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ON THE REPRESENTATION OF IOST SOLUTIONS FOR DIRAC'S EQUATION SYSTEM WITH DISCONNECTED COEFFICIENTS

Abstract

The representation of Iost solution of the Dirac's equation system with disconnected coefficients is given, and the kernel of this representation is studied completely.

Consider the system of Dirac equations

$$By' + \Omega(x)y = \lambda\rho(x)y, \quad 0 \le x < \infty, \tag{1}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad \rho(x) = \begin{cases} 1, & x \ge a \\ \alpha, & 0 \le x < a \end{cases} \tag{2}$$

$1 \ne \alpha > 0$, $p(x)$ and $q(x)$ are complex-valued measurable functions, and the Euclidean norm satisfies the condition

$$\int_0^{+\infty} \|\Omega(x)\| dx < \infty. \tag{3}$$

Denote by $f(x,\lambda)$ a solution of the Iost equation (1), i.e. the solution with conditions

$$\lim_{x \to +\infty} f(x,\lambda)e^{-i\lambda x} = \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{4}$$

In the case $\rho(x) \equiv 1$ for the solution $f(x,\lambda)$ there exists a triangle representation having an important significance in the theory of inverse scattering problems (see for instance [1], [2]).

In the present paper it is shown that when $\rho(x)$ has the form (2), then and in this case for the Iost's solution, there exists integral representation.

In spite of the non-triangular property of this representation, the characters of the kernel show that it may be used also in the inverse problems theory.

It is easy to show that the vector-function

$$f^0(x,\lambda) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda\mu(x)}, \text{ where } \mu(x) = \begin{cases} x, & x > a \\ \alpha x - \alpha a, & 0 \le x \le a \end{cases},$$

is the solution of the equation (1), when $\Omega(x) \equiv 0$.

The basic result of the paper is the following

Theorem. *If the conditions (2), (3) are fulfilled, the equation (1) for all λ from the closed upper half-plane has a unique solution $f(x,\lambda)$, satisfying the condition (4) and representable in the form*

$$f(x,\lambda) = f^0(x,\lambda) + \int_{\mu(x)}^{+\infty} K(x,t) \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda t} dt,$$

and the matrix function $K(x,t)$, whose elements belong to $L_1(\mu(x),\infty)$, possesses the following properties:

a) $\int_{\mu(x)}^{+\infty}\|K(x,t)\|dt \le e^{\sigma(x)}-1$, *where* $\sigma(x)=\int_x^{+\infty}\|\Omega(t)\|dt$;

b) $\rho(x)\{BK(x,\mu(x))-K(x,\mu(x))B\}=\Omega(x)$;

c) if $\Omega(x)$ *is absolutely continuous, then*

$$B\frac{\partial}{\partial x}K(x,t)+\Omega(x)K(x,t)+\rho(x)\frac{\partial}{\partial t}K(x,t)B=0.$$

Proof. Consider the equation (1) and let $F(x,\lambda)$ be a matrix solution of this equation satisfying the condition

$$\lim_{x\to+\infty}F(x,\lambda)e^{\lambda B\mu(x)}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

It is sufficient to prove that such a solution exists, unique and representable in the form

$$F(x,\lambda)=e^{-\lambda B\mu(x)}+\int_{\mu(x)}^{+\infty}K(x,t)e^{-\lambda Bt}dt \tag{5}$$

since it holds the formula

$$F(x,\lambda)\begin{pmatrix}1\\ -i\end{pmatrix}=f(x,\lambda),\qquad e^{-\lambda B\mu(x)}\begin{pmatrix}1\\ -i\end{pmatrix}=\begin{pmatrix}1\\ -i\end{pmatrix}e^{i\lambda\mu(x)}.$$

Consider the integral equation for the solution $F(x,\lambda)$:

$$F(x,\lambda)=e^{-\lambda B\mu(x)}-\int_x^{+\infty}B\Omega(t)e^{\lambda B\mu(x)-\lambda B\mu(t)}F(t,\lambda)dt.$$

By substituting here instead of $F(x,\lambda)$ its representation (5), after simple transformation for matrix-functions

$$K_\pm(x,t)=\frac{1}{2}[K(x,t)\pm BK(x,t)B]\qquad (K(x,t)=K_+(x,t)+K_-(x,t))$$

we get the following integral equations:

$$K_+(x,t)=-\frac{1}{2\alpha}B\Omega\left(\frac{t+\alpha x+\alpha a-a}{2\alpha}\right)-\int_x^{\frac{t+\alpha x+\alpha a-a}{2\alpha}}B\Omega(\xi)K_-(\xi,t-\alpha\xi+\alpha x)d\xi,$$

if $0<x<a,\ \alpha x-\alpha a+a<t<-\alpha x+\alpha a+a$;

$$K_+(x,t)=-\frac{1}{2}B\Omega\left(\frac{t+\alpha x-\alpha a+a}{2}\right)-\int_x^a B\Omega(\xi)K_-(\xi,t-\alpha\xi+\alpha x)d\xi-$$

$$-\int_a^{\frac{t+\alpha x-\alpha a+a}{2}}B\Omega(\xi)K_-(\xi,t-\xi+\alpha x-\alpha a+a)d\xi,$$

if $0<x<a,\ t>-\alpha x+\alpha a+a$;

$$K_-(x,t)=-\int_x^a B\Omega(\xi)K_-(\xi,t+\alpha\xi-\alpha x)d\xi-$$

$$-\int_a^{+\infty}B\Omega(\xi)K_+(\xi,t+\xi-\alpha x+\alpha a-a)d\xi,$$

if $0<x<a,\ t>\alpha x-\alpha a+a$;

$$K_+(x,t)=-\frac{1}{2}B\Omega\left(\frac{x+t}{2}\right)-\int_x^{\frac{x+t}{2}}B\Omega(\xi)K_-(\xi,t+x-\xi)d\xi\,,$$

$$K_-(x,t)=-\int_x^{+\infty}B\Omega(\xi)K_+(\xi,t-x+\xi)d\xi\,,$$

if $t>x>a$.

We solve the system of integral equations by sequential approximations method. First of all we assume that $x>0$. Supposing

$$K_0^-(x,t)=0,\ \ K_0^+(x,t)=-\frac{1}{2}B\Omega\left(\frac{x+t}{2}\right),$$

$$K_n^-(x,t)=-\int_x^{+\infty}B\Omega(\xi)K_{n-1}^+(\xi,t-x+\xi)d\xi,$$

$$K_n^+(x,t)=-\int_x^{\frac{x+t}{2}}B\Omega(\xi)K_{n-1}^-(\xi,t+x-\xi)d\xi,\quad n=1,2,...$$

and applying mathematical induction method it is easy to show that

$$K_{2n-1}^+(x,t)=0,\ \ K_{2n}^-(x,t)=0,$$

$$\int_x^{+\infty}\left\|K_{2n}^+(x,t)\right\|dt\le\frac{\{\sigma(x)\}^{2n+1}}{(2n+1)!},\quad \int_x^{+\infty}\left\|K_{2n-1}^-(x,t)\right\|dt\le\frac{\{\sigma(x)\}^{2n}}{(2n)!}.$$

Therefore, the series

$$K^{\pm}(x,\cdot)\overset{def}{=}\sum_{n=0}^{+\infty}K_n^{\pm}(x,\cdot)$$

converge in the space $L_1(x,\infty)$ uniformly in $x>a$ and

$$\int_x^{+\infty}\left\|K^+(x,t)\right\|dt\le sh\sigma(x),\quad \int_x^{+\infty}\left\|K^-(x,t)\right\|dt\le ch\sigma(x)-1\,.$$

Consequently

$$\int_x^{+\infty}\left\|K(x,t)\right\|dt\le\int_x^{+\infty}\left\|K^+(x,t)\right\|dt+\int_x^{+\infty}\left\|K^-(x,t)\right\|dt\le e^{\sigma(x)}-1\,.$$

Now assume that $0<x<a$. In this case, we can found a solution of the integral equations system by the sequential approximations method, by putting

$$K^{\pm}(x,\cdot)=\sum_{n=0}^{+\infty}K_n^{\pm}(x,\cdot),$$

where

$$K_0^-(x,t)=0,\ \ t>\alpha x-a\alpha+a$$

$$K_0^+(x,t)=\begin{cases}-\dfrac{1}{2\alpha}B\Omega\left(\dfrac{t+\alpha x+\alpha a-a}{2\alpha}\right), & \alpha x-a\alpha+a<t<-\alpha x+a\alpha+a,\\ -\dfrac{1}{2}B\Omega\left(\dfrac{t+\alpha x-\alpha a+a}{2\alpha}\right), & t>-\alpha x+a\alpha+a,\end{cases}$$

$$K_n^-(x,t)=-\int_x^{a}B\Omega(\xi)K_{n-1}^+(\xi,t+\alpha\xi-\alpha x)d\xi-$$

$$-\int_a^{+\infty} B\Omega(\xi)K_{n-1}^{+}(\xi,t+\xi-\alpha x+\alpha a-a)d\xi,$$

$$K_n^{+}(x,t)=\begin{cases}-\int_x^{\frac{t+\alpha x+\alpha a-a}{2a}} B\Omega(\xi)K_{n-1}^{-}(\xi,t-\alpha\xi+\alpha x)d\xi, & \alpha x-\alpha a+a<t<-\alpha x+a\alpha+a,\\ -\int_x^{a} B\Omega(\xi)K_{n-1}^{-}(\xi,t-\alpha\xi+\alpha x)d\xi- & \\ -\int_a^{\frac{t+\alpha x+\alpha a+a}{2}} B\Omega(\xi)K_{n-1}^{-}(\xi,t-\xi+\alpha x-\alpha a+a)d\xi, & t>-\alpha x+a\alpha+a,\end{cases}$$

$n=1,2,\dots$

Applying mathematical induction method we have

$$K_{2n-1}^{+}(x,t)=K_{2n}^{-}(x,t)=0,\quad n=1,2,\dots,$$

$$\int_{\alpha x-\alpha a+a}^{+\infty}\left\|K_{2n}^{+}(x,t)\right\|dt\le\frac{\{\sigma(x)\}^{2n+1}}{(2n+1)!},\qquad \int_{\alpha x-\alpha a+a}^{+\infty}\left\|K_{2n-1}^{-}(x,t)\right\|dt\le\frac{\{\sigma(x)\}^{2n}}{(2n)!}$$

Consequently for $0<x<a$ for the representation kernel (5) we get

$$\int_{\alpha x-\alpha a+a}^{+\infty}\left\|K(x,t)\right\|dt\le e^{\sigma(x)}-1.$$

Thus, the statements of the theorem, and estimates have been proved. The remaining statements of the theorem are directly established by starting from the equations system with respect to $K^{\pm}$.

References

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Received November 3, 1999; Revised December 30, 1999.
Translated by Aliyeva E.T.