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ON BEHAVIOUR OF SOLUTION OF THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE SOBOLEV EQUATION IN CYLINDRICAL DOMAIN AT $t \to +\infty$

Abstract

At studying small oscillation of rotating fluid by S.L. Sobolev in [1] was distinguished the class of equation of the form

$$\frac{\partial^2}{\partial t^2}\Delta u + \frac{\partial^2 u}{\partial x_1^2} = 0 \tag{S}$$

and for its he studied Cauchy problem, first and second boundary value problems. In article [2] he investigated behavior of solution of Cauchy problem at $t \to +\infty$ too. At present there are great numbers of articles devoted to studying different boundary value problems for Sobolev's equations. More detailed literature can be found in [3]-[6].

§1. Definitions, notations and uniqueness of solution of initial-boundary value problem for Sobolev's equation.

Let $R_m(y)$ be m-dimensional Euclidean space with element $y = (y_1, y_2, \ldots, y_m)$ and $R_n(x)$ is the same space with element $x = (x_1, x_2, \ldots, x_n)$. Let $\amalg = R_n(x) \times \Omega$ be a cylindrical domain in $R_n(x) \times R_m(y)$, where Ω is a bounded domain in $R_m(y)$ with smooth boundary $\partial\Omega$. Let $Q = \amalg \times (0, \infty)$. We consider in Q the next problem

$$\frac{\partial^2}{\partial t^2}\Delta_{n+m} u(x,y,t) + \Delta\ u(x,y,t) = f(x,y)e^{i\omega t} \tag{1}$$

with the initial conditions

$$u(x,y,0) = 0, \quad \frac{\partial ux,y,0}{\partial t} = 0, \tag{2}$$

and the boundary condition

$$u(x,y,t)_{\partial\amalg} = 0, \tag{3}$$

where ω is a real number, Δ_{n+m} is the Laplacian on (x,y), Δ_n- on (x), $f(x,y) \in C_0^\infty(\amalg)$. We denote that, if symbol of operator at higher derivative on time variable has not real roots (non-singular case) initial-boundary value problem for this equation were studied in [3]-[6]. The singular case, with the exception of Cauchy problem and initial-boundary value problem in a quarter space, is not studied. In given article the singular case for problem (1)-(3) is studied.

Definition. *The function $u(x,y,t)$ we shall call a classical solution of problem (1)-(3), if $u(x,y,t) \in C^{2,2,2}(\amalg,(0,\infty)) \cap C^{1,0,2}(\overline{\amalg},[0,\infty))$, satisfies the equation (1), conditions (2), (3) in ordinary sense and*

$$\left|D_x^\alpha D_y^\beta D_t^\nu u(x,y,t)\right| \le C(t)e^{-\gamma|x|} \tag{4}$$

uniformly with respect to (y), where $|x|$ is Euclidean norm of (x) in R_n, $\gamma > 0$ some constant, $0 \le \alpha \le 2$, $0 \le \beta \le 2$, $0 \le \nu \le 2$, $|C(t)| \le Ct^q$, $q > 0$, continuous function.

Theorem 1. *The classical solution of the problem (1)-(3) is unique.*

Proof. We show that homogeneous problem, corresponding to the problem (1)-(3) has only trivial solution. Let $\sigma_R(x)$ be the sphere with center at origin of coordinates and radius R in $R_n(x)$, $\amalg_R = \Omega \times \sigma_R(x)$. Then

$$\partial \amalg_R = \partial\Omega \times \sigma_R(x) \cup \Omega \times \partial\sigma_R(x).$$

Multiplying homogeneous equation (1) to $u_t(x,y,t)$ integrating on $\amalg_R$ by part in this using first Green's formula and the boundary condition (3) we receive

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\amalg_R}\!\!\dots\!\int \sum_{i=1}^{n}\left[\left(\frac{\partial u_t}{\partial x_i}\right)^2 + \left(\frac{\partial u}{\partial x_i}\right)^2\right] d\amalg -$$
$$- \int\limits_{\Omega\times\partial\sigma_R}\left[u_t\left(\frac{\partial}{\partial n}\left(\frac{\partial^2 u}{\partial t^2}\right) + \frac{\partial u}{\partial n}\right)\right] ds = 0, \tag{5}$$

where ds is the element of the surface $\partial\amalg_R$. In (5) tending $R \to \infty$ by virtue of conditions (4) we receive

$$\frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\amalg}\!\!\dots\!\int \sum_{i=1}^{n}\left(\frac{\partial u_t}{\partial x_i}\right)^2 d\amalg + \frac{1}{2}\frac{\partial}{\partial t}\int\limits_{\amalg}\!\!\dots\!\int \sum_{i=1}^{\infty}\left(\frac{\partial u}{\partial x}\right)^2 d\amalg = 0. \tag{6}$$

Denoting by

$$\int\limits_{\amalg}\!\!\dots\!\int \sum_{i=1}^{n}\left(\frac{\partial \vartheta}{\partial x_i}\right)^2 d\amalg = \|\nabla\vartheta\|^2_{L_2(\amalg)},$$

energy integral

$$E(t) == \|\nabla u_t\|^2_{L_2(\amalg)} + \|\nabla u\|^2_{L_2(\amalg)},$$

and integrating (6) on $[0,t]$ we obtain

$$E(t) = E(0).$$

Since for homogeneous problem $E(0) = 0$, then

$$E(t) \equiv 0$$

for all $t > 0$. From this follows, that $u(x,y,y) \equiv 0$. Theorem 1 is proved.

§2. Construction of Green's function of stationary problem.

By virtue of estimation (4) we accomplish Laplace transformation with respect to t in problem (1)-(3). Then obtain next boundary value problem with complex parameter k

$$k^2(\Delta_{n+m} + \Delta_n)\hat{u}(x,y,k) = \frac{f(x,y)}{k - i\omega}, \tag{7}$$

$$\hat{u}(x,y,k)\Big|_{\partial\amalg} = 0, \tag{8}$$

where $\operatorname{Re} k > 0$, $\hat{u}(x,y,k)$ is Laplace transformation of $u(x,y,t)$. Problem (7)-(8) is called stationary problem corresponding to the problem (1)-(3). Now we construct Green's function for the problem (7)-(8). Accomplishing in (7)-(8) Fourier transformation with respect to x we have

$$k^2\left(\Delta_m-|s|^2\right)\hat{\hat{u}}-|s|^2\hat{\hat{u}}=\frac{\tilde{f}(s,y)}{k-i\omega}, \tag{9}$$

$$\hat{\hat{u}}(s,y,k)\Big|_{\partial\Omega}=0, \tag{10}$$

$\tilde{f}(s,y)$ denotes Fourier transformation of $f(x,y)$. For simplicity we put

$$\hat{\hat{u}}(s,y,k)=V(s,y,k).$$

We consider the differential expression $L=\Delta_m$ with the domain of definition

$$D(h)=\left\{\vartheta(y):\vartheta(y)\in C^2(\Omega)\cap C(\overline{\Omega}),\ \Delta_m\vartheta(y)\in L_2(\Omega),\ \vartheta|_{\partial\Omega}=0\right\}.$$

The differential expression L with domain of definition $D(L)$ generates a negative-definite self-adjoint operator L_1. It is known [7] (p. 177-178) that spectrum of operator L_1 is discrete and for its eigen-values λ_l are such that

$$0>\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_l\ge\cdots,\quad \lim_{l\to\infty}\lambda_l=-\infty.$$

The eigen-functions $\varphi_l(y)$ of the operator L_1, corresponding to eigen-values λ_l, forms a basis in the space $L_2(\Omega)$. Now we can prove the following theorem.

Theorem 2. *Green's function of problem (7)-(8) is an analytic function of the parameter k, without pole-point $k=0$, singular points $k=\pm i$ and for its take place the next representation*

$$G(k,x,y,z)=-\frac{i(2\pi)^{-\frac{n}{2}}}{4k^2}|x|^{1-\frac{n}{2}}\sum_{l=1}^{\infty}\frac{1}{\lambda_l}\left(\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right)^{\frac{n}{2}+1}\times$$

$$\times H^{(1)}_{\frac{n}{2}-1}\left(|x|\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right)\varphi_l(y)\varphi_l(z), \tag{11}$$

where $H^{(1)}_{\nu}(z)$ is Hankel function first kind, order ν. At $\operatorname{Re}k\ge\varepsilon>0$, $|x|\ge\delta>0$ the series in (11) converges uniformly with respect to k and (x,y,z) in every compact subset from Π.

Proof. Using theorem 3.6. from [7] (p.177) for solution of problem (9)-(10) we have

$$V(s,y,k)=\sum_{l=1}^{\infty}\frac{C_l(s)\varphi_l(y)}{\lambda_l k^2-|s|^2(k^2+1)}, \tag{12}$$

where

$$C_l(s)=\frac{1}{k-i\omega}\int_{\Omega}\tilde{f}(s,z)\varphi_l(z)dz.$$

The solution of problem (7)-(8) is determined as the inverse Fourier transformation of $V(s,y,k)$ with respect to s:

$$\hat{u}(x,y,k)=\frac{1}{(2\pi)^n}\int_{R^n}V(s,y,k)e^{-i(x,s)}ds=$$

$$=\frac{1}{(2\pi)^n}\sum_{l=1}^{\infty}\varphi_l(y)\int_{R_n}\frac{C_l(s)e^{-(x,s)}}{\lambda_l k^2-(k^2+1)|s|^2}ds\ . \tag{13}$$

The intergration here is allowed because the series (12) and its derivatives converge uniformly [8] (p.231). Taking into account

$$\tilde{f}(s,z)=\mathcal{F}(f(x,y)),$$

where $\mathcal{F}$ is Fourier transformation, from (13) we obtain

$$\hat{u}(x,y,k)=\frac{1}{(2\pi)^n(k-i\omega)}\sum_{l=1}^{\infty}\varphi_l(y)\int_{R_n}f_l(\xi)\left[\int_{R_n}\frac{e^{i(s,\xi-x)}ds}{\lambda_l k^2-(k^2-1)|s|^2}\right]d\xi, \qquad (14)$$

where

$$f_l(\xi)=\int_{\Omega}f(\xi,z)\varphi_l(z)d\xi.$$

Denote $\eta=\xi-x$,

$$J_l(k,\eta)=\frac{1}{(2\pi)^2}\lim_{N\to\infty}\int_{|S|\le N}\frac{e^{i(\eta,s)}}{\lambda_l k^2-(k^2+1)|s|^2}\equiv\frac{1}{(2\pi)^n}\lim_{N\to\infty}J_{l,N}(k,\eta).$$

Generally speaking, this integral does not exist in the usual sense. Hence, applying the theory of distribution and passing on to spherical coordinates, in this taking into account spherically symmetry of $\lambda_l k^2-(k^2+1)|s|^2$, we obtain

$$J_{l,N}(k,\eta)=(2\pi)^{-\left(\frac{n}{2}+1\right)}|\eta|^{1-\frac{n}{2}}\int_0^N\frac{|s|^{\frac{n}{2}}j_{\frac{n}{2}-1}(|\eta||s|)}{\lambda_l k^2-(k^2+1)|s|^2}d|s|, \qquad (15)$$

where $j_\nu(z)$ is the Bessel function of ν order. We now calculate integral in (15). Let n be an odd number. Then $z^{\frac{n}{2}}j(z)$ is an even integral function. Expressing the Bessel functions $j_\nu(z)$ in terms of the Henkel functions $H_\nu^{(1,2)}(z)$ [9] (p. 175) we represent $J_{l,N}(k,\eta)$ in the next form

$$J_{l,N}(k,\eta)=\frac{1}{4}(2\pi)^{-\left(\frac{n}{2}+1\right)}|\eta|^{1-\frac{n}{2}}\left\{\int_{-N}^{N}\frac{|s|^{\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(|\eta||s|)}{\lambda_l k^2-(k^2+1)|s|^2}d|s|+\right.$$
$$\left.+\int_{-N}^{N}\frac{|s|^{\frac{n}{2}}H^{(2)}_{\frac{n}{2}-1}(|\eta||s|)}{\lambda_l k^2-(k^2+1)|s|^2}d|s|\right\}\equiv J^{(1)}_{l,N}(k,\eta)+J^{(2)}_{l,N}(k,\eta). \qquad (16)$$

Poles of the integrand are

$$|s|_{1,2}=\pm\sqrt{\frac{\lambda_l k^2}{k^2+1}}.$$

For square root we choose such a branch, which is a pure imaginary number for the negative arguments. Taking into account analyticity of integrand in $J^{(1)}_{l,N}(k,\eta)$, the asymptotic behavior of Hankel functions as $z\to\infty$ [9], (p.219), applying theorem of residues and tending $N\to\infty$, we obtain

$$J_{l,N}^{(1)}(k,\eta)=\lim_{N\to+\infty}J_{l,N}^{(1)}(k,\eta)=-\frac{i(2\pi)^{-\frac{n}{2}}}{8\lambda_l k^2}|\eta|^{1-\frac{n}{2}}\left(\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right)^{\frac{n}{2}+1}H_{\frac{n}{2}-1}^{(1)}\left(|\eta|\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right). \qquad (17)$$

By analogy form

$$J_l^{(2)}(k,\eta)=-\frac{i(2\pi)}{8\lambda_l k^2}|\eta|^{1-\frac{n}{2}}\left(-\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right)^{\frac{n}{2}+1}H_{\frac{n}{2}-1}^{(1)}\left(-\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right). \qquad (18)$$

Taking into account [9] (p. 218)

$$H_{\frac{n}{2}-1}^{(2)}(-z)=(-1)^{\frac{n}{2}}H_{\frac{n}{2}-1}^{(1)}(-z) \qquad (19)$$

from (16)-(18), we obtain

$$J_l(k,\eta)=-\frac{i(2\pi)^{-\frac{n}{2}}}{4\lambda_l k^2}|\eta|^{1-\frac{n}{2}}\left(\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right)^{\frac{n}{2}+1}H_{\frac{n}{2}-1}^{(1)}\left(|\eta|\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right). \qquad (20)$$

Now let n be an odd number. Then $z^{\frac{n}{2}}j_{\frac{n}{2}-1}(z)$ is an even integral function. Making section $(-\infty,0)$, expressing the Bessel function by the Hankel functions $H_\nu^{(1,2)}(z)$ and taking into account formula (19), we obtain

$$\int_0^N\frac{|s|^{\frac{n}{2}}H_{\frac{n}{2}-1}^{(2)}(|\eta||s|)}{\lambda_l k^2-(k^2+1)|s|^2}d|s|=\int_{-N}^0\frac{|s|^{\frac{n}{2}}H_{\frac{n}{2}-1}^{(1)}(|\eta||s|)}{\lambda_l k^2-(k^2+1)|s|^2}d|s|$$

and from (15)

$$j_{l,N}(k,\eta)=\frac{(2\pi)^{-\left(\frac{n}{2}+1\right)}}{2}|\eta|^{1-\frac{n}{2}}\int_{-N}^N\frac{|s|^{\frac{n}{2}}H_{\frac{n}{2}-1}^{(1)}(|\eta||s|)}{\lambda_l k^2-(k^2+1)|s|^2}d|s|.$$

Proceedings as before, for even n again we obtain formula (20). Putting this expression of $J_l(k,\eta)$ in series (14), changing order of integration and summation, we have

$$\hat{u}(x,y,k)=\frac{i}{4(2\pi)^{\frac{n}{2}}k^2(k-i\omega)}\int\limits_{\Pi}\cdots\int\sum_{l=1}^{\infty}\frac{1}{\lambda_l}\left(\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right)^{\frac{n}{2}+1}|x-\xi|^{1-\frac{n}{2}}\times$$
$$\times H_{\frac{n}{2}-1}^{(1)}\left(|x-\xi|\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right)\varphi_l(y)\varphi_l(z)f(\xi,z)d\Pi.$$

From this for the Green's function $G(x,y,z,k)$ we receive expression (11). Now we show the convergence of series in (11).

In [10] is shown that

$$\|\varphi_l(y)\|_{\overset{\left(\left[\frac{m}{2}\right]+1\right)}{H(\Omega)}}\le C|\lambda_l|^{\frac{1}{2}\left(\left[\frac{m}{2}\right]+1\right)}.$$

From this by S.L. Sobolev's imdedding theorem we get

$$\|\varphi_l(y)\|_{C(\overline{\Omega})} \le C|\lambda_l|^{\frac{1}{2}\left(\left[\frac{m}{2}\right]+1\right)}. \quad (21)$$

It is known, that [8] (p.190)

$$c_0 l^{\frac{2}{m}} \le |\lambda_l| \le c_1 l^{\frac{2}{m}}, \quad (22)$$

where c_0, c_1 are constants, which does not depend of l. Then from (21), (22), it follows that

$$\|\varphi_l(y)\|_{C(\overline{\Omega})} \le C l^{\frac{\left[\frac{m}{2}\right]+1}{m}}. \quad (23)$$

It can be shown, that at $\operatorname{Re} k \ge \varepsilon > 0$

$$\operatorname{Re}\sqrt{\frac{k^2}{k^2+1}} \ge \frac{\sqrt{2}}{2}\frac{\varepsilon}{\sqrt{\varepsilon^2+1}}. \quad (24)$$

Considering the asymptotic behavior of the function $H_\nu^{(1)}(z)$ at $z \to \infty$ from (22)-(24) we receive that the series in (11) converges uniformly with respect to k and (x, y) in every compact subset of Π. Theorem 2 is proved.

Remark 1. Properly from the construction of Green's function $G(x, y, z, k)$, it at $x \ne 0$ satisfies homogeneous boundary value problem, corresponding to the problem (7), (8) and infinitely differentiable with respect to x, y, z.

Remark 2. From the proof of Theorem 2 it follows too, that for solution of problem (7), (8) takes place formula

$$\hat{u}(x, y, k) = \frac{1}{k - i\omega}\int \cdots \int_{\Pi} G(x-\xi, y, z, k) f(\xi, z) d\Pi, \quad (25)$$

where $G(x, y, z, k)$ is defined by (11).

§3. Behavior of solution of homogeneous problem at $t \to +\infty$.

We consider the following initial-boundary value problem for homogeneous equation in Q:

$$\frac{\partial^2}{\partial t^2}\Delta_{n+m} u(x, y, t) + \Delta_n u(x, y, t) = 0, \quad (26)$$

$$u(x, y, 0) = \varphi_0(x, y), \quad u_t'(x, y, 0) = \varphi_1(x, y), \quad (27)$$

$$u(x, y, t)\big|_{\partial\Pi} = 0, \quad (28)$$

where $\varphi_0(x, y),\ \varphi_1(x, y) \in C_0^\infty(\Pi)$. We shall study behavior of solution of problem (26)-(28) at $t \to +\infty$. The problem (26)-(28) by Laplace transformation is reduced, as well as in §2, to the next stationary boundary value problem

$$k^2\Delta_{n+m}\hat{u}(x, y, k) + \Delta_n \hat{u}(x, y, k) = k f_0(x, y) + f_1(x, y), \quad (29)$$

$$\hat{u}(x, y, k)\Big|_{\partial\Pi} = 0, \quad (30)$$

where

$$-\Delta_{n+m}\varphi_0(x, y) = f_0(x, y), \quad -\Delta_{n+m}\varphi_1(x, y) = f_1(x, y).$$

Using uniformly convergence of series in (11), for solution of problem (26)-(28) we have the next representation

$$u(x,y,t)=-\frac{i}{4(2\pi)^{\frac{n}{2}}}\int\cdots\int_{\Pi}\sum_{l=1}^{\infty}\frac{|x-\xi|^{-\frac{n}{2}+1}}{\lambda_l}\varphi_l(y)\varphi_l(z)\times$$

$$\times\left[\frac{1}{2\pi i}\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty}\frac{e^{kt}}{k^2}\left(\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right)^{\frac{n}{2}+1}H^{(1)}_{\frac{n}{2}-1}\left(|x-\xi|\sqrt{\frac{\lambda_l k^2}{k^2+1}}\right)\right](kf_0(\xi,z)+f_1(\xi,z))d\Pi, \qquad (31)$$

where ε_1 is an arbitrary positive number. The next theorem takes place:

Theorem 3. *Let* $n=1$, $\varphi_0(x,y)$, $\varphi_1(x,y)\in C_0^{\infty}(\Pi)$. *Then for solution of initial-boundary value problem (26)-(28) takes place the next asymptotic representation at* $t\to+\infty$

$$u(x,y,t)=-\frac{i}{\pi}\sum_{l=1}^{\infty}\frac{\varphi_l(y)}{\lambda_l^{1/2}}f_{1l}-O(1)\sum_{l=1}^{\infty}\varphi_l(y)\int_{R_1}|x-\xi|f_{1l}(\xi)d\xi,$$
$$f_{1l}(\xi)=\int_{\Omega}f_1(\xi,z)\varphi_l(z)dz,\quad f_{1l}=\int_{R_1}f_{1l}(\xi)d\xi, \qquad (32)$$

where both series in (32) converges uniformly with respect to (x,y) in every compact subset of Π.

Proof. Using the expression of function $H^{(1)}_{-\frac{1}{2}}(z)$, for interior integral in (31) we have

$$\mathcal{F}_l^{(1)}(\eta,t)=\frac{1}{\sqrt{2|\eta|}\pi^{3/2}i}\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty}\frac{e^{kt}}{k}\sqrt{\frac{\lambda_l}{k^2+1}}e^{i|\xi|k\sqrt{\frac{\lambda_l}{k^2+1}}}dk. \qquad (33)$$

Denote in the complex plane k by $\widetilde{C}_\varepsilon^{(1)}$, $\widetilde{C}_\varepsilon^{(2)}$ circles of radius ε with centers at points $k=-i$, $k=+i$ respectively and

$$C_\varepsilon^{(1)}=\left\{k:k\in\widetilde{C}_\varepsilon^{(1)},-\pi\le\arg(k+i)\le\frac{\pi}{2}\right\},$$

$$C_\varepsilon^{(2)}=\left\{k:k\in\widetilde{C}_\varepsilon^{(2)},-\frac{\pi}{2}\le\arg(k-i)\le\pi\right\},$$

$$\varphi=\arg\frac{k}{\sqrt{k^2+1}},$$

C_δ is a half circle in the left complex half-plane with radius δ and center at point $k=0$. Making section $(-\infty-i,-i)$, $(-\infty+i,-i)$, for square root we choose a branch, which is positive for positive value radicand.

We put

$$\Gamma_\varepsilon^\delta=(-\infty-i,-i-\varepsilon)\cup C_\varepsilon^{(1)}\cup(i(\varepsilon-1),i\delta)\cup C_\delta\cup(i\delta,i(1-\varepsilon))\cup C_\varepsilon^{(2)}\cup(-\varepsilon+i,i-\infty).$$

Taking into account behavior of integrand in (33), using Cauchy's theorem, it can be shown that

$$\mathcal{F}_l^{(1)}(\eta,t)=\left(\frac{2\lambda_l}{\pi|\eta|}\right)^{\frac{1}{2}}-\frac{i\lambda_l^{\frac{1}{2}}}{(2|\eta|)^{\frac{1}{2}}\pi^{3/2}}\int_{\Gamma_\varepsilon^\delta}\frac{e^{k\left(t+i|\eta|\sqrt{\frac{\lambda_l}{k^2+1}}\right)}}{k(k^2+1)}dk. \qquad (34)$$

Taking into account variation of φ at round k on contours $C_\varepsilon^{(1)}$, $C_\varepsilon^{(2)}$, we obtain that $\cos\varphi\geq 0$, therefore the integrand in (34) has summable singularities. Therefore we can tend to limit at $\varepsilon\to 0$ in (34). Then

$$\mathcal{F}_l^{(1)}(\eta,t)=\left(\frac{2\lambda_l}{\pi|\eta|}\right)^{\frac{1}{2}}-\frac{i\lambda_l^{\frac{1}{2}}}{(2|\eta|)^{\frac{1}{2}}\pi^{3/2}}\int_{\Gamma^\delta}\frac{e^{k\left(t+i|\eta|\sqrt{\frac{\lambda_l}{k^2+1}}\right)}}{k(k^2+1)}dk, \qquad (35)$$

where $\Gamma^\delta=\lim\limits_{\varepsilon\to 0}\Gamma_\varepsilon^\delta$. We study now asymptotics of integral on countour Γ^δ in (35) at $t\to+\infty$.

Consider the integral

$$\mathcal{A}_l^{(2)}(\eta,t)=\int_{-\infty+i}^{i}\frac{e^{k\left(t+i|\eta|\sqrt{\frac{\lambda_l}{k^2+1}}\right)}}{k(k^2+1)}dk. \qquad (36)$$

At $k\to i$ on ray $(-\infty+i,i)$, $\varphi\to-\dfrac{\pi}{4}$. Therefore the integrand in (36) exponentially tends to zero together with derivatives of arbitrary order. This function tends to zero exponentially at $\operatorname{Re}k\to-\infty$ too. Therefore integrating in (36) one time by parts (integrating in this e^{kt}) at $t\to+\infty$ we obtain

$$\mathcal{A}_l^{(2)}(\eta,t)=|\eta|\lambda_l^{\frac{1}{2}}O(t^{-1}) \qquad (37)$$

uniformly with respect to η.

By analogy is estimated the integral on ray $(-\infty-i,-i)$. On this ray at $k\to -i$, $\varphi\to\dfrac{\pi}{4}$, therefore

$$\mathcal{A}_l^{(1)}(\eta,t)=\int_{-\infty-i}^{-i}\frac{e^{k\left(t+i|\eta|\sqrt{\frac{\lambda_l}{k^2+1}}\right)}}{k\sqrt{(k^2+1)}}dk=|\eta|\lambda_l^{1/2}O(t^{-1}) \qquad (38)$$

uniformly with respect to η.

Now consider integrals

$$\mathcal{B}_1^\delta(\eta,t)=\int_{-i}^{-\delta i}\frac{e^{k\left(t+i|\eta|\sqrt{\frac{\lambda_l}{k^2+1}}\right)}}{k\sqrt{k^2+1}}dk,\quad \mathcal{B}_1^2(\eta,t)=\int_{\delta i}^{-i}\frac{e^{k\left(t+i|\eta|\sqrt{\frac{\lambda_l}{k^2+1}}\right)}}{k\sqrt{k^2+1}}dk.$$

On contour $(-i,-\delta i)$ $\varphi=-\dfrac{\pi}{2}$. We represent $\mathcal{B}_1^\delta(\eta,t)$ in form

$$\mathcal{B}_1^\delta(\eta,t)=\left(\int_{-i}^{-i(1-\varepsilon)}+\int_{-i(1-\varepsilon)}^{-\delta i}\right)\frac{e^{k\left(t+i|\eta|\sqrt{\frac{\lambda_l}{k^2+1}}\right)}}{k\sqrt{k^2+1}}dk,$$

where ε is a sufficiently small number. Estimating modulo of the first integral and second one integrating one time by parts at $t \to +\infty$ we obtain

$$\mathcal{B}_1^{\delta}(\eta,t) = |\eta|\lambda_l^{1/2}o(1) \tag{39}$$

uniformly with respect to η.

By analogy is proved, that at $t \to +\infty$

$$\mathcal{B}_2^{\delta}(\eta,t) = |\eta|\lambda_l^{1/2}o(1) \tag{40}$$

uniformly with respect to η.

Now consider the integral

$$\mathcal{A}_l^{\delta}(\eta,t) = \int_{C_\delta} \frac{e^{k\left(t+i|\eta|\sqrt{\frac{\lambda_l}{k^2+1}}\right)}}{k\sqrt{(k^2+1)}}dk.$$

Since integrand in $\mathcal{A}_l^{\delta}(\eta,t)$ is continuously differentiable and on contour C_δ $\operatorname{Re} k < 0$, then integrating one time by parts at $t \to +\infty$ we obtain

$$\mathcal{A}_l^{\delta}(\eta,t) = |\eta|\lambda_l^{1/2}O(t^{-1}), \tag{41}$$

uniformly with respect to η.

From (35), (37)-(41) it follows, at $t \to +\infty$

$$\mathcal{F}_l^{\mathrm{I}}(\eta,t) = \left(\frac{2\lambda_l}{\pi|\eta|}\right)^{\frac{1}{2}} + \lambda_l|\eta|^{1/2}o(1) \tag{42}$$

uniformly with respect to η.

Consider the integral

$$\mathcal{F}_l^{(0)}(\eta,t) = \sqrt{\frac{\lambda_l}{2|\eta|}}\frac{1}{\pi^{3/2}i}\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty} e^{k\left(t+i|\eta|\sqrt{\frac{\lambda_l}{k^2+1}}\right)}\frac{dk}{\sqrt{k^2+1}}.$$

By analogy it is proved, that at $t \to +\infty$

$$\mathcal{F}_l^{(0)}(\eta,t) = \lambda_l|\eta|^{1/2}o(1), \tag{43}$$

uniformly with respect to η.

Putting (42), (43) into (31), we receive for the solution of problem (26)-(28) the representation (32). Now we establish the convergence of series in (32). By virtue of (21)

$$\left\|\sum_{l=1}^{\infty}\frac{\varphi_l(y)}{\lambda_l^{\frac{1}{2}}}f_{1l}\right\|_{C(\bar\Omega)} \le \sum_{l=1}^{\infty}|\lambda_l|^{-\frac{1}{2}}\|\varphi_l(y)\|_{C(\bar\Omega)}|f_{1l}| \le \sum_{l=1}^{\infty}|\lambda_l|^{\frac{1}{2}\left[\frac{m}{2}\right]}|f_{1l}|.$$

By virtue of condition on $\varphi_1(x,y)$ the function

$$\mathcal{F}_1(y) = \int_{R_1} f_1(\xi,y)d\xi \tag{44}$$

belongs to $C_0^{\infty}(\Omega)$. Then according to theorem 8 from [8] (p. 230)

$$\sum_{l=1}^{\infty}|\lambda_l|^{\nu}|\mathcal{F}_{1l}|^2 \le C\|\mathcal{F}_1(y)\|^2_{H^{\nu}(\Omega)}, \tag{45}$$

where

$$\mathcal{F}_{1l} = \int_{\Omega}\mathcal{F}_1(y)\varphi_l(y)dy.$$

Putting in the expression of $\mathfrak{F}_{1l}$ instead of $\mathfrak{F}_1(y)$ its expression from (44) and according to Fibini theorem, change order of integration, we obtain

$$\mathfrak{F}_{1l} = \int_\Omega \left\{ \int_{R_1} f_1(\xi, y) d\xi \right\} \varphi_l(y) dy = \int_{R_1} \left\{ \int_\Omega f_1(\xi, y) \varphi_l(y) dy \right\} d\xi = \int_{R_1} f_{1l}(\xi) d\xi = f_{1l}.$$

Putting in (45) f_{1l} instead of $\mathfrak{F}_{1l}$, we obtain

$$\sum_{l=1}^{\infty} |\lambda_l|^{\nu} |f_{1l}|^2 \le C \|\mathfrak{F}_1(y)\|_{H^{\nu}(\Omega)}. \tag{46}$$

Then

$$\sum_{l=1}^{\infty} |\lambda_l|^{\frac{1}{2}\left[\frac{m}{2}\right]} |f_{1l}| \le \sum_{l=1}^{\infty} |\lambda_l|^{-m} + \sum_{l=1}^{\infty} |\lambda_l|^{m+\left[\frac{m}{2}\right]} |f_{1l}|^2. \tag{47}$$

Taking $\nu = m + \left[\frac{m}{2}\right]$ by virtue of estimations (22) and (46) we received, that both series in (47) converge.

Now we establish the convergence of the second series in (32). Putting in its the expression of $f_{1l}(\xi)$ from (32), we receive

$$\sum_{l=1}^{\infty} \varphi_l(y) \int_{R_1} |x-\xi| f_{1l}(\xi) d\xi = \sum_{l=1}^{\infty} \varphi_l(y) \left(\int_{R_1} |x-\xi| \int_\Omega f_1(\xi, z) \varphi_l(z) dz \right) d\xi =$$

$$= \sum_{l=1}^{\infty} \left(\int_\Omega \Phi(x,z) \varphi_l(z) dz \right) \varphi_l(y) = \sum_{l=1}^{\infty} \Phi_l(x) \varphi_l(y). \tag{48}$$

The function $\Phi(x,z)$ is infinite differentiable with respect to (x,z) and is finite on z. As above we estimate the series in (48)

$$\left\| \sum_{l=1}^{\infty} \Phi_l(x) \varphi_l(y) \right\|_{C(\overline{\Omega})} \le \sum_{l=1}^{\infty} |\Phi_l(x)| \|\varphi_l(y)\|_{C(\overline{\Omega})} \le$$

$$\le C \left(\sum_{l=1}^{\infty} |\lambda_l^{-m}| + \sum_{l=1}^{\infty} |\lambda_l|^{-m+\left[\frac{m}{2}\right]+1} |\Phi_l(x)|^2 \right). \tag{49}$$

The first series in (49) converges by virtue of estimation (22) and for the second series is true the estimation

$$\sum_{l=1}^{\infty} |\lambda_l|^{m+\left[\frac{m}{2}\right]+1} |\Phi_l(x)|^2 \le C \|\Phi(x,z)\|_{H^{m+\left[\frac{m}{2}\right]+1}(\Omega)}.$$

Therefore the series in (48) converges uniformly with respect to (x,y) in every compact subset of Π. Theorem 3 is proved.

Substituting in the interior integral in (31) contour of integration by $\Gamma_\varepsilon^\delta$ and doing as well as at proof the convergence of series in (32), it is possible to receive estimation (4) for solution of the non-stationary problem (26)-(28).

Remark 3. If $n \ge 3$ is odd number, then using the method of proof of the theorem 3 it can be shown, that at $t \to +\infty$

$$\left(1 + \frac{\partial^2}{\partial t^2}\right)^{\left[\frac{n+1}{4}\right]} u(x,y,t) = o(1),$$

uniformly with respect to (x,y) in every compact subset of Π.

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