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THEOREM ON FINITENESS OF NEGATIVE SPECTRUM OF QUASIELLIPTIC OPERATOR

Abstract

The operator $L_1 = L + Q(x)$, where L is quasielliptic operator with constant coefficients, and $Q(x)$ is measurable bounded function was considered. By the help of anisotropic Khardy and Poincare inequalities a theorem on finitness of negative spectrum of operator L_1 was proved.

Consider linear differentiable operator defined on $\overset{\circ}{C}{}^{\infty}(R^n)$

$$Lu = \sum_{(\alpha,\lambda)=2} a_\alpha D^\alpha u,$$

coefficients of which are constant, and $\sum_{(\alpha,\lambda)=2} a_\alpha (i\xi)^\alpha > 0$ for $\xi \in R^n \setminus \{0\}$. Here $\lambda = (p_1^{-1},\dots,p_n^{-1})$, $\alpha = (\alpha_1,\dots,\alpha_n)$ are vectors from R^n, $\alpha_1,\dots,\alpha_n$ are integer, $p_1,\dots,p_n$ are natural numbers, (α,λ) is scalar product, $D^\alpha u$ are generalized derivatives.

Let $Q(x)$ be measurable bounded function $x \in R^n$. Consider operator $L_1 = L + Q$ on $\overset{\circ}{C}{}^{\infty}(R^n)$. Its closure is self-adjoint operator $L_2(R^n) \to L_2(R^n)$. In many papers [1], [2], [3] the question on spectrum of such operator in elliptic case was studied. The inequalities of Khardy type were essentially used for investigation of spectrum of operator L [4]. One of such inequalities is represented in the following lemma.

Lemma. *If $U(x) \in \overset{\circ}{C}{}^{\infty}(R^n)$, $\sum_{i=1}^{n} p_i^{-1} > 2$, then*

$$\int_{R^n} \left(\sum_{i=1}^{n} x_i^{2p_i} \right)^{-1} U^2 dx \le \gamma \int_{R^n} \sum_{i=1}^{n} \left| D^{p_i} u \right|^2 dx,$$

where $\gamma = const$ and doesn't depend on U.

Using this lemma it is easy to prove following statement.

Theorem. *Let $\sum_{i=1}^{n} \frac{1}{p_i} > 2$, $q_i > 0$, $i = 1,\dots,n$,*

$$\sum_{1}^{n} \frac{1}{q_i} = \sum_{1}^{n} \frac{1}{p_i}.$$

There exits $\eta_q = const > 0$ such, that if

$$\overline{\lim_{|x|\to\infty}} \left(\sum_{1}^{n} x_i^{2q_i} \right) |Q_-(x)| \le \eta_q,$$

then negative spectrum of operator L_q consists of at most finite number of eigen values, which have finite multiplicity. (Q_- is negative part of function $Q(x)$).

Proof. At first, we should note, that there exists constant c such, that for any $\xi \in R^n$

$$\sum_{(\alpha,\lambda)=2} a_\alpha (i\xi)^\alpha \geq c\sum_{i=1}^{n} \xi_i^{2p_i}. \tag{1}$$

Let $\varphi \in \overset{\circ}{C}{}^\infty(R^n)$ and $\tilde{\varphi}$ is Fourier transformation φ. Let $\left(\sum_1^n x_i^{2q_i}\right)|Q_-(x)| \leq \overline{\eta}_q$, or $|Q_-(x)| < \overline{\eta}_q\left(\sum_1^n x_i^{2q_i}\right)^{-1}$.

It is clear, that (taking (1) into account)

$$\begin{aligned}(L_1\varphi,\varphi) &= (L\varphi,\varphi)+(Q\varphi,\varphi) = (\tilde{L}\tilde{\varphi},\tilde{\varphi})+(Q\varphi,\varphi) = \\ &\int_{R^n}\sum_{(\alpha,\lambda)=2} a_\alpha (i\xi)^\alpha |\tilde{\varphi}|^2 d\xi + \int_{R^n} Q|\varphi|^2 dx \geq \\ &\geq \int_{R^n} c\left(\sum_1^n \xi_i^{2p_i}\right)|\tilde{\varphi}|^2 d\xi - \int_{R^n} Q_-|\varphi|^2 dx \geq \\ &\geq c\int_{R^n}\sum_1^n |D^{p_i}u|^2 dx - \overline{\eta}_q \int_{R_n}\left(\sum x_i^{2q_i}\right)^{-1}|\varphi|^2 dx.\end{aligned} \tag{2}$$

Denote by $\Pi_\mu = \left\{x : x\in R^n, |x_i| < \mu^{\frac{1}{p_i}}\right\}$ and $\Pi'_\mu = \Pi_{2\mu}\setminus\Pi_\mu$.

By Hölder inequality

$$\int_{\Pi'_\mu}\left(\sum_1^n x_i^{2q_i}\right)^{-1}|\varphi|^2 dx \leq \left(\int_{\Pi'_\mu}|\varphi|^l dx\right)^{\frac{2}{l}}\left(\int_{\Pi'_\mu}\left(\sum_1^n x_i^{2q_i}\right)^{-\left(\frac{l}{2}\right)^*} dx\right)^{\frac{1}{\left(\frac{l}{2}\right)^*}} \tag{3}$$

$$\frac{1}{\left(\frac{l}{2}\right)^*}+\frac{1}{\frac{l}{2}} = 1, \qquad l = \frac{2\sum_1^n \frac{1}{p_i}}{\sum_1^n \frac{1}{p_i}-2}\cdot\left(\frac{l}{2}\right)^* = \frac{1}{2}\sum_1^n \frac{1}{p_i}.$$

Estimate second multiplier in right-hand side of (3). For this aim we estimate integral

$$K_1 = \int_{\substack{\mu^{\frac{1}{p_1}} < x_1 < (2\mu)^{\frac{1}{p_1}} \\ |x_i| < (2\mu)^{\frac{1}{p_i}}}}\left(\sum_1^n x_i^{2q_i}\right)^{-\frac{1}{2}\sum_1^n \frac{1}{p_i}} dx$$

$$K_1 \leq \int_{\substack{\mu^{\frac{1}{p_1}} < x_1 < (2\mu)^{\frac{1}{p_1}} \\ -\infty < x_i < +\infty}}\left(\sum_1^n x_i^{2q_i}\right)^{-\frac{1}{2}\sum_1^n \frac{1}{p_i}} dx$$

Last integral we denote by K_2. The set $\left\{x : \mu^{\frac{1}{p_1}} < x_1 < (2\mu)^{\frac{1}{p_1}},\ -\infty < x_i < +\infty,\ i = \overline{1,n}\right\}$ we denote by $\Pi'_\mu(1)$. Other similar sets we denote by $\Pi'_\mu(2),\ldots,\Pi'_\mu(n)$ correspondingly.

$$K_2 = \int\limits_{\mu^{\frac{1}{p_1}} < x_1 < (2\mu)^{\frac{1}{p_1}}} dx_1 \int\limits_{R^{n-1}} \left(x_1^{2q_1} + \sum_2^n x_i^{2q_i} \right)^{-\frac{1}{2}\sum_1^n \frac{1}{p_i}} dx_2 ... dx_n .$$

Here we change variables and remind condition of theorem $\sum_1^n q_i^{-1} = \sum_1^n p_i^{-1}$.

It is easy to see

$$\int\limits_{R^{n-1}} \left(x_1^{2q_1} + \sum_2^n x_i^{2q_i} \right)^{-\frac{1}{2}\sum_1^n \frac{1}{p_i}} dx_2 ... dx_n = \int\limits_{R^{n-1}} x_1^{-1} \left(1 + \sum_2^n y_i^{2q_i} \right)^{-\frac{1}{2}\sum_1^n \frac{1}{p_i}} dy .$$

From here

$$K_2 = \frac{1}{2} \ln 2 \int\limits_{R^{n-1}} \left(1 + \sum_1^n y_i^{2q_i} \right)^{-\frac{1}{2}\sum_1^n \frac{1}{p_i}} dy .$$

Making substitution $y_i = t_i^{\frac{1}{q_i}}$ $(i = 2,..,n)$ and passing to spherical coordinates $t_i = rf_i(\varphi_1,...,\varphi_{n-1})$ $(i = 2,...,n)$ we obtain

$$K = \frac{1}{p_1} \ln 2 \int\limits_{R^{n-1}} \left(1 + \sum_2^n t_i^2 \right)^{-\frac{1}{2}\sum \frac{1}{p_i}} \frac{1}{q_2} .... \frac{1}{q_n} t_2^{\frac{1}{q_2}-1} .. t_n^{\frac{1}{q_n}-1} dt_2 ... dt_n =$$

$$= \frac{1}{p_1} \ln 2 \cdot c_1 \int\limits_0^\infty \left(1 + r^2 \right)^{-\frac{1}{2}\sum_1^n \frac{1}{p_i}} r^{\sum_2^n \frac{1}{q_i} - 1} dr =$$

$$= c \int\limits_0^\infty \frac{\left(r^2 \right)^{\frac{1}{2}\sum_1^n \frac{1}{p_i} - \frac{1}{2q_1} - \frac{1}{2}}}{\left(1 + r^2 \right)^{\frac{1}{2}\sum_{i=1}^n \frac{1}{p_i}}} dr.$$

From condition $\left| \frac{r^2}{1 + r^2} \right| < 1$ on $[0, \infty)$ we obtain

$$K \le c \int\limits_1^\infty r^{-\frac{1}{q_1} - 1} dr + c_1, \text{ where } c_1 = \int\limits_0^1 \frac{\left(r^2 \right)^{\frac{1}{2}\sum_1^n \frac{1}{p_i} - \frac{1}{2q_1} - \frac{1}{2}}}{\left(1 + r^2 \right)^{\frac{1}{2}\sum_1^n \frac{1}{p_i}}} dr$$

and $\int\limits_1^\infty r^{-\frac{1}{q_1} - 1} dr$ converges.

Estimating all integrals $\int\limits_{\Pi'_\mu(j)} \left(\sum_1^n x_i^{2q_i} \right)^{-\frac{1}{2}\sum_1^n \frac{1}{p_i}} dx$ consequently, we estimate integral

$$\int\limits_{\Pi'_\mu} \left(\sum_1^n x_i^{2q_i} \right)^{-\frac{1}{2}\sum_1^n \frac{1}{p_i}} dx ,$$

$$\text{i.e. } \int_{\Pi'_\mu} \left(\sum_1^n x_i^{2q_i} \right)^{-\frac{1}{2}\sum_1^n \frac{1}{p_i}} dx < C,$$

where C doesn't depend on μ.

Now estimate first multiplier in right-hand side o (3).

For integral $\int_{\Pi'_1} |\varphi|^l dx$ we apply theorem on embedding 10.2 from [5]. Let us verify conditions of this theorem. Here $\alpha = (0,\dots,0)$, $p_1 = p_2 = \dots = p_n = 2$ $\vec{p} = (p_1,\dots,p_n)$, $\vec{q} = (l,\dots,l)$, $l = \frac{2t}{t-2}$, $t = \sum_1^n \frac{1}{p_i}$. Vector $\vec{l} = (l_1,\dots,l_n)$ in theorem 10.2 in our case is $\vec{p}$.

Thus,

$$\chi = \left| \left(\alpha + \frac{1}{\vec{p}} - \frac{1}{q} \right) : \vec{l} \right| = 1 \qquad (\chi \text{ in theorem 10.2}).$$

$\frac{2t}{t-2} > 2$ means, that $1 < p_n < \infty$, i.e. conditions of theorem 10.2 [5] are valid. By this theorem

$$\left(\int_{\Pi'_1} |\varphi|^l dx \right)^{\frac{1}{l}} \le c_1 \left(\int_{\Pi'_1} |\varphi|^2 dx \right)^{\frac{1}{2}} + c_1 \left(\int_{\Pi_1} \sum_1^n \left| \frac{\partial^{p_i} u}{\partial x_i^{p_i}} \right|^2 dx \right)^{\frac{1}{2}}$$

where c_1 doesn't depend on φ, but on diameter of Π'_1.

Taking into account that in Π'_1 $\bar{c}_3 < \sum_1^n x_i^{2p_i} < \bar{c}_4$ we obtain

$$\left(\int_{\Pi'_1} |\varphi|^l dx \right)^{\frac{l}{2}} \le c_3 \int_{\Pi'_1} \frac{|\varphi|^2}{\sum_1^n x_i^{2p_i}} dx + c_3 \int_{\Pi'_1} \sum_{k=1}^n \left| \frac{\partial^{p_i} \varphi}{\partial x_i^{p_i}} \right|^2 dx. \tag{4}$$

We make substitution $x_j = \mu^{-\frac{1}{p_j}} y_j$. For this, Π'_1 passes to Π'_μ. From (4) we obtain:

$$\left(\int_{\Pi'_\mu} |\varphi|^l dy \right)^{\frac{2}{l}} \le c_3 \int_{\Pi'_\mu} \frac{|\varphi|^2}{\sum_1^n y^{2p_i}} dy + c_3 \int_{\Pi'_\mu} \sum_1^n \left| \frac{\partial^{p_i} \varphi}{\partial y_i^{p_i}} \right|^2 dy,$$

where c_3 doesn't depend on φ and μ.

Thus, taking (3) into account and summing up last inequalities $\mu = 2^{-m}$, $m = \pm 1, \pm 2, \dots$ we obtain

$$\int_{R^n} \left(\sum_1^n x_i^{2q_i} \right)^{-1} |\varphi|^2 dx \le c \int_{R^n} \frac{|\varphi|^2}{\sum_1^n y^{2p_i}} dy + c \int_{R^n} \sum_1^n \left| \frac{\partial^{p_i} \varphi}{\partial y_i^{p_i}} \right| dy.$$

From lemma we obtain

$$\int_{R^n}\left(\sum_1^n x_i^{2q_i}\right)^{-1}|\varphi|^2 dx \le c\int_{R^n}\sum_1^n\left|\frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}}\right|^2 dx. \qquad (5)$$

From inequality (2) we obtain

$$(L_1\varphi,\varphi)\ge \bar{c}_1\int_{R^n}\sum_1^n\left|\frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}}\right|^2 dx-\bar{c}_2\bar{\eta}_q\int_{R^n}\sum_{i=1}^n\left|\frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}}\right|^2 dx=$$

$$=\int_{R^n}\sum_{i=1}^n\left|\frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}}\right|^2 dx \quad (\bar{c}_1-\bar{c}_2\bar{\eta}_q) \text{ and if } \bar{\eta}_q<\frac{\bar{c}_1}{\bar{c}_2},$$

then $(L_1\varphi,\varphi)>0$ for any $\varphi\in \mathring{C}^\infty(R^n)$.

Thus, there exists such $\bar{\eta}_q$, that if $\left(\sum_{i=1}^n x_i^{2q_i}\right)|Q_-(x)|<\bar{\eta}_q$, then operator L_1 is positive $L_1: L_2(R^n)\to L_2(R^n)$.

Now we return to statement of theorem.

Let

$$\overline{\lim_{|x|\to\infty}}\left(\sum_1^n x_i^{2p_i}\right)|Q_-(x)|\le\bar{\eta}_q$$

where $\bar{\eta}_q$ is already chosen.

It means, that there exist enough big parallelepiped Π with center at the origin, outside of which

$$\left(\sum_{i=1}^n x_i^{2q_i}\right)|Q_-(x)|\le\bar{\eta}_q.$$

By condition of theorem $|Q(x)|<M$ on all R^n. We divide parallelepiped Π onto small parallelepipeds with sides equal to $\mu^{\frac{1}{p_i}}$ $(i=1,\dots,n)$ and denote it by Π_μ^j. Let $\varphi\in\mathring{C}^\infty(R^n)$ and

$$\int_{\Pi_\mu^j}\varphi(x)x^\alpha_{\Pi_\mu^j}dx=0$$

on each parallelepiped of division. Here by $x^\alpha_{\Pi_\mu^j}$ we denote function, which coincides with function x^α on Π_μ^j and equal to zero outside of Π_μ^j, and $(\alpha,\lambda)<1$. From this condition we obtain

$$\int_\Pi\varphi(x)x^\alpha_\Pi dx=0 \text{ for each } (\alpha,\lambda)<1.$$

By Poincare inequality [6]

$$\int_\Pi|\varphi|^2dx=c\int_\Pi\sum_1^n\left|\frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}}\right|^2dx,$$

where c dependents on diameter of Π, but not on φ.

Let $\Pi_\mu^0 = \left\{ x : |x_i| < \mu^{\frac{1}{p_i}},\ \mu > 0 \right\}$ and $\int\limits_{\Pi_\mu^0} \varphi(x) x^\alpha dx = 0$.

If we make substitution $x_i = \mu^{\frac{1}{p_i}} y_i$, then Poincare inequality transforms to the following form

$$\int\limits_{\Pi_\mu^0} |\varphi|^2 dx \le c\mu^2 \int\limits_{\Pi_\mu^0} \sum_1^n \left| \frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}} \right|^2 dx, \tag{6}$$

where c is the same constant, doesn't depend on μ. Further, let

$$\overset{\bullet}{Q}_- = \begin{cases} Q_- & x \,\overline{\in}\, \Pi \\ 0 & x \in \Pi \end{cases}, \quad \overset{\bullet\bullet}{Q}_- = \begin{cases} 0 & x \,\overline{\in}\, \Pi \\ Q_- & x \in \Pi \end{cases}.$$

We have

$$(L_1\varphi, \varphi) \ge c \int\limits_{R^n \setminus \Pi} \sum_1^n \left| \frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}} \right|^2 dx + c \int\limits_{\Pi} \sum_1^n \left| \frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}} \right|^2 dx -$$

$$- \int\limits_{R^n \setminus \Pi} \overset{\bullet}{Q}_- |\varphi|^2 dx - \int\limits_{\Pi} \overset{\bullet\bullet}{Q}_- |\varphi|^2 dx.$$

The member

$$c \int\limits_{R^n \setminus \Pi} \sum_1^n \left| \frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}} \right|^2 dx - \int\limits_{R^n \setminus \Pi} \overset{\bullet}{Q}_- |\varphi|^2 dx > 0$$

because there is place, where

$$\left(\sum_{i=1}^n x_i^{2q_i} \right) \left| \overset{\bullet}{Q}_- \right| < \overline{\eta}_q.$$

We have already prove it. Therefore in this case

$$(L_1\varphi, \varphi) \ge c \int\limits_{\Pi} \sum_1^n \left| \frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}} \right|^2 dx - \int\limits_{R^n} \overset{\bullet\bullet}{Q}_- |\varphi|^2 dx \ge$$

$$\ge c \int\limits_{\Pi} \sum_1^n \left| \frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}} \right|^2 dx - M \int\limits_{\Pi} |\varphi|^2 dx =$$

$$= \sum_j \left[c \int\limits_{\Pi_\mu^j} \sum_1^n \left| \frac{\partial^{p_i}\varphi}{\partial x_i^{p_i}} \right|^2 dx - M \int\limits_{\Pi_\mu^j} |\varphi|^2 dx \right].$$

Further, from (6) we have

$$(L_1\varphi, \varphi) = \sum_j \left[\frac{c}{2c_1\mu} \int\limits_{\Pi_\mu^j} |\varphi|^2 dx - M \int\limits_{\Pi_\mu^j} |\varphi|^2 dx \right] > 0$$

for enough small μ. It is clear, that the number of functions $x_{\Pi_\mu^j}^\alpha$ is finite. Its linear cover is finite-dimensional subspace, and $\varphi \in \overset{\circ}{C}{}^\infty(R^n)$ is orthogonal to this subspace. Self-adjoint operator L_1 is positive on orthogonal complement of this finite-dimensional subspace. Therefore its negative spectrum is finite.

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