E)\to H^2$), where $\dim E=\dim(H^2\Theta\theta H^2)>1$. On the other hand, it is known that each subspace of the form $H^2(E_1),E_1\underset{\neq}{\subset}E$, reduces S (see [2], Theorem 3.22) and hence, each subspace $UH^2(E_1)$ reduces T_θ. It is obvious that each nontrivial subspace $UH^2(E_1)$ is invariant subspace for the $alg(T_\theta,T_{\bar\theta})$. Combining this with (3), we conclude that the operators $T_{\varphi\circ\theta},T^*_{\varphi\circ\theta}$ leave invariant $UH^2(E_1)$, that is, $T_{\varphi\circ\theta}$ has a nontrivial reducing subspace. This proves the theorem.

3. Now, in the case $\theta(0)=0$, using the Hardy space expectation operator [3] we give the alternative proof of Theorem 1.

Let Λ denote the σ-algebra of Lebesgue measurable sets in the unit circle $\mathbf{T}$ and $\mathscr{B}\subset\Lambda$ be the smallest σ-subalgebra with respect to which θ is measurable. If $f\in L^p=L^p(\mathbf{T},\Lambda,m)$ then $\nu(B)=\int_B fdm$ is the measure on $\mathscr{B}$ which is absolutely continuous with respect to the restricted measure $m|\mathcal{B}$. By Radon-Nikodym theorem there exists a unique $\mathcal{B}$- measurable function g such that $\nu(B)=\int_B g\,dm,\ B\in\mathcal{B}$. Put $P_{\theta,\mathcal{B}}f \overset{def}{=} g$. Hence, the function $P_{\theta,\mathcal{B}}f$ is the Radon-Nikodym derivative of f with respect to $\mathcal{B}$. The operator $P_\theta \overset{def}{=} P_{\theta,\mathcal{B}}$ is the conditional expectation operator of probability theory and has the following properties [3], [4]:

1) $P_\theta 1=1$;
2) $P_\theta(f\cdot P_\theta g)=(P_\theta f)(P_\theta g)$;
3) $\|P_\theta f\|_p\le\|f\|_p$, that is, P_θ is a contraction operator on L^P;
4) P_θ is a projection, $P_\theta^2=P_\theta$;
5) $\int\limits_{\mathbf{B}}(P_\theta f)\cdot g\,dm=\int\limits_{\mathbf{B}} f\cdot(P_\theta f)dm$ for $f\in L^P$, $q\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$;
6) the range of P_θ is $L^P(\mathcal{B})$, the set of functions in L^P measurable with respect to $\mathcal{B}$, where P_θ is considered as an operator on L^P.

Let Φ be the set of all functions of the form $P(\theta,\overline{\theta})$, P a complex polynomial in two variables. Then, for $1\le p<+\infty$, $L^P(\mathcal{B})$ is the closure in L^P of Φ [5]. Since θ is inner function, $\overline{\theta}^n\theta^m=\theta^{m-n}$, and a polynomial $p(\theta,\overline{\theta})$ in θ and $\overline{\theta}$ has the form $\sum\limits_{i=-n}^{n}a_i\theta^i$. Using this we can show that P_θ leaves H^2 invariant. Really, since θ is inner with $\theta(0)=0$, $\{\theta^n:n\in\mathbf{Z}\}$ is an orthonormal set in L^2. As noted above, this orthonormal set spans $L^2(\mathcal{B})=R(P_\theta)$. By properties 4) and 5) P_θ is the orthogonal projection onto $L^2(\mathcal{B})$. By the form of the spanning orthonormal set,

$$P_+P_\theta=P_\theta P_+. \tag{4}$$

Really, for each $f\in L^2$

$$P_+P_\theta f=P_+\lim_k\sum_{j=-n}^{n}C_{j,k}^f\theta^j=\lim_k P_+\sum_{j=-n}^{n}C_{j,k}^f\theta^j=$$

$$=\lim_k\sum_{j=0}^{n}C_{j,k}^f\theta^j=\lim_k P_\theta\sum_{j=0}^{n}C_{j,k}^f\theta^j=$$

$$=\lim_k P_\theta P_+\sum_{j=0}^{n}C_{j,k}^f\theta^j=\lim_k P_\theta P_+\sum_{j=-n}^{n}C_{j,k}^f\theta^k=$$

$$=P_\theta P_+\lim_k\sum_{j=-n}^{n}C_{j,k}^f\theta^j=P_\theta P_+P_{\theta f},$$

that is,

$$P_+P_\theta=P_\theta P_+P_\theta.$$

Then, since $P_\theta^*=P_\theta$, $P_+^*=P_+$, we obtain $P_\theta P_+=(P_+P_\theta)^*=(P_\theta P_+P_\theta)^*=$ $=P_\theta P_+P_\theta=P_+P_\theta$, therefore, the equality (4) is valid and consequently, P_θ leaves H^2 invariant.

Moreover, using again the fact that $\theta^{m-n}=\overline{\theta}^n\theta^m$, and that the composition operator $C_\theta, C_\theta f = f\circ\theta$ is an isometry in H^2 ,we conclude that P_θ is a orthogonal projection of H^2 onto $R(C_\theta)=H^2(\theta)\overset{def}{=}\{f\circ\theta : f\in H^2\}$. Hence, if $\varphi\in L^\infty$ then by (4) and (2) for every $f\in H^2$ we have

$$P_\theta T_{\varphi\circ\theta}f = P_\theta P_+(\varphi\circ\theta)f = P_+P_\theta((\varphi\circ\theta)f) = \\ = P_+(\varphi\circ\theta)(P_\theta f) = T_{\varphi\circ\theta}P_\theta f,$$

so

$$P_\theta T_{\varphi\circ\theta} = T_{\varphi\circ\theta}P_\theta.$$

Consequently, $H^2(\theta)$ is a nontrivial reducing subspace of the operator $T_{\varphi\circ\theta}$. This completes the proof.

4. In the next theorem we consider the cyclic set of the analytic Toeplitz operator and in terms of expectation operators derive a necessary condition.

Let X be a banach space and $A: X\to X$ be a linear bounded operator. The set $E\subset X$ is cyclic for A if $X = span\{A^nE : n\ge 0\}$, where $span\{...\}$ is the closed linear hull of the set $\{...\}$. We denote the cyclic set of operator A by $Cyc(A)$.

Theorem 2. *Let θ be a nonconstant inner function, $\theta(0)=0$ and φ in H^∞ such that $Cyc(T_\varphi)=Cyc(T_z)$. Let $\{f_1, f_2,...,f_\mu\}\in Cyc(T_{\varphi\circ\theta})$, where $\mu=\mu(T_{\varphi\circ\theta})$ is a multiplicity of spectrum of the operator $T_{\varphi\circ\theta}$. Then $GCD\{(C_\theta^*P_\theta f_k)_{inn} : 1\le k\le\mu\}\equiv 1$, where $GCD\{...\}$ is the greatest common divisor of the set $\{...\}$.*

Proof. Since $\{f_1, f_2,...,f_\mu\}\in Cyc(T_{\varphi\circ\theta})$ then

$$H^2 = Span\{T^n_{\varphi\circ\theta}f_k : n\ge 0, 1\le k\le\mu\} = \\ = Span\{(\varphi\circ\theta)^n f_k : n\ge 0, 1\le k\le\mu\} = \\ = Span\{(\varphi^n\circ\theta)f_k : n\ge 0, 1\le k\le\mu\}$$

If P_θ corresponding expectation operator and C_θ composition operator, then we have

$$C_\theta H^2 = H^2(\theta) = P_\theta H^2 = P_\theta Span\{(\varphi^n\circ\theta)f_k : n\ge 0, 1\le k\le\mu\} = \\ = Span\{P_\theta(\varphi^n\circ\theta)f_k : n\ge 0, 1\le k\le\mu\} = \\ = Span\{(\varphi^n\circ\theta)(P_\theta f_k) : n\ge 0, 1\le k\le\mu\} = \\ = Span\{(\varphi^n\circ\theta)(g_k\circ\theta) : n\ge 0, 1\le k\le\mu\} = \\ = Span\{C_\theta(\varphi^n g_k) : n\ge 0, 1\le k\le\mu\} = \\ = CS_\theta pan\{\varphi^n g_k : n\ge 0, 1\le k\le\mu\}.$$

So

$$C_\theta H^2 = C_\theta Span\{\varphi^n g_k : n\ge 0, 1\le k\le\mu\}.$$

Since by condition $\theta(0)=0$ C_θ isometry in H^2, from last equality we have

$$H^2 = Span\{\varphi^n g_k : n\ge 0, 1\le k\le\mu\},$$

that is, $\{g_1,\dots,g_\mu\}\in Cyc(T_\varphi)=Cyc(T_z)$. Therefore, since $g_k=C_\theta^*P_\theta f_k, 1\le k\le\mu$, we assert that (by well known Beurling criteria) $GCD\{(C_\theta^*P_\theta f_k)_{inn}:1\le k\le\mu\}\equiv 1$. This completes the proof.

We remark that actually, $C_\theta^*P_\theta f_k=C_\theta^*f_k$, because $P_\theta=C_\theta C_\theta^*$.

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Received November 4, 1999; Revised December 30, 1999.
Translated by author.