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DESCRIPTION OF THE SCATTERING DATA FOR THE HYPERBOLIC SYSTEM OF THREE EQUATIONS ON THE SEMI-AXIS

Abstract

In this work the description of the scattering data for the hyperbolic system of three first order equations on the semi-axis is given.

Let's consider on the semi-axis $x \geq 0$ the hyperbolic system of three first order equations of the form

$$\sigma \frac{\partial \psi(x,t)}{\partial t} - \frac{\partial \psi(x,t)}{\partial x} = \text{U}(x,t)\psi(x,t), \tag{1}$$

where $\psi = (\psi_1, \psi_2, \psi_3)$ is the seeked function; $\sigma = diag(\xi_1, \xi_2, \xi_3)$ is the constant diagonal-matrix, moreover $\xi_1 > \xi_2 > 0 > \xi_3$; $\text{U}(x,t)$ is the matrix potential with the zeros elements on the diagonal $u_{ii}(x,t) = 0$ and the integrated non-diagonal elements

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} |u_{ij}(x,t)|^2 dxdt < \infty, i, j = 1,2,3. \tag{2}$$

In this work by the results on the inverse scattering problem for the system (1) on the semi-axis [1-3], we have managed to choose the minimal information (scattering data) sufficient for solving of the inverse scattering problem.

Let's consider the generalized solutions of (1), which are ordinary, measured by x and t functions. By variable t they belong to the space $L_2(E)$ $(E = (-\infty, \infty))$, and their L_2-norm by x is restricted. We name such solutions admitable.

Every admitible solution of (1) with the potential satisfying the conditions (2) admits on the semi-axis $x. \geq 0$ the asymptotically representation

$$\begin{aligned} \psi_i(x,t) &= a_i(t + \xi_i x) + 0(1), i = 1,2, \\ \psi_3(x,t) &= b(t + \xi_3 x) + 0(1), x \to +\infty, \end{aligned} \tag{3}$$

where functions $a_1(s), a_2(s) L_2$ determine the problems of the falling waves and the function $b(s) \in L_2$ is the profile of the scattering wave.

The scattering problem on the semi-axis is in finding of the solution of the system (1) by the given falling waves a_1, a_2 and the boundary conditions for $x = 0$.

Let's consider two problems. The first problem $(k = 1)$, the second problem $(k = 2)$ are in finding of the solution of the system (1) by the given falling waves $a = (a_1, a_2)$ determining for $x \to +\infty$ the asymptotics of the solutions φ_1, φ_2 of the form (3), and by the boundary condition

$$\psi_3^k(0,t) = \psi_k^k(0,t), \quad k = 1,2. \tag{4}$$

The joint consideration of these two problems we name the scattering problem for the system (1) on the semi-axis.

In [2] it was proved that there exists only solution of the scattering problem on the semi-axis for (1) with arbitrary given falling waves $a_1, a_2 \in L_2(E)$. Moreover

$$\psi_3^k(x,t)=b_k(t+\xi_3 x)+0(1),\ b_k(s)\in L_2(E),\ k=1,2. \qquad (5)$$

Therefore, for the system of equations (1) on the semi-axis the scattering operator S is determined which connects the asymptotics $\psi_1^k(x,t)$, $\psi_2^k(x,t)$ (a_1,a_2) and $\psi_3^k(x,t)$ (b_1,b_2) for $x\to+\infty$ and is the matrix operator $S=(s_{ij})_{i,j=1}^2$ in the space $L_2(-\infty;+\infty;E^2)$.

The inverse scattering problem for the system of equations (1) is in finding the potential $U(x,t)$ by the given scattering operator S or the scattering data.

Let's introduce into consideration the passage operator Π connecting the asymptotics (a_1,a_2,b) of the admitable solution of (1) with the boundary values of the solution for $x=0$, that is, with $(\psi_1,(0,t),\psi_2(0,t),\psi_3(0,t))$

$$\Pi\begin{pmatrix}a_1\\ a_2\\ b\end{pmatrix}=\begin{pmatrix}\psi_1(0,t)\\ \psi_2(0,t)\\ \psi_3(0,t)\end{pmatrix}. \qquad (6)$$

In [1-2] it was also proved that the passage operator Π by operator S is one-to-one determined and inversely. The passage operator Π is two-side factored on the matrix three angle operators:

$$\Pi=\begin{pmatrix}1+D_{11-} & 0 & 0\\ D_{21-} & I+D_{22-} & 0\\ D_{31-} & D_{32-} & I+D_{33-}\end{pmatrix}\begin{pmatrix}I & \Phi_{12} & \Phi_{13}\\ 0 & I & \Phi_{23}\\ 0 & 0 & I\end{pmatrix}^{-1}=$$
$$=\begin{pmatrix}I & A_{12+} & A_{13+}\\ 0 & I & A_{23+}\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix}1+B_{11+} & 0 & 0\\ B_{21} & 1+B_{22+} & 0\\ B_{31-} & B_{32} & 1+B_{33+}\end{pmatrix}. \qquad (7)$$

In (7) the operators $D_{11-},D_{21-},D_{31-},D_{22-},D_{32-},D_{33-},A_{12+},A_{13+},A_{23+},B_{11+}$, B_{22+},B_{33+}, B_{31-} are the volterra integral operators of the corresponding polarity and the operators $\Phi_{12},\Phi_{13},\Phi_{23},B_{21},B_{23}$ are Fredholm operators. By the factorization components of the operator Π let's construct the new operator S^{Φ}:

$$S^{\Phi}=\begin{pmatrix}I+D_{11-} & 0 & 0\\ D_{21-} & I+D_{22-} & 0\\ D_{31-} & D_{32-} & 1+D_{33-}\end{pmatrix}^{-1}\begin{pmatrix}I & A_{12+} & A_{13+}\\ 0 & I & A_{23+}\\ 0 & 0 & I\end{pmatrix}=$$
$$=\begin{pmatrix}I & \Phi_{12} & \Phi_{13}\\ 0 & I & \Phi_{23}\\ 0 & 0 & I\end{pmatrix}^{-1}\begin{pmatrix}1+B_{11+} & 0 & 0\\ B_{21} & 1+B_{22+} & 0\\ B_{31-} & B_{32} & 1+B_{33+}\end{pmatrix}. \qquad (8)$$

From (7) and (8) it follows that operators S^{Φ} and Π are one-to-one connected with each other.

In [1-3] it was proved

$$\mathcal{F}_x S^{\Phi}\mathcal{F}_{-x}=(1+w_-(x))^{-1}(1+w_+(x)) \qquad (9)$$

where, $\mathcal{F}_x = diag\left(T_{\xi_1 x}, T_{\xi_2 x}, T_{\xi_3 x}\right)$, $T_x f(t) = f(t+x)$, $w_\pm(x)$ are the matrix volterra integral operators of the corresponding polarity and their kernels are connected with the potential $U(x,t)$ by the correlation

$$\left[\sigma, w_\pm(t,x,x)\right] = \pm U(x,t), \qquad (10)$$

where the quadratic square brackets mean the commutation.

Let's denote

$$\begin{pmatrix} I & \Phi_{12} & \Phi_{13} \\ 0 & I & \Phi_{23} \\ 0 & 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & f_{12} & f_{13} \\ 0 & I & f_{23} \\ 0 & 0 & I \end{pmatrix},$$

$$\begin{pmatrix} I + D_{11-} & 0 & 0 \\ D_{21-} & I + D_{22-} & 0 \\ D_{31-} & D_{32-} & 1 + D_{33-} \end{pmatrix}^{-1} = \qquad (11)$$

$$= \begin{pmatrix} (I + D_{11-})^{-1} & 0 & 0 \\ 0 & (I + D_{22-})^{-1} & 0 \\ 0 & 0 & (1 + D_{33-})^{-1} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ d_{21-} & I & 0 \\ d_{31-} & d_{32-} & I \end{pmatrix}.$$

Then from (8) we obtain the equality

$$\begin{pmatrix} I & 0 & 0 \\ d_{21-} & I & 0 \\ d_{31-} & d_{32-} & I \end{pmatrix} \begin{pmatrix} I & A_{12+} & A_{13+} \\ 0 & I & A_{23+} \\ 0 & 0 & I \end{pmatrix} =$$

$$\begin{pmatrix} 1 + D_{11-} & (I + D_{11-}) f_{12} & (I + D_{11-}) f_{13} \\ 0, & I + D_{22-} & (I + D_{22-}) f_{23} \\ 0 & 0 & 1 + D_{33-} \end{pmatrix}^{-1} \begin{pmatrix} I + B_{11+} & 0 & 0 \\ B_{21} & 1 + B_{22+} & 0 \\ B_{31-} & D_{32} & 1 + B_{33+} \end{pmatrix} \qquad (12)$$

The scattering data for the system of equation (1) we will call the operators $\{d_{21-}, d_{31-}, d_{32-}, A_{12+}, A_{13+}, A_{23+}\}$.

Theorem. *Let the system of equations (1) satisfies the condition (2). Then therefore passage operator is determined uniquely.*

Proof. Really, from (12)

I. $(I + D_{33-})(1 + B_{33+})$ is known. Hence by Gohberg-Kreyn factorization theory the operators D_{33-} and B_{33+} are also known. $(1 + D_{22-}) f_{23} (1 + B_{33+})$ and $(1 + D_{11-}) f_{13} (1 + B_{33+})$ are known, then $(1 + D_{22-}) f_{23}$ and $(1 + D_{11-}) f_{13}$ are determined one-to-one.

II. Taking into account point I from the known operators

$(1 + D_{33-}) B_{32}$

$(1 + D_{22-})(1 + B_{22+}) + (1 + D_{22-}) f_{23} B_{32}$

we find B_{32} and from factorization $(1 + D_{22-})(1 + B_{22+})$ we find D_{22-} and B_{22+}. Since the operator $(1 + D_{11-}) f_{12} (1 + B_{22}) + (1 + D_{11-}) f_{13} B_{32}$ is known, hence it is known $(1 + D_{11-}) f_{12}$.

III. The operators $(1+D_{33-})B_{31-}, (1+D_{22-})B_{21}+(1+D_{22-})f_{23}B_{31-}$ are known so B_{31-} and B_{21} are known.

And from the known operator $(1+D_{11-})(1+B_{11+})+(1+D_{11-})f_{12}B_{21}+(1+D_{11-})f_{13}B_{31-}$ we also find D_{11-} and B_{11+}. Then the operators f_{13}, f_{12}, f_{23} are also found.

Thus, by the scattering data we can construct the passage operator Π or S^{Φ}.

The algorithm of solving of the inverse problem is in that the constructed by the scattering data operator S^{Φ} admits factorization (9) by whose factorization components the potential is found according to (10).

Note, that description of the scattering on the whole axis for (1) was solved in [4].

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