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ON BESOV SPACE OF VECTOR-VALUED FUNCTIONS

Abstract

In present paper the space $B^s_{p,q}(R^n : E_0, E)$, $s = (s_1...s_n)$ for $-\infty < s_j < \infty$ was investigated. Earlier author considers the same space, which was considered for $s_j > 0$. Some embedding theorems characterizing considered space depending on parameters p, q and s were obtained.

One of the main intensive studying class of deformated functions is space $B^s_{p,q}(R^n)(s > 0, 1 \le p \le \infty))$ which was introduced by O.V.Besov [1]. The space $B^s_{p,q}$ was determined earlier and for $s \le 0$ [4]. At paper [6] was considered space $B^s_{p,q}(\Omega : E_0, E)$, where $s = (s_1,...,s_n)$, $s_j > 0$, $1 < p < \infty$, $1 \le q \le \infty$. The aim of present paper is expansion of definition of space $B^s_{p,q}(R^n : E_0, E)$, for $s = (s_1,...,s_n)$, and determination of some theorems on embedding. Note, that for $E = E_0$ some questions on spaces $B^s_{p,q}\ (R^n : E_0, E)$ was considered in paper [3], $1 \le p < \infty$.

Denote by $L_p(R^n : E)$ the space of strong measurable functions f, determined on R^n with values from E, for which norm is defined by the following way

$$\|f\|_{L_p(R^n:E)} = \left(\int_{R^n} \|f(x)\|^p_E dx \right)^{\frac{1}{p}} < \infty. \tag{1}$$

It is known, that if E is Banach space, then $L_p(R^n : E)$ is also Banach space.

Let N be a set of integer numbers, $N_0 = N \cup \{0\}$, $\alpha = (\alpha_1,...,\alpha_n)$ is multi-index with integer-valued coefficients, $S(R^n)$ is a class of infinitely differentiable functions φ, for which for $\forall m > 0$, $\forall \alpha = (\alpha_1,...,\alpha_n)$ inequality takes place:

$$\sup_x \left(1 + |x|^m\right) \left|D^{|\alpha|}\varphi\right| < C, \tag{2}$$

where $D^{|\alpha|}\varphi = D^{\alpha_1}_{x_1}\left(...\left(D^{\alpha_n}_{x_n}\varphi\right)\right)$, $|\alpha| = \alpha_1 + ... + \alpha_n$.

For $\varphi \in S(R^n)$ the Fourier transformation and inverse Fourier transformation are determined by formulae

$$(F\varphi)(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{-i<x,\xi>}\varphi(x)dx, \quad \varphi \in S(R^n), \tag{3}$$

$$\left(F^{-1}\varphi\right)(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{i<x,\xi>}\varphi(x)dx, \quad \varphi \in S(R^n), \tag{4}$$

where $<x,\xi> = \sum_{j=1}^n x_j \xi_j$.

$S'(R^n : E)$ is space of linear bounded mappings from $S(R^n)$ in E. For $S'(R^n : E)$ direct and inverse Fourier transformations are determined by formulas (3), (4), but integral have sense of Bochner. For $f\in S'(R^n:E)$ the derivative $D^\alpha f$ is determined by formula:

$$\left(D^\alpha f\right)(\varphi)=(-1)^{|\alpha|}f\left(D^\alpha\varphi\right),\ \forall\varphi\in S\left(R^n\right).$$

For $f\in S'(R^n:E)$ and $\varphi\in S(R^n)$ its convoluiton is defined from relation :

$$(f*\varphi)(x)=f_y(\varphi(x-y))$$

it is obvious, that $f*\varphi\in S'(R^n:E)$ and therefore following identities are valid:

$$F(f*\varphi)=(2\pi)^{n/2}Ff\cdot F\varphi, \tag{5}$$

$$F^{-1}(f*\varphi)=(2\pi)^{n/2}F^{-1}f\cdot F^{-1}\varphi. \tag{6}$$

Let E be Banach space $\sigma=(\sigma_1,...,\sigma_n)$, $-\infty<\sigma_j<\infty$, $j=1,..,n$ $1<p\le\infty$.

We define Banach space $l_p^\sigma(E)$ by the following rule:

$$l_p^\sigma(E)=\left\{\varphi:\varphi=\{\varphi_k\}_{k=0}^\infty,\varphi_k\in E, where\|\varphi\|_{l_p^\sigma(E)}=\left(\sum_{k=1}^\infty 2^{\frac{k}{n}\sum_{j=1}^n\sigma_j p}\|\varphi_k\|_E^p\right)^{\frac{1}{p}}<\infty\right\}$$

for $1<p<\infty$ and $\|\varphi\|_{l_\infty^\sigma}=\sup_k 2^{k\frac{1}{n}\sum_{j=1}^n\sigma_j p}\|\varphi_k\|_E<\infty$ for $p=\infty$.

Theorem 1. *Let* $\sigma'=(\sigma_1',...,\sigma_n')$, $\sigma''=(\sigma_1'',...,\sigma_n'')$, $-\infty<\sigma_j',\sigma_j''<\infty$, $\sigma_j'\ne\sigma_j''$, $1\le p,p_1,p_2<\infty$.

Then $l_{p_1}^{\sigma'}(E)$ *and* $l_{p_2}^{\sigma_2''}(E)$ *are interpolational pair and for* $0<\theta<1$ *the following equality takes place:*

$$\left(l_{p_1}^{\sigma'}(E),l_{p_2}^{\sigma_2''}(E)\right)_{\theta,p}=l_p^\sigma(E), \tag{7}$$

where $\left(l_{p_1}^{\sigma'}(E),l_{p_2}^{\sigma_2''}(E)\right)_{\theta,p}$ *is interpolational space between* $l_{p_0}^{\sigma'}(E)$ *and* $l_{p_2}^{\sigma_1'}(E)$, $\sigma=(\sigma_1,...,\sigma_n)$, $\sigma_j=\theta\sigma_j'+(1-\theta)\sigma_j''$, $j=1,...,n$.

In the proof of theorem we use method of work [4], but taking into account the vectors of σ',σ''.

Definition 1 ([4]). *For arbitrary natural* m *the set of all systems of functions* $\{\varphi_k\}_{k=0}^\infty$ *with following properties, we denote by* Φ_m:

1. $\varphi_k(x)\in S(R^n)$, $(F\varphi_k)(\xi)\ge 0$ *for* $k\in N_0$;
2. $suppF\varphi_k\subset\{\xi:\xi\in R^n,2^{k-m}\le|\xi|\le 2^{k+m}\}$, $k\in N$, $suppF\varphi_0\subset\{\xi\in R^n,|\xi|\le 2^m\}$;
3. *There exists positive number* C_1 *such, that*
$$\sum(F\varphi_k)(\xi)\ge C_1;$$
4. *For any multi-index* α *there exists positive number* $C_2(\alpha)$ *such, that*
$$|\xi|^{-|\alpha|}\left|D^\alpha F\varphi_k(\xi)\right|\le C_2(\alpha).$$

Denote $\Phi = \bigcup_{m=1}^{\infty} \Phi_m$.

Definition 2. *Let* $s = (s_1, \ldots, s_n)$, $-\infty < s_j < \infty$, $1 < p < \infty$, $1 \le q \le \infty$, E_0 *and* E *are Banach spaces and* E_0 *is continuously embedded into* E. *Denote by* $B^s_{p,q}(R^n : E_0, E)$ *the set of functions from* $S'(R^n : E_0)$, *for which is following norm is finite:*

$$\|f\|_{B^s_{p,q}(R^n:E_0,E)} = \left\|\{f * \varphi_k\}\right\|_{l_q(L_p(R^n:E_0))} + \left\|\{f * \varphi_k\}\right\|_{l^s_q(L_p(R^n:E))} < \infty, \tag{8}$$

where $\{\varphi_k\}_{k=0}^{\infty} \in \Phi$ *and norm doesn't depend on choice of* $\{\varphi_k\}_{k=0}^{\infty}$.

Let's $s = (s_1, \ldots, s_n)$, $-\infty < s_j < \infty$, consider operator

$$J_s f = F^{-1} \sum_{j=1}^{n} \left(1 + x_j^2\right)^{s_j/2} F f.$$

It is easy to show, that [4] in this case also J_s makes continuous mutual point-to-point mapping $S(R^n)$ onto $S(R^n)$and $S'(R^n : E)$ onto $S'(R^n : E)$. The following equality holds:

$$J_s^{-1} - J_{-s}.$$

Theorem 2. *Let* $\sigma = (\sigma_1, \ldots, \sigma_n)$, $s = (s_1, \ldots, s_n)$, $-\infty < s_j < \infty$ *and* $1 < p < \infty$, $1 \le q \le \infty$. *Then* J_s *makes continuous mutual point-to-point mapping* $B^{\sigma}_{p,q}(R^n : E)$ *onto* $B^{\sigma-s}_{p,q}(R^n : E)$.

Proof. Let $\{\varphi_k\}_{k=0}^{\infty} \in \Phi$. Consider sequence

$$\psi_k = \varphi_k * F^{-1}\left(\frac{2^{k \cdot \frac{1}{n}\sum_{j=1}^{n} s_j}}{\sum_{j=1}^{n}\left(1 + \xi_j^2\right)^{s_j/2}}\right).$$

So as $F\psi_k = (2\pi)^{n/2} F(\varphi_k) \cdot \dfrac{2^{k \cdot \frac{1}{n}\sum_{j=1}^{n} s_j}}{\sum_{j=1}^{n}\left(1 + \xi_j^2\right)^{s_j/2}}$ to $\{\varphi_k\}_{k=0}^{\infty} \in \Phi$.

Consequently,

$$F_s f * \psi_k = F^{-1}\left((2\pi)^{n/2} F(\psi_k) \cdot F J_s f\right) =$$

$$= F^{-1}\left((2\pi)^{n/2} F(\varphi_k) \cdot \frac{2^{\frac{k}{n}\sum_{j=1}^{n} s_j}}{\sum_{j=1}^{n}\left(1 + \xi_j^2\right)^{s_j/2}} \cdot F\left(F^{-1}\left(\sum_{j=1}^{n}\left(1 + \xi_j^2\right)\right)\right) F f\right) =$$

$$= F^{-1}(2\pi)^{n/2} \cdot 2^{\frac{k}{n}\sum_{j=1}^{n} s_j} F(\varphi_k) \cdot F f = (2\pi)^{n/2} \cdot 2^{\frac{k}{n}\sum_{j=1}^{n} s_j} (f * \varphi_k).$$

Then

$$\left\|J_s f * \psi_k\right\|_{l_q^{\sigma-s}(ER^n:E)} = \left\|(2\pi)^{n/2}\cdot\sum_{k=1}^{\infty}2^{\frac{k}{n}\sum_{j=1}^{n}s_j}\left(f*\varphi_k\right)\right\|_{l_q^{\sigma-s}\left(L_p\left(R^n:E\right)\right)} =$$

$$=(2\pi)^{n/2}\left\|\sum_{k=1}^{\infty}2^{\frac{k}{n}\sum_{j=1}^{n}s_j}\cdot 2^{\frac{k}{n}\sum_{j=1}^{n}(\sigma_j-s_j)q}\left(f*\varphi_k\right)\right\|_{\left(L_p\left(R^n:E\right)\right)}\sim$$

$$\sim(2\pi)^{n/2}\left\|\sum_{k=1}^{\infty}2^{\frac{k}{n}\sum_{j=1}^{n}\sigma_j q}\left(f*\varphi_k\right)\right\|_{\left(L_p\left(R^n:E\right)\right)}\sim\left\|\left(f*\varphi_k\right)\right\|_{l_q^{\sigma}\left(L_p\left(R^n:E\right)\right)}.$$

Taking into account, that for $\psi_k\in\Phi\left\|\left(f*\psi_k\right)\right\|_{l_p\left(L_p\left(R^n:E_0\right)\right)}\sim\|f\|_{L_p\left(R^n:E_0\right)}$ we obtain statement of the theorem.

Let E be Banach space. one-parameter family $\{G(t)\}_{0\le t<\infty}$ of linear bounded operators, mappings from E to E_0 is called strong continuous semi-group, if

1) $G(t_1)G(t_2)=G(t_1+t_2)$, $0\le t_1,\ t_2<\infty$, $G(0)=J$ is unit operator.
2) For all $x\in E$ and all $t\in[0,\infty)$ correlation $\lim\limits_{\tau\to t}G(\tau)x=G(t)x$.

Λ is generating operator of half-group $\{G(t)\}_{0\le t<\infty}$, $D(\Lambda)$ is domain of definition of operator Λ, and is determined by rule

$$D(\Lambda)=\left\{x:x\in E,\ \exists\lim_{t\to 0}\frac{G(t)x-x}{t}\right\}.$$

$\Lambda_x=\lim\limits_{t\to 0}\dfrac{G(t)x-x}{t}$ for $\forall x\in D(\Lambda)$.

Consider in space $L_p\left(R^n:E\right)$ strong continuous commutative group $\{G_j(t)\}_{0\le t<\infty}$, $j=1,\dots,n$ of isometric operators

$$\{G_j(t)f\}(x)=f\left(x_1,\dots,x_{j-1},x_j+t,x_{j+1},\dots,x_n\right). \tag{9}$$

The generating operator of group $\{G_j(t)\}$ we denote by Λ_j, its domain of definition by $D(\Lambda_j)$. Then for $x\in R^n$, t, $j=1,\dots,n$ and coordinate vector e_j we have

$$\Delta_j(t)f(x)=f\left(x+te_j\right)-f(x)=\left\{G_j(t)f\right\}x-f(x).$$

$$\Delta_j^l=\Delta_j^{l-1}\left(\Delta_j\right)\ \ for\ \ l=2,3,\dots$$

Theorem 3. *Let* $m=(m_1,\dots,m_n)$, $k=(k_1,\dots,k_n)$, $l=(l_1,\dots,l_n)$, $k_i\in N$, $m_i\in N_0$, $l_i>0$, $m_i>l_i-k_i$, $i=1,\dots,n$ $1<p<\infty$, $1\le q\le\infty$, $0<\delta<\infty$, *then* $B_{p,q}^l\left(R^n:E_0,E\right)=$
$=\left\{f;f\in L_p\left(R^n:E_0\right)\right\}$, $D_j^{k_j f}\in L_p\left(R^n:E\right)$

$$\|f\|^{*}_{B^{l}_{p,q}}=\|f\|_{L_p(R^n:E_0)}+\left\{\sum_{j=1}^{n}\int_0^{\delta}t^{-(l_j-k_j)}\left\|\Delta_j^{m_j}(t)D_j^{k_j}f\right\|_{L\,(R^n:E)}dt\right\}^{1/q}<\infty\Big\}$$

for $1<q<\infty$. *(Corresponding to changings for* $q=\infty$*). And* $\|f\|^{*}_{B^{l}_{pq}}$ *is equivalent to the norm, determined from (8).*

Proof. Consider continuous semi-groups of operators $(G_j(t)f)(x)$, determined in (9), where $f\in L_p(R^n:E_0)$. Corresponding generating operator denote by Λ_j and its domain of definition by $D(\Lambda_j)$ and correspondingly, for Λ_j^n and domain of definition by $D(\Lambda_j^n)$.

Consider space

$$W_p^{s_j}(R^n:E)=\{f:f\in S'(R^n:E_0)\}\|f\|_{W_p^{s_j}(R^n:E_0,E)}=$$
$$=\|f\|^{p}_{L_p(R^n:E_0)}+\left\|\frac{\partial^{s_j}f}{\partial x^{s_j}}\right\|^{p}_{L_p(R^n:E_0)}\Bigg\}<\infty,$$

$W_p^{s_1,\dots,s_n}(R^n:E_0,E)=\bigcap\limits_{j=1}^{n}W_p^{s_j}(R^n:E_0,E)$, taking into account that generating operator of semi-group (9) is $\Lambda_j^{s_j}=\dfrac{\partial^{s_j}}{\partial x_j^{s_j}}$, $D(\Lambda_j^{s_j})=W_p^{s_j}(R^n:E_0,E)$. By virtue of theorem 1.1.5, 1.1.6 for $l=(l_1..l_n)$ $l_j=\theta s_j$, $0<\theta<1$ of [6] we obtain

$$B^{l}_{p,q}(R^n:E_0,E)=\left(L_p(R^n:E_0),W_p^{s_1,\dots,s_n}(R^n:E_0,E)\right)_{\theta,q}=$$
$$=\left\{f:f\in L_p(R^n:E_0)\right\}\|f\|_{L_p(R^n:E_0)}+\|f\|_{(L_p(R^n:E_0),W_p^{s_1,\dots,s_n}(R^n:E_0,E))_{\theta,q}}=$$
$$=\|f\|_{L_p(R^n:E_0)}+\sum_{j=1}^{n}\left\|t_1^{-(l_j-k_j)}(G(t)-J)^{m_j}\Delta_j^{k_j}f\right\|_{L_q^{s}(R^n:E)}=$$
$$=\|f\|_{L_p(R^n:E_0)}+\left(\sum_{j=1}^{n}\int_0^{\delta}t^{-(l_j-k_j)q+1}\left\|\Delta^{m_j}(R^n:t)D^{k_j}f\right\|^{q}_{L_p(R^n:E)}dt\right)^{1/q}\Bigg\},$$

where $0<\delta<\infty$.

Theorem 4. *Let* $s=(s_1,\dots,s_n)$, $-\infty<s_j<\infty$, $1<p<\infty$, $1\le q<\infty$. *For function*

$$f(x)=\sum_{j=0}^{\infty}Q_j(x)\qquad(10)$$

to belong to space $B^{s}_{p,q}(R^n:E_0,E)$, *where* $Q_j(x)$ *are* E*-valued integer functions of power* 2^{k_j} *by* x_j, *convergence of series (10) is in the sense* $S'(R^n:E)$, *it is necessary and sufficient for norm*

$$\|f\|_{B^s_{p,q}(R^n:E_0,E)} = \|f\|_{L_p R^n:E_0} + \left(\sum_{j=0}^{\infty} 2^{k\frac{1}{n}\sum_{j=1}^{n} s_j} \|Q_j\|_{L_p(R^n:E)}\right) \tag{11}$$

to be finite, moreover, norm $\|f\|^*$ is equivalent to the norm (8).

Theorem 5. *Let $s=(s_1,\dots,s_n)$, $-\infty < s_j < \infty$, $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)$, $\varepsilon_j > 0$, $1<p<\infty$, $1 \le q_1 \le q_2 \le \infty$, E_0, E are Banach spaces, $E_0 \subset E$, then following continuous embeddings take place*

$$B^{s+\varepsilon}_{p,\infty}(R^n:E_0,E) \subset B^s_{p,q}(R^n:E_0,E) \subset B^s_{p,\infty}(R^n:E_0,E)$$

$$B^s_{p,q_1}(R^n:E_0,E) \subset B^s_{p,q_2}(R^n:E_0,E) \subset B^s_{p,\infty}(R^n:E_0,E)$$

$$B^s_{p,q}(R^n:E_0,E) \subset B^s_{p,p}(R^n:E_0,E), \quad 1<q\le p<\infty$$

$$B^s_{p,p}(R^n:E_0,E) \subset B^s_{p,q}(R^n:E_0,E), \quad 1<p\le q<\infty.$$

Theorem 6. *Let H_0 and H be Hilbert spaces, H_0 is continuously imbedded in H, $s=(s_1,\dots,s_n)$, $0<s_j<\infty$, $l=(l_1,\dots,l_n)$, $0<l_j<\infty$, $1<p_1<p_2<\infty$, $1<q<\infty$, $\alpha = (\alpha_1,\dots,\alpha_n)$ is multi-index with integer value components, $\chi_k = \sum_{j=1}^{n} \frac{\alpha_j + \frac{1}{p_2} - \frac{1}{p_1}}{l_j} + \frac{s_k}{l_k}$, $\chi = \max_{0\le k\le 1} \chi_k$. Then for $\chi<1$ take place following continuous embedding*

$$D^\alpha B^l_{p_1,q}(R^n:H_0,H) \subset B^s_{p_2,q}(R^n:(H_0,H)_\chi),$$

where $(H_0,H)_\chi$ is interpolation space between H_0 and H.

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