ISMAILOV M.I.

THE INVERSE NON-STATIONARY SCATTERING PROBLEM FOR THE SYMMETRIC HYPERBOLIC SYSTEMS ON THE SEMI-AXIS

Abstract

In the work the scattering problems on the semi-axis for the symmetric hyperbolic system of three equations for joint consideration of two problems with different boundary conditions are considered and the scattering operator is constructed. The possibility of the one-to-one restoration of the coefficients of the system by the scattering operator is proved.

Let's consider on the semi-axis $x \ge 0$ the system of differential equations of the first order of a view

$$\xi_i \frac{\partial \psi_i(x,t)}{\partial t} - \frac{\partial \psi_i(x,t)}{\partial x} = \sum_{j=1}^{3} (\xi_i - \xi_j) u_{ij}(x,t) \psi_j(x,t), \quad i = 1,2,3 \tag{1}$$

supposing that the coefficients are the functions summed with square and $\xi_1 \ge \xi_2 > 0 > \xi_3$.

The non-stationary direct and inverse scattering problem for (1) in the case $\xi_1 > \xi_2 > 0 > \xi_3$ on the semi-axis and on the whole axis were investigated in [3,4], and for the hyperbolic system of two equations of the first order on the whole axis and on the semi-axis were studied in [1].

In the present work the direct and inverse scattering problems on the semi-axis in the case $\xi_1 = \xi_2 > 0 > \xi_3$ are considered.

For simplicity we suppose that $\xi_1 = \xi_2 = 1,\ \xi_3 = -1$. Then the system (1) is reduced to the following system

$$\begin{cases} \dfrac{\partial \psi_i(x,t)}{\partial t} - \dfrac{\partial \psi_i(x,t)}{\partial x} = v_i(x,t)\psi_3(x,t), \quad i = 1,2 \\ \dfrac{\partial \psi_3(x,t)}{\partial t} + \dfrac{\partial \psi_3(x,t)}{\partial x} = v_3(x,t)\psi_1(x,t) + v_4(x,t)\psi_2(x,t), \end{cases} \tag{2}$$

where $x \ge 0,\ -\infty < t < +\infty$ and

$$\int_{-\infty}^{+\infty} \int_{0}^{+\infty} |v_i(x,t)|^2 \, dx dt < +\infty, i = \overline{1,4}. \tag{3}$$

1. The scattering problem.

Every admissible solution ([3]) of the system (2) with the potential satisfying the conditions (3) admits on the semi-axis $x \ge 0$ the asymptotic representation

$$\begin{aligned} \psi_i(x,t) &= a_i(t+x) + 0(1), i = 1,2 \\ \psi_3(x,t) &= b(t-x) + 0(1), x \to \infty, \end{aligned} \tag{4}$$

where the functions $a_1(s), a_2(s)$ and $b(s)$ are the functions from space $L_2(R)$.

Let's consider two problems. The first problem consists in finding of the solution of (2) by the given functions $a = (a_1, a_2)$ determining for $x \to +\infty$ the asymptotics of solutions ψ_1, ψ_2 of view (4) and satisfying the boundary conditions

$$\psi_3(0,t)=\psi_1(0,t). \tag{5}$$

The second problem consists in finding of the solution of (2) by the given functions $a=(a_1,a_2)$ and the boundary solution

$$\psi_3(0,t)=\psi_2(0,t). \tag{6}$$

Theorem 1. *Let the coefficients of (2) satisfy the conditions (3). Then there is the only admissible solution of the first and the second scattering problems on the semi-axis for the system of equations (2) with arbitrary given functions* $a_1,a_2\in L_2(R)$.

Proof. Of the k-th problem is equivalent to the following system of integral equations

$$\begin{aligned}\psi_i^k(x,t)&=a_i(x+t)+\int_x^\infty (v_i\psi_3^k)(s,x+t-s)ds, i=1,2\\ \psi_3^k(x,t)&=b_k(t-x)-\int_x^\infty (v_3\psi_1^k+v_4\psi_2^k)(s,t-x+s)ds,\end{aligned} \tag{7}$$

where

$$b_k(t)=a_k(t)+\int_0^\infty [(v_k\psi_3^k)(s,t-s)+(v_3\psi_1^k)(s,t+s)+(v_4\psi_2^k)(s,t+s)]ds, k=1,2 \tag{8}$$

Existence and uniqueness of solutions of system (7) follow from its volterrity by variable t by virtue of the theorem 4.1.1 [2].

By virtue of conditions (2) from (7) we obtain the asymptotical representations for $\psi_3^k(x,t)$ for $x\to+\infty$ of the form (4)

$$\psi_3^k(x,t)=b_k(t-x)+0(1), b_k\in L_2(R), k=1,2. \tag{9}$$

On the base of theorem 1 according to (9) two solutions of (2) correspond to the each vector function $a=(a_1,a_2)\in L_2$ the solutions of the first and the second problems with the boundary conditions correspondingly to (5) and (6). These two solutions determine by (9) two functions $b=(b_1,b_2)\in L_2$. And in the space $L_2(R,C^2)$ the operator S has been determined which determines $a(s)$ into $b(s)$:

$$\begin{pmatrix}b_1(s)\\ b_2(s)\end{pmatrix}=S\begin{pmatrix}a_1(s)\\ a_2(s)\end{pmatrix},\quad S=(S_{ij})_{i,j=1}^2 \tag{10}$$

We will name this operator the scattering operator for the system (2) on the semi-axis.

2. Transformation operators.

Admissible solution of the system (2) with the given asymptotic $a_2(x+t), a_2(x+t), b(t-x)$ for $x\to+\infty$ satisfies the system of integral equations

$$\begin{aligned}\psi_i(x,t)&=a_i(x+t)+\int_x^\infty (v_i\psi_3)(s,x+t-s)ds, i=1,2,\\ \psi_3(x,t)&=b(t-x)-\int_x^\infty (v_3\psi_1+v_4\psi_2)(s,t-x+s)ds.\end{aligned} \tag{11}$$

Lemma 1. *Every admissible solution of the system (1) with the condition (3) admits the representation*

$$\psi_i(x,t)=a_i(x+t)+\sum_{j=1}^{2}\int_t^{+\infty}A_{ij}(x,t,\tau)a_j(x+\tau)d\tau+\int_{-\infty}^{t}A_{i3}(x,t,\tau)b(\tau-x)d\tau, i=1,2,$$

$$\psi_3(x,t)=b(t-x)+\sum_{j=1}^{2}\int_t^{+\infty}A_{3j}(x,t,\tau)a_j(x+\tau)d\tau+\int_{-\infty}^{t}A_{33}(x,t,\tau)b(\tau-x)d\tau, \qquad (12)$$

where the kernels for the fixed x *are the Hilbert-Schmidt kernels and connected with the potential by the equalities*

$$A_{i3}(x,t,t)=\frac{1}{2}v_i(x,t), i=1,2,$$

$$A_{31}(x,t,t)=-\frac{1}{2}v_3(x,t), A_{32}(x,t,t)=-\frac{1}{2}v_4(x,t),$$

$$A_{i1}(x,t,t)=-\frac{1}{2}\int_x^{+\infty}v_i(s,x+t-s)v_3(s,t+x-s)ds\, i=1,2, \qquad (13)$$

$$A_{i2}(x,t,t)=-\frac{1}{2}\int_x^{+\infty}v_i(s,x+t-s)v_4(s,t+x-s)ds\, i=1,2,$$

$$A_{33}(x,t,t)=-\frac{1}{2}\int_x^{+\infty}v_1(s,s-x+t)v_3(s,s-x+t)ds-\frac{1}{2}\int_x^{+\infty}v_2(s,s-x+t)v_4(s,s-x+t)ds.$$

Proof. If the solution of the system of equations (11) is represented in the form (12) for any $a_1,a_2,b\in L_2$, then substituting (12) into (11) we obtain the system of equations for the kernels:

$$A_{ij}(x,t,\tau)=\int_x^{+\infty}v_i(s,x+t-s)A_{3j}(s,x+t-s,\tau+x-s)ds,$$

$$A_{3j}(x,t,\tau)=-\frac{1}{2}v_{2+j}\left(\frac{\tau+2x-t}{2},\frac{t+\tau}{2}\right)-$$

$$-\int_x^{\frac{\tau+2x-t}{2}}\sum_{k=1}^{2}v_{2+k}(s,t-x+s)A_{kj}(s,t-x+s,\tau+x-s)ds, \qquad (14)$$

$$A_{i3}(x,t,\tau)=\frac{1}{2}v_i\left(\frac{t+2x-\tau}{2},\frac{t+\tau}{2}\right)-\int_x^{\frac{t+2x-\tau}{2}}v_i(s,x+t-s)A_{33}(s,x+t-s,\tau-x+s)ds,$$

$$A_{33}(x,t,\tau)=-\int_x^{+\infty}\sum_{k=1}^{2}v_{2+k}(s,t-x+s)A_{k3}(s,t-x+s,\tau-x+s)ds, \tau\le x, i,j=1,2.$$

Therefore, for proof of representation (12) it is sufficient to prove that (14) has the only solution, it follows from volterrity 4.1.1 [2]. The equalities (13) immediately follow from (14) for $\tau=t$.

3. The properties of the scattering operator.

Using the representation (12), the boundary conditions (5), (6) and definition (10) of the scattering operator, we obtain

$$b_k(t)+A_{31-}(0)a_1(t)+A_{32-}(0)a_2(t)+A_{33+}(0)b_k(t)=$$
$$=a_k(t)+A_{k1-}(0)a_1(t)+A_{k2-}(0)a_2(t)+A_{k3+}(0)b_k(t), k=1,2,$$

that is

$$S_{11}=\left(I+A_{33+}(0)-A_{13+}(0)\right)^{-1}\left(I+A_{11-}(0)-A_{31-}(0)\right),$$
$$S_{12}=\left(I+A_{33+}(0)-A_{13+}(0)\right)^{-1}\left(A_{21-}(0)-A_{32-}(0)\right),$$
$$S_{21}=\left(I+A_{33+}(0)-A_{23+}(0)\right)^{-1}\left(A_{21-}(0)-A_{31-}(0)\right),$$
$$S_{22}=\left(I+A_{33+}(0)-A_{23+}(0)\right)^{-1}\left(I+A_{22-}(0)-A_{32-}(0)\right). \qquad (15)$$

The supplementary properties of the scattering operators are obtained with the help of the following lemmas:

Lemma 2. *If* $a_1(s)=0$ *and* $a_2(s)=0$ *for* $s\le\lambda$, *then for* $x+t\le\lambda$ $\psi_1^k(x,t)=\psi_2^k(x,t)= =\psi_3^k(x,t)=0,\quad k=1,2$.

Proof. Let's consider the system (7)-(8) for $x+t\le\lambda$. If $a_1(s)=0$ and $a_2(s)=0$ for $s\le\lambda$, then the free term $b(t)$ is equal to zero for $x+t\le\lambda$ and so for $x+t\le\lambda$ $\psi_i^k(x,t)=0,\ i=1,2,3,\ k=1,2$.

Lemma 3. *If* $a_2(s)=0$ *for* $s\in R$ *and* $b_1(s)=0$ *for* $s\ge\lambda$, *then for* $t-x\ge\lambda$ $\psi_1^1(x,t)=\psi_2^1(x,t)=\psi_3^1(x,t)=0$.

Lemma 4. *If* $a_1(s)=0$ *for* $s\in R$ *and* $b_1(s)=0$ *for* $s\ge\lambda$, *then for* $t-x\ge\lambda$ $\psi_1^2(x,t)=\psi_2^2(x,t)=\psi_3^2(x,t)=0$.

Proofs of Lemma 3 and Lemma 4 are analogous to Lemma 2.

Theorem 2. *The matrix elements* S_{11} *and* S_{22} *of the scattering operator of the non-stationary problem for the system (2) on the semi-axis admits the two-side factorization.*

Proof. By virtue of (15) it is sufficient to prove only the left factorization S_{11} and S_{22}. From (15) we obtain that there are the operators S_{11}^{-1} and S_{22}^{-1} which differ from the Hilbert-Shmidt unit operator. Assume

$$S_{kk}=I+F_{kk},S_{kk}^{-1}=I+G_{kk},k=1,2, \qquad (16)$$

where F_{kk} and $G_{kk}(k=1,2)$ are Hilbert-Shmidt integral operators whose kernels we denote by $F_{kk}(t,s),G_{kk}(t,s)(k=1,2)$.

From lemmas 2,3 and 4 taking into account (12) we obtain the following connection of the transform operators with the scattering operator:

$$A_{kk}(x,t,s)+\int_{-\infty}^{t}A_{k3}(x,t,\tau)F_{kk}(\tau-x,s+x)d\tau=0,$$
$$A_{3k}(x,t,s)+F_{kk}(t-x,s+x)+\int_{-\infty}^{t}A_{33}(x,t,\tau)F_{kk}(\tau+x,s+x)d\tau=0,\quad t\le s,k=1,2.$$
$$A_{33}(x,t,s)+\int_{t}^{+\infty}A_{3k}(x,t,\tau)G_{kk}(x+\tau,s-x)d\tau=0, \qquad (17)$$
$$A_{k3}(x,t,s)+G_{kk}(x+t,s-x)+\int_{t}^{+\infty}A_{kk}(x,t,\tau)G_{kk}(x+\tau,s-x)d\tau=0,\quad s\ge t,k=1,2.$$

Determine Hilbert-Shmidt volterra operators $A_{k+},B_{k+},A_{k-},B_{k-}$ $(k=1,2)$ with the help of the kernels

$$A_{k+}(t,s)=\int_{-\infty}^{t}A_{k3}(0,t,\tau)F_{kk}(\tau,s)d\tau+A_{k3}(0,t,s),\quad t\ge s,$$

$$B_{k+}(t,s)=F_{kk}(t,s)+\int_{-\infty}^{t}A_{33}(0,t,\tau)F_{kk}(\tau,s)d\tau+A_{33}(0,t,s),\quad t\ge s,\tag{18}$$

$$A_{k-}(t,s)=G_{kk}(t,s)+\int_{t}^{+\infty}A_{kk}(0,t,\tau)G_{kk}(\tau,s)d\tau+A_{kk}(0,t,s),\quad t\le s,$$

$$B_{k-}(t,s)=\int_{t}^{+\infty}A_{3k}(0,t,\tau)G_{kk}(\tau,s)d\tau+A_{3k}(0,t,s),\quad t\le s.$$

Supposing $x=0$ from (17) taking into account (18) we obtain

$$\begin{aligned}&A_{kk-}(0)+A_{k3+}(0)F_{kk}=A_{k+}-A_{k3+}(0),\\&F_{kk}+A_{3k-}(0)+A_{33+}(0)F_{kk}=B_{k+}-A_{33+}(0),\\&G_{kk}(0)+A_{k3+}(0)+A_{kk-}(0)G_{kk}=A_{k-}-A_{kk-}(0),\\&A_{33+}(0)+A_{3k-}(0)G_{kk}=B_{k-}-A_{3k-}(0).\end{aligned}\tag{19}$$

From (19) taking into account $F_{kk}G_{kk}=G_{kk}F_{kk}=-F_{kk}-G_{kk}$, $k=1,2$ admit the left factorization.

4. The inverse scattering problem.

We will understand as the inverse scattering problem the problem of restoration of equation (2), that is of function $v_i(x,t)\left(i=\overline{1,4}\right)$ by the known scattering operator S.

From (12) and (13) we can determine by S, more exactly by S_{11} and S_{22}, the value of the potential for $x=0$. For finding of the potential for any values x it is natural to consider the scattering problem for the system (2) with the displaced potential

$$\begin{pmatrix}0&0&v_1(x+x_0,t)\\0&0&v_2(x+x_0,t)\\v_3(x+x_0,t)&v_4(x+x_0,t)&0\end{pmatrix}.\tag{20}$$

Denote by $S(x_0)$ the scattering operator of the problem on the semi-axis with potential (20). On the base of theorem 2 we conclude that for any $x\ge 0$ the operators $S_{11}(x)$ and $S_{22}(x)$ admit the two-side factorization and

$$\begin{aligned}&S_{11}(x)=\left(I+A_{33+}(x)-A_{13+}(x)\right)^{-1}\left(I+A_{11-}(x)-A_{31-}(x)\right),\\&S_{22}(x)=\left(I+A_{33+}(x)-A_{23+}(x)\right)^{-1}\left(I+A_{22-}(x)-A_{32-}(x)\right).\end{aligned}\tag{21}$$

Moreover, the operators

$$F(x)=S(x)-I,\quad G_{kk}(x)=S_{kk}^{-1}(x)-I$$

are Hilbert-Shmidt integral operators whose kernels we denote by $F(x,t,s),G_{kk}(x,t,s)$.

For $x=0$, $F(0,t,s)=F(t,s)=\left(F_{ij}(t,s)\right)_{i,j=1}^{2}$, $G_{kk}(0,t,s)=G_{kk}(t,s), k=1,2$.

Lemma 5. *For any* $x\ge 0$

$$F_{ij}(x,t,s)=F_{ij}(t-x,s+x),t\le s,$$
$$G_{ii}(x,t,s)=G_{ii}(t+x,s-x),t\ge s,i,j=\overline{1,2} \tag{22}$$

Proof. From (17) by subtraction we obtain for $t\le s$

$$A_{11}(x,t,s)-A_{31}(x,t,s)-F_{11}(t-x,s+x)+$$
$$+\int_{-\infty}^{t}[A_{13}(x,t,\tau)-A_{33}(x,t,\tau)]F_{11}(\tau+x,s+x)d\tau=0\ . \tag{23}$$

Taking for $t\ge s$ the left-hand side of (23) as the kernel of some operator $R_+(x)$ and $F_{11}(t-x,s+x)$ as the kernel of operator F_{11}^x, we rewrite (23) in the operator form:

$$A_{11-}(x)-A_{31-}(x)-(I+A_{33+}(x)-A_{13+}(x))F_{11}^x=R_+(x),$$

hence

$$F_{11}(x)-F_{11}^x=(I+A_{33+}(x)-A_{13+}(x))^{-1}(I+R_+(x))-I\ .$$

Taking into account that the right-hand side is the volterra operator with the valuable upper limit, we conclude for $t\le s$

$$F_{11}(x,t,s)-F_{11}^x(t,s)=0\Rightarrow F_{11}(x,t,s)=F_{11}(t-x,s+x).$$

By analogy the rest of the parts of lemma are proved.

Theorem 3. *Let $S=I+F$ be the scattering operator for the system (2) on the semi-axis. Then the potential in equation (2) is one-to-one restored by S.*

Proof. If the operator S is known, then with help of (22) we can find the volterra cuttings $F_{kk-}(x)$ and $G_{kk+}(x)$ and consequently, we can one-to-one determine $S_{kk}(x)=I+F_{kk}(x)$, so as these operators are two-side factorizied (theorem 4.3.3 [2]). From (21) and (23) the coefficients are one-to-one restored by $S_{kk}(x)$, $k=1,2$.

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References

[1]. Нижник Л.П. *Обратная нестационарная задача рассеяния.* Киев: Наук. Думка, 1973, 182 с.

[2]. Нижник Л.П. *Обратная задача рассеяния для гиперболических уравнений.* Киев: Наук. Думка, 1991, 232 с.

[3]. Нижник Л.П., Искендеров Н.Ш. *Обратная нестационарная задача рассеяния для гиперболической системы трех уравнений первого порядка на полуоси.* Укр.мат. журн., 1990, т.42, №7, с.931-938.

[4]. Искендеров Н.Ш. *Прямая и обратная задачи рассеяния для системы трех гиперболических уравнений первого порядка на полуоси с заданными рассеянны волнами.* Киев, 1985, 20 с., (Препринт/ АН УССР Ин-та математики; 85-87).

Iasmailov M.I.
Institute of Mathematics and Mechanics of AS Azerbaijan,
9, F. Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

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